Handbook of Differential Equations

Stationary Partial Differential Equations

VOLUME 4

Edited by

Michel Chipot

HANDBOOK OF DIFFERENTIAL EQUATIONS

STATIONARY PARTIAL DIFFERENTIAL EQUATIONS

VOLUME IV

This page intentionally left blank

HANDBOOK OF DIFFERENTIAL EQUATIONS

STATIONARY PARTIAL DIFFERENTIAL EQUATIONS

Volume IV

Edited by

M. CHIPOT
Institute of Mathematics, University of Zürich, Zürich, Switzerland

Amsterdam • Boston • Heidelberg • London • New York • Oxford
Paris • San Diego • San Francisco • Singapore • Sydney • Tokyo

North-Holland is an imprint of Elsevier
Radarweg 29, PO Box 211, 1000 AE Amsterdam, The Netherlands
The Boulevard, Langford Lane, Kidlington, Oxford OX5 1GB, UK

First edition 2007

Copyright © 2007 Elsevier B.V. All rights reserved

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means electronic, mechanical, photocopying, recording or otherwise without the prior written permission of the publisher

Permissions may be sought directly from Elsevier's Science & Technology Rights Department in Oxford, UK: phone (+44) (0) 1865 843830; fax (+44) (0)1865 853333; email: permissions@elsevier.com. Alternatively you can submit your request online by visiting the Elsevier web site at http://elsevier.com/locate/permissions, and selecting *Obtaining permission to use Elsevier material*

Notice
No responsibility is assumed by the publisher for any injury and/or damage to persons or property as a matter of products liability, negligence or otherwise, or from any use or operation of any methods, products, instructions or ideas contained in the material herein. Because of rapid advances in the medical sciences, in particular, independent verification of diagnoses and drug dosages should be made

Library of Congress Cataloging-in-Publication Data
A catalog record for this book is available from the Library of Congress

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library

ISBN-13: 978-0-444-53036-3
Set ISBN: 0 444 51743-x

For information on all North-Holland publications
visit our web site at books.elsevier.com

Printed and bound in The Netherlands

07 08 09 10 11 10 9 8 7 6 5 4 3 2 1

Preface

This handbook is the volume IV in a series devoted to stationary partial differential equations. As the preceding volumes, it is a collection of self contained, state-of-the-art surveys written by well-known experts in the field.

The topics covered by this volume include rearrangements techniques and applications, Liouville-type theorems, similarity solutions of degenerate boundary layer equations, monotonicity and compactness methods for nonlinear variational inequalities, stationary Navier–Stokes flow in two dimensional channels, the investigation of singular phenomena in nonlinear elliptic problems. It includes also a very complete study of the maximum principles for elliptic partial differential equations. I hope that these surveys will be useful for both beginners and experts and help to the diffusion of these recent deep results in mathematical science.

I would like to thank all the contributors for their elegant articles. I also thank Arjen Sevenster and Andy Deelen at Elsevier for the excellent editing work of this volume.

M. Chipot

This page intentionally left blank

List of Contributors

Brock, F., *Departamento de Matemáticas, Universidad de Chile, Casilla 653, Chile* (Ch. 1)

Farina, A., *LAMFA, CNRS UMR 6140, Université de Picardie Jules Verne, Faculté de Mathématiques et d'Informatique, 33, rue Saint-Leu, 80039 Amiens, France* (Ch. 2)

Guedda, M., *LAMFA, CNRS UMR 6140, Université de Picardie Jules Verne, Faculté de Mathématiques et d'Informatique, 33, rue Saint-Leu 80039 Amiens, France* (Ch. 3)

Kenmochi, N., *Department of Mathematics, Chiba University, L-33 Yayoi-cho, Inage-Ku 263, Chiba, 263-8222 Japan* (Ch. 4)

Morimoto, H., *Department of Mathematics, Meiji University, 1-1-1 Higashi-mita, Tanaka-ku, Kanagawa, Kawasaki, 214 8571, Japan* (Ch. 5)

Pucci, P., *Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, Perugia, Italy* (Ch. 6)

Rădulescu, V.D., *Department of Mathematics, University of Craiova, 200585 Craiova, Romania and Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania* (Ch. 7)

Serrin, J., *Department of Mathematics, University of Minnesota, Minneapolis, MN, USA* (Ch. 6)

This page intentionally left blank

Contents

Preface v
List of Contributors vii
Contents of Volume I xi
Contents of Volume II xiii
Contents of Volume III xv

1. Rearrangements and Applications to Symmetry Problems in PDE 1
   F. Brock
2. Liouville-Type Theorems for Elliptic Problems 61
   A. Farina
3. Similarity and Pseudosimilarity Solutions of Degenerate Boundary Layer Equations 117
   M. Guedda
4. Monotonicity and Compactness Methods for Nonlinear Variational Inequalities 203
   N. Kenmochi
5. Stationary Navier–Stokes Flow in 2-D Channels Involving the General Outflow Condition 299
   H. Morimoto
6. Maximum Principles for Elliptic Partial Differential Equations 355
   P. Pucci and J. Serrin
7. Singular Phenomena in Nonlinear Elliptic Problems: From Blow-Up Boundary Solutions to Equations with Singular Nonlinearities 485
   V.D. Rădulescu

Author Index 595

Subject Index 603

This page intentionally left blank

Contents of Volume I

Preface v
List of Contributors vii

1. Solutions of Quasilinear Second-Order Elliptic Boundary Value Problems via Degree Theory 1
   C. Bandle and W. Reichel
2. Stationary Navier–Stokes Problem in a Two-Dimensional Exterior Domain 71
   G.P. Galdi
3. Qualitative Properties of Solutions to Elliptic Problems 157
   W.-M. Ni
4. On Some Basic Aspects of the Relationship between the Calculus of Variations and Differential Equations 235
   P. Pedregal
5. On a Class of Singular Perturbation Problems 297
   I. Shafrir
6. Nonlinear Spectral Problems for Degenerate Elliptic Operators 385
   P. Takáč
7. Analytical Aspects of Liouville-Type Equations with Singular Sources 491
   G. Tarantello
8. Elliptic Equations Involving Measures 593
   L. Véron

Author Index 713

Subject Index 721

This page intentionally left blank

Contents of Volume II

Preface v
List of Contributors vii
Contents of Volume I xi

1. The Dirichlet Problem for Superlinear Elliptic Equations 1
   T. Bartsch, Z.-Q. Wang and M. Willem
2. Nonconvex Problems of the Calculus of Variations and Differential Inclusions 57
   B. Dacorogna
3. Bifurcation and Related Topics in Elliptic Problems 127
   Y. Du
4. Metasolutions: Malthus versus Verhulst in Population Dynamics. A Dream of Volterra 211
   J. López-Gómez
5. Elliptic Problems with Nonlinear Boundary Conditions and the Sobolev Trace Theorem 311
   J.D. Rossi
6. Schrödinger Operators with Singular Potentials 407
   G. Rozenblum and M. Melgaard
7. Multiplicity Techniques for Problems without Compactness 519
   S. Solimini

Author Index 601

Subject Index 609

This page intentionally left blank

Contents of Volume III

Preface v
List of Contributors vii
Contents of Volume I xi
Contents of Volume II xiii

1. Elliptic Equations with Anisotropic Nonlinearity and Nonstandard Growth Conditions 1
   S. Antontsev and S. Shmarev
2. A Handbook of Γ-Convergence 101
   A. Braides
3. Bubbling in Nonlinear Elliptic Problems Near Criticality 215
   M. del Pino and M. Musso
4. Singular Elliptic and Parabolic Equations 317
   J. Hernández and F.J. Mancebo
5. Schauder-Type Estimates and Applications 401
   S. Kichenassamy
6. The Dam Problem 465
   A. Lyaghfouri
7. Nonlinear Eigenvalue Problems for Higher-Order Model Equations 553
   L.A. Peletier

Author Index 605

Subject Index 613

This page intentionally left blank

CHAPTER 1

Rearrangements and Applications to Symmetry Problems in PDE

F. Brock

Departamento de Matemáticas, Universidad de Chile, Casilla 653, Chile
E-mail: fbrock@uchile.cl

In honor of the 65th birthday of Albert Baernstein II

Contents

1. Introduction . . . 3
2. Basic definitions . . . 6
3. Rearrangements . . . 6
3.1. General properties . . . 7
3.2. Two-point rearrangement . . . 15
3.3. Symmetrizations . . . 19
4. Inequalities for symmetrizations . . . 24
5. Symmetry results . . . 37
5.1. Uniformly elliptic case . . . 37
5.2. Degenerate elliptic case . . . 42
6. Other symmetry results . . . 51
List of notations . . . 54
Acknowledgement . . . 55
References . . . 55

Abstract

This survey is meant as a general introduction to the theory of rearrangements on $\mathbb{R}^N$. We give proofs of all the basic integral inequalities for symmetrization, and a number of applications to symmetry problems in PDE. A particular rôle plays the method of two-point rearrangement.

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 4
Edited by M. Chipot
© 2007 Elsevier B.V. All rights reserved

This page intentionally left blank

1. Introduction

Consider a variational problem of the following form

$$\text{(P)} \quad J(v) \equiv \int_\Omega \left(\frac{1}{p} |\nabla v|^p - F(x, v) \right) \mathrm{d}x \longrightarrow \text{Stat.!}, \quad v \in K, \tag{1.1}$$

where K is a closed subset of $W_0^{1,p}(\Omega)$, $p \geqslant 1$, and Ω is a domain in $\mathbb{R}^N$. The *nonnegative* minimizers of problems like (P) describe stable – so-called *ground* – states of equilibria, as they appear in plasma physics, heat conduction and chemical reactors (see [55,63,88,89, 78,107], and the references cited therein). We ask for symmetries of the solutions of (P), if F and Ω have certain "symmetries". A well-known result is the following.

Let $v^\star$ denote the Schwarz symmetrization of v (for a definition see Section 3.3). Assume that $\Omega = \mathbb{R}^N$, or $\Omega = B_R$ $(R > 0)$, $F = F(v)$ and F is continuous, and K contains only nonnegative functions and has the property that, if $v \in K$, then also $v^\star \in K$. Then

$$J(v^\star) \leqslant J(v). \tag{1.2}$$

If, in addition, problem (P) has a *unique global* minimizer u, then we obtain from (1.2) that $u = u^\star$. (Note that this means that u is radially symmetric and radially nonincreasing, i.e.

$$u = u\big(|x|\big) \text{ and } u \text{ is nonincreasing in } r \ (r = |x|).)$$

However, if the global minimizer is not unique, then the question arises whether there still holds equality in (1.2) if $v \neq v^\star$. Unfortunately, this cannot be excluded in general, as the following simple example shows (see [38]).

EXAMPLE 1.1. Let $F \equiv 0$. Then there are nonnegative smooth functions v with compact support which are *not* radially symmetric and satisfy $J(v) = J(v^\star)$. Their superlevel sets $\{v > c\}$, $c > 0$, are nested balls which, however, might be *nonconcentric*, and the set $\{\nabla v = 0\}$ has nonempty interior, that is the graph of v has "plateaus".

Physically relevant are not only the global minima but also the *local* minima and *critical points* of (P). To show symmetry properties of these functions, the above argument fails, because in general the Schwarz symmetrization $v^\star$ is not close to v. Even though one expects symmetric solutions in many cases, there are again exceptions. Here is another typical example.

EXAMPLE 1.2 (Semilinear problem for the p-Laplacian). Let B be a ball in $\mathbb{R}^N$ with center 0, $f \in C(\mathbb{R}_0^+)$, $p > 1$, and let $u \in C^2(\overline{B})$ satisfy

$$\begin{aligned} &-\Delta_p u \equiv -\nabla\big(|\nabla u|^{p-2}\nabla u\big) = f(u), \quad u > 0 \quad \text{in } B, \\ &u = 0 \quad \text{on } \partial B. \end{aligned} \tag{1.3}$$

Note that the associated variational problem is

$$\int_B \left(\frac{1}{p} |\nabla v|^p - F(u) \right) \mathrm{d}x \longrightarrow \text{Stat.!}, \quad v \in W_0^{1,p}(B), \tag{1.4}$$

where

$$F(v) := \int_0^v f(z)\,\mathrm{d}z.$$

If $p = 2$ and f is smooth then it is well known (see [64]) that

$$\begin{aligned} &u = u^\star \quad \text{and} \\ &\frac{\partial u}{\partial r} < 0 \quad \text{in } B \setminus \{0\} \ \big(r = |x|\big). \end{aligned} \tag{1.5}$$

However, if $p > 2$ or if f is not smooth, then the conclusion (1.5) holds only under some additional assumptions. Below we give a short (but not complete) list of sufficient criteria for (1.5).

(i) $p = 2$ and $f = f_1 + f_2$, where f_1 is smooth and f_2 is increasing, [64];
(ii) $p = N$ and $f(v) > 0$ for $v > 0$, [75] (see also [87] for the case $p = N = 2$);
(iii) $f \in C^1(\mathbb{R}_0^+)$ and ∇u vanishes only at 0, [17];
(iv) $f \in C^1(\mathbb{R}_0^+)$ and $1 < p \leqslant 2$, [50] and [103].

The proofs for (i), (iii) and (iv) use the so-called *Moving Plane Method* (MPM) which turned out to be a very powerful technique in proving symmetry results for positive solutions of elliptic and parabolic problems in symmetric domains during the last two decades. Important references for uniformly elliptic equations are [102,64,65,24–26,82,83,80,81], and [42]. There have also been obtained symmetry results for some degenerate elliptic equations including in particular the case of the p-Laplacian operator for $p \in (1, 2)$ during the last 5 years, see [49–51,67,52,53,103], and [54].

The (MPM) exploits comparison principles for elliptic equations while using the invariance of the equation with respect to reflections. We emphasize that the nonnegativity assumption on the solution is needed to make the method work. Furthermore, if the differential operator of the problem *degenerates* and/or the nonlinearity f in (1.3) is not smooth, then the (MPM) is often applicable only under additional assumptions on the solution. This concerns, for instance, the p-Laplacian operator for $p > 2$ (compare the cases (iii) and (iv) above).

Notice that the result (ii) was proved by combining an isoperimetric inequality and a Pohozaev-type identity. However this method is not applicable if $p \neq N$.

One can construct radially symmetric solutions of (1.3) for which the second condition in (1.5) fails if either $p > 2$ and f is smooth, or if $p \in (1, 2]$ and f is Hölder continuous (see [67]). Moreover, if $p = 2$ and f is only continuous and changes sign, then we cannot hope that the solution of (1.3) is radially symmetric. Below we give examples of solutions in the case $p \geqslant 2$ which have a plateau and two radially symmetric "shifted bumps" on it (see [33]). Note that similar examples can also be found in the recent paper [103].

Let $p \geqslant 2$, $s > 2$,

$$w(x) = \begin{cases} (1-|x|^2)^s & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| > 1, \end{cases} \quad \text{and}$$

$$v(x) = \begin{cases} 1 & \text{if } |x| < 5, \\ 1 - ((|x|^2 - 25)/11)^s & \text{if } 5 \leqslant |x| \leqslant 6. \end{cases}$$

We choose $x^1, x^2 \in B_4$ with $|x^1 - x^2| \geqslant 2$ and set

$$u(x) := v(x) + w\big(x - x^1\big) + w\big(x - x^2\big), \quad x \in B_6.$$

The graph of u is built up by three radially symmetric "mountains", one of them having a "plateau" at height 1 while the other two are congruent to each other with their "feet" lying on the plateau. After a short computation we see that u is a solution of (1.4) with $B = B_6$ and

$$f(u) := \begin{cases} (2s/11)^{p-1}(25 + 11(1-u)^{1/s})^{(p/2)-1}(1-u)^{p-(p/s)-1} \\ \quad \times \{(50/11)(p-1)(s-1) + (2ps - 2s - p + n)(1-u)^{1/s}\} \\ \qquad \text{if } 0 \leqslant u \leqslant 1, \\ (2s)^{p-1}(1-(u-1)^{1/s})^{(p/2)-1}(u-1)^{p-(p/s)-1} \\ \quad \times \{-2(s-1)(p-1) + (2ps - 2s - p + n)(u-1)^{1/s}\} \\ \quad \text{if } 1 \leqslant u \leqslant 2. \end{cases}$$

If $p = 2$ and $s > 2$ then we have $f \in C^\infty([0,2] \setminus \{1\}) \cap C^{1-(2/s)}([0,2])$. The difference quotient of f is not bounded below near $u = 1$, i.e. $f \notin C^1([0,2])$. In contrast, if $p > 2$ and $s > p/(p-2)$, then we have $f \in C^1([0,2])$.

On the other hand, the functions in the above examples are distinguished by some "local" symmetry which can be described as follows:

(LS) *Every connected component of* $\{x \in B\colon u(x) > 0,\ \nabla u \neq 0\}$ *is an annulus* A *of the form* $B_{R_2}(z) \setminus \overline{B_{R_1}(z)}$ $(R_2 > R_1 \geqslant 0,\ z \in B)$, u *is radially symmetric in* A, *that is* $u(x) = v(|x - z|)$ *and* $\partial v/\partial r < 0$ *for* $R_2 > r > R_1$.

Our aim is to obtain those weak – and also other – symmetries for stationary solutions of problems like (P), using rearrangement arguments. Our approach is closely related to the corresponding variational problems of the differential equations. But in contrast to the (MPM), we can deal with a large class of degenerate operators, and also with nonsmooth nonlinearities in the equations. Furthermore, we may sometimes drop the nonnegativity assumption provided that the solution is a *minimizer* of the corresponding variational problem.

We would like to refer the interested reader to the nice survey of H. Brezis [29], where a lot of challenging open symmetry problems in PDE can be found.

We now outline the content of our work. In Section 3.1 we investigate a general class of rearrangements on $\mathbb{R}^N$. There is no doubt that slight modifications of these results hold as well in other situations. In Section 3.2 we study the two-point rearrangement. This very

simple type of rearrangement will then be helpful in Sections 3.3 and 4 to show many properties of the Schwarz, Steiner and cap symmetrizations, including all the basic inequalities that compare an integral of some given functions with the same integral of their symmetrizations. In Section 5 we obtain symmetry properties for minimizers of elliptic variational problems, where we combine the two-point rearrangement with two important tools in PDE: the Principle of Unique continuation (Section 5.1) and the Strong Maximum Principle (Section 5.2). These results are contained in two recent articles of the author, [35,36]). Finally we briefly report on other symmetry results in Section 6, in particular those which have been obtained using the method of continuous Steiner symmetrization (see [30,32,33].

2. Basic definitions

We will assume $N \in \mathbb{N}$, $N \geqslant 2$ throughout our work. We write $x = (x_1, \dots, x_N) = (x_1, x')$, $y = (y_1, \dots, y_N) = (y_1, y'), \dots$, for points in $\mathbb{R}^N$, and $x \cdot y = \sum_{i=1}^N x_i y_i$, $|x| = \sqrt{x \cdot x}$ for the Euclidean scalar product and norm, respectively.

Let $\mathcal{L}^k$ denote k-dimensional Lebesgue measure, $1 \leqslant k \leqslant N$. and let $\mathcal{M}$ denote the set of all $\mathcal{L}$-measurable – measurable in short – sets of $\mathbb{R}^N$. By $\|\cdot\|_{p,M}$ we denote the usual norm in the space $L^p(M)$ ($M \in \mathcal{M}$), and we write $\|\cdot\|_p := \|\cdot\|_{p,\mathbb{R}^N}$ ($1 \leqslant p \leqslant +\infty$). If Ω is an open set in $\mathbb{R}^N$, and if $u \in L^\infty(\Omega)$ we define the modulus of continuity $\omega_{u,\Omega}$ by

$$\omega_{u,\Omega}(t) := \sup\big\{\big|u(x) - u(y)\big|\colon\ x, y \in \Omega,\ |x - y| < t\big\}, \quad t > 0. \tag{2.1}$$

(Here and in the following sup (inf) means ess sup (ess inf).) We also write $\omega_{u,\mathbb{R}^N} = \omega_u$. Notice that if $u \in C(\Omega)$ then u is equicontinuous on Ω iff $\lim_{t \searrow 0} \omega_{u,\Omega}(t) = 0$.

By $W^{1,p}(\Omega)$ we denote the Sobolev space of functions $u \in L^p(\Omega)$ having generalized partial derivatives $\partial u/\partial x_i \in L^p(\Omega)$, $i = 1, \dots, N$, and we denote by $W_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ in the space $W^{1,p}(\Omega)$. Usually we extend measurable functions $u : \Omega \to \mathbb{R}_0^+$ by zero outside Ω, so that $W_0^{1,p}(\Omega) \subset W^{1,p}(\mathbb{R}^N)$ in that sense (see [1]).

By $C_0^{0,1}(\Omega)$ we denote the space of Lipschitzean functions with compact support in Ω.

For any of the above function spaces, let the lower index "+" indicate the corresponding subset of nonnegative functions, e.g. $L_+^p(\mathbb{R}^N)$, $W_{0+}^{1,p}(\Omega)$, $C_{0+}^{0,1}(\Omega), \dots$.

Finally, a function $G : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called a Young function if G is continuous and convex with $G(0) = 0$.

3. Rearrangements

Since the times of Steiner [106], Hardy, Littlewood and Polya [68] rearrangements have been used to prove isoperimetric inequalities in mathematical physics. The monographs [68,96,45,39,44,93,20,72,69,85] and the surveys [101,5,76,59,18,112,113,37,84] provide many results and further references.

3.1. *General properties*

In this section we introduce a general concept for rearrangements which is appropriate for all the symmetrizations on $\mathbb{R}^N$, and we prove general properties.

We often treat measurable sets only in a.e. sense, that is we identify a set M with its equivalence class given by all measurable sets $\widetilde{M}$ with $\mathcal{L}^N(M \Delta \widetilde{M}) = 0$. If $M_1, M_2 \in \mathcal{M}$, we will then write

$$M_1 = M_2 \iff \mathcal{L}^N(M_1 \Delta M_2) = 0, \quad \text{and} \tag{3.1}$$

$$M_1 \subset M_2 \iff \mathcal{L}^N(M_1 \setminus M_2) = 0. \tag{3.2}$$

A set transformation $T : \mathcal{M} \to \mathcal{M}$ is called a rearrangement if it is monotone and measure-preserving, that is, ($M, M_1, M_2 \in \mathcal{M}$),

$$\text{if } M_1 \subset M_2, \text{ then } TM_1 \subset TM_2, \tag{3.3}$$

$$\mathcal{L}^N(M) = \mathcal{L}^N(TM), \quad \text{and} \tag{3.4}$$

$$T\mathbb{R}^N = \mathbb{R}^N. \tag{3.5}$$

Also, for any $M \in \mathcal{M}$, the set TM is called a rearrangement of M.

Some rearrangements satisfy, in addition to (3.3)–(3.5),

$$\text{if } M_1, M_2 \in \mathcal{M},\ M_1 \subset M_2, \text{ then } \mathcal{L}^N(M_2 \setminus M_1) = \mathcal{L}^N(TM_2 \setminus TM_1). \tag{3.6}$$

REMARK 3.1.

(1) The notion of rearrangement is reserved for *measurable* sets. If a rearrangement has certain smoothing properties, as for instance the symmetrizations and the two-point rearrangement (see Sections 3.2 and 3.3), then one can introduce pointwise representatives for the set TM when M is open or compact.

(2) (3.3) and (3.4) imply ($M_1, M_2 \in \mathcal{M}$),

$$T(M_1 \cap M_2) \subset TM_1 \cap TM_2, \tag{3.7}$$

$$T(M_1 \cup M_2) \supset TM_1 \cup TM_2, \tag{3.8}$$

$$\mathcal{L}^N(M_1 \setminus M_2) \geqslant \mathcal{L}^N(TM_1 \setminus TM_2) \quad \text{and} \tag{3.9}$$

$$\mathcal{L}^N(M_1 \Delta M_2) \geqslant \mathcal{L}^N(TM_1 \Delta TM_2). \tag{3.10}$$

(3) We emphasize that generally rearrangements *may or may not* satisfy (3.6) – although that property is obviously satisfied when M_1 has finite measure. The properties (3.3), (3.4) and (3.6) imply in particular, that T is *continuous from the outside*, that is, if $\{M_n\}$ is a nonincreasing sequence in $\mathcal{M}$, then

$$T\left(\bigcap_{n=1}^{\infty} M_n\right) = \bigcap_{n=1}^{\infty} TM_n. \tag{3.11}$$

Next we introduce rearrangements of measurable functions $u:\mathbb{R}^N \to \mathbb{R}$. We will usually not distinguish between u and its equivalence class given by all measurable functions which differ from u on a nullset only. We say that u belongs to $\mathcal{S}$ – the set of *'symmetrizable functions'* – if

$$\mathcal{L}^N\big(\{u > \lambda\}\big) < +\infty \quad \forall \lambda > \inf u,$$

and we set $\mathcal{S}_+ := \{u \in \mathcal{S}\colon \operatorname{ess\,inf} u = 0\}$. (Here and in the following we use the abbreviation $\{u > \lambda\} = \{x \in \mathbb{R}^N\colon\ u(x) > \lambda\}$.) Notice that $L^p_+(\mathbb{R}^N)$, $W^{1,p}_+(\mathbb{R}^N)$ $(1 \leqslant p < +\infty)$, and $C^{0,1}_{0+}(\mathbb{R}^N)$ are subsets of $\mathcal{S}_+$. On the other hand, if $u \in L^p(\mathbb{R}^N)$ for some $p \in [1,\infty)$, and if $\mathcal{L}^N(\{u > 0\}) > 0$ and $\mathcal{L}^N(\{u < 0\}) > 0$, then $u \notin \mathcal{S}$.

If $u \in \mathcal{S}$, its *distribution function* μ_u is given by

$$\mu_u(\lambda) := \mathcal{L}^N\big(\{u > \lambda\}\big), \quad \lambda \in \mathbb{R}. \tag{3.12}$$

Obviously, μ_u is a nonincreasing, right-continuous function with $\mu_u(\lambda) = +\infty\ \forall \lambda < \inf u$, $\mu_u(\lambda) = 0\ \forall \lambda > \sup u$, and $\mu_u(\lambda) < \infty\ \forall \lambda \in (\inf u, \infty)$.

We will say that two functions $u, v \in \mathcal{S}$ are *equidistributed*, $u \sim v$, if $\mu_u(\lambda) = \mu_v(\lambda)$ $\forall \lambda \in \mathbb{R}$.

The distribution function of μ_u – that is, the right-continuous inverse of μ_u – is called the *symmetric decreasing rearrangement of* u and is denoted by $u^\sharp$. It is easy to see that $u^\sharp$ is a nonincreasing, right-continuous function on $\mathbb{R}^+_0$ with $u^\sharp(0) = \sup u$, $\lim_{s\to+\infty} u^\sharp(s) = \inf u$, and

$$u^\sharp(s) = \inf\big\{\lambda \in \mathbb{R}\colon\ \mu_u(\lambda) \leqslant s\big\}, \quad s \in \mathbb{R}. \tag{3.13}$$

Let T be a rearrangement and $u:\mathbb{R}^N \to \mathbb{R}$ measurable. We define a measurable function $Tu:\mathbb{R}^N \to \mathbb{R}$ by

$$Tu(x) := \sup\big\{\lambda \in \mathbb{R}\colon\ x \in T\{u > \lambda\}\big\}, \quad x \in \mathbb{R}^N. \tag{3.14}$$

From (3.14) one obtains using monotonicity

$$\{Tu \geqslant \lambda\} \subset \bigcap_{t\colon\, t<\lambda} T\{u > t\},$$

$$\{Tu > \lambda\} \subset \bigcup_{\lambda'\colon\, \lambda'>\lambda} \bigcap_{t\colon\, t<\lambda'} T\{u > t\}, \quad \text{and}$$

$$T\{u \geqslant \lambda\} \subset \{Tu \geqslant \lambda\},$$

which implies

$$\{Tu > \lambda\} = T\{u > \lambda\} \quad \text{and} \tag{3.15}$$

$$\{Tu \geqslant \lambda\} = \bigcap_{t\colon\, t<\lambda} T\{u > t\} \quad \forall \lambda \in \mathbb{R}. \tag{3.16}$$

Furthermore, if $u \in \mathcal{S}$, or if T satisfies (3.6), then we obtain from (3.15) and (3.16) that also

$$T\{u \geqslant \lambda\} = \{Tu \geqslant \lambda\} \quad \forall \lambda \in \mathbb{R}. \tag{3.17}$$

The mapping T defined by (3.14) is again called a *rearrangement*, and the function Tu a *rearrangement of* u.

REMARK 3.2.

(1) Notice that (3.15), (3.16) together with (3.6) imply the monotonicity of T for *functions*,

$$u \leqslant v \quad \Longrightarrow \quad Tu \leqslant Tv. \tag{3.18}$$

(2) In the special case that u is a characteristic function, i.e. $u = \chi(M)$, (3.14) reduces to

$$T\chi(M) = \chi(TM). \tag{3.19}$$

Similarly, if u is a step function,

$$u = c_0 + \sum_{i=1}^{N} c_i \chi(M_i), \tag{3.20}$$

where $c_0 \in \mathbb{R},\ c_i \geqslant 0,\ M_i \in \mathcal{M},\ i = 1, \ldots, k,\ M_1 \supset M_2 \supset \cdots \supset M_k,$

then

$$Tu = c_0 + \sum_{i=1}^{N} c_i \chi(TM_i). \tag{3.21}$$

The definition of Tu can be written in a more compact form using the so-called *layer-cake formula*,

$$u = \int_0^{+\infty} \chi\big(\{u > \lambda\}\big)\,\mathrm{d}\lambda - \int_{-\infty}^{0} \chi\big(\{u \leqslant \lambda\}\big)\,\mathrm{d}\lambda, \tag{3.22}$$

i.e. u is the superposition of the characteristic functions of its level sets. Notice that the integrals in (3.22) are à la Bochner, i.e., if $0 = t_0^k < t_1^k < \cdots < t_k^k$, $0 = s_0^k > s_1^k > \cdots > s_k^k$, $k \in \mathbb{N}$, $\max\{\max\{s_{i-1}^k - s_i^k;\ t_i^k - t_{i-1}^k\}\colon\ 1 \leqslant i \leqslant k\} \to 0$, $t_k^k \to +\infty$ and $s_k^k \to -\infty$, as $k \to +\infty$, then

$$\sum_{i=1}^{k} \chi\big(\{u > t_i^k\}\big)\big(t_i^k - t_{i-1}^k\big) + \sum_{i=1}^{k} \chi\big(\{u \leqslant s_i^k\}\big)\big(s_i^k - s_{i-1}^k\big) \longrightarrow u \quad \text{in measure.}$$

Using (3.15) and (3.22), we obtain

$$Tu = \int_0^{+\infty} \chi\big(T\{u > \lambda\}\big)\,\mathrm{d}\lambda - \int_{-\infty}^{0} \chi\big(\mathbb{R}^N \setminus T\{u > \lambda\}\big)\,\mathrm{d}\lambda. \tag{3.23}$$

Notice that the second integral in (3.22) and (3.23) is zero when $u \geqslant 0$.

(3) If $u \in L^1(\mathbb{R}^N)$ then we have that

$$\int_{\mathbb{R}^N} u\,\mathrm{d}x = \int_0^{\infty} \mathcal{L}^N\big(\{u > \lambda\}\big)\,\mathrm{d}\lambda - \int_{-\infty}^{0} \mathcal{L}^N\big(\{u \leqslant \lambda\}\big)\,\mathrm{d}\lambda, \tag{3.24}$$

a variant of Fubini's Theorem. From the formulas (3.23), (3.24) we then immediately obtain the following

LEMMA 3.1. *Let T be a rearrangement and $u \in L^1(\mathbb{R}^N)$. Furthermore, let either $u \geqslant 0$, or let T satisfy property* (3.6). *Then*

$$\int_{\mathbb{R}^N} u\,\mathrm{d}x = \int_{\mathbb{R}^N} Tu\,\mathrm{d}x. \tag{3.25}$$

The next result seems to be well known as well, but I could not find a reference.

LEMMA 3.2. *Let T be a rearrangement, $u : \mathbb{R}^N \to \mathbb{R}$ measurable, and $\varphi : \mathbb{R} \to \mathbb{R}$ a non-decreasing function. Furthermore, assume that either $u \in \mathcal{S}$ or T satisfies* (3.6). *Then*

$$\varphi(Tu) = T\varphi(u). \tag{3.26}$$

PROOF. We have to show that for every $\lambda \in \mathbb{R}$,

$$\big\{T\varphi(u) > \lambda\big\} = \big\{\varphi(Tu) > \lambda\big\}.$$

There are numbers $t, \lambda_1, \lambda_2 \in \mathbb{R}$, such that

$$\lim_{s \nearrow t} \varphi(s) = \lambda_1 \leqslant \lambda \leqslant \lambda_2 = \lim_{s \searrow t} \varphi(s),$$

and obviously $\lambda_1 \leqslant \varphi(t) \leqslant \lambda_2$. We consider three cases:

(i) $\lambda = \lambda_2$. Setting $t' = \max\{s \geqslant t\colon\ \varphi(\sigma) = \lambda_2\ \forall \sigma \in [t, s]\}$, we have then, for every $s \in \mathbb{R}$,

$$s > t' \quad \Longleftrightarrow \quad \varphi(s) > \lambda.$$

By (3.16) this means that

$$\big\{T\varphi(u) > \lambda\big\} = T\big\{\varphi(u) > \lambda\big\} = T\{u > t'\} = \{Tu > t'\} = \big\{\varphi(Tu) > \lambda\big\}. \tag{3.27}$$

(ii) $\varphi(t) \leqslant \lambda < \lambda_2$. Then we have for every $s \in \mathbb{R}$,

$$s > t \quad \Longleftrightarrow \quad \varphi(s) > \lambda,$$

so that (3.27) holds with t' replaced by t.

(iii) $\lambda_1 \leqslant \lambda < \varphi(t)$. Then we have for every $s \in \mathbb{R}$,

$$s \geqslant t \quad \Longleftrightarrow \quad \varphi(s) > \lambda,$$

Together with (3.15)–(3.17) this leads to

$$\big\{T\varphi(u) > \lambda\big\} = T\big\{\varphi(u) > \lambda\big\} = T\{u \geqslant t\} = \{Tu \geqslant t\} = \big\{\varphi(Tu) > \lambda\big\}.$$

The lemma is proved. □

The following important theorem can be seen as a form of *Cavalieri's principle* (see e.g. [72]).

THEOREM 3.1. *Let T be a rearrangement, $f : \mathbb{R} \to \mathbb{R}$ continuous or nondecreasing, $u : \mathbb{R}^N \to \mathbb{R}$ measurable, and $f(u) \in L^1(\mathbb{R}^N)$. Furthermore, assume either that $u \in \mathcal{S}$, or that T satisfies* (3.6). *Then $f(Tu) \in L^1(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} f(u)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(Tu)\,\mathrm{d}x. \tag{3.28}$$

PROOF. If f is nondecreasing, then (3.28) follows from Lemmata 3.1 and 3.2.

Next let f be continuous. For every $\lambda > 0$, the set $G := \{t \in \mathbb{R}\colon\ f(t) > \lambda\}$ is open. Hence there is representation $G = \bigcup_{i=1}^{\infty} I_i$, with mutually disjoint intervals $I_i = (a_i, b_i)$, $i = 1, 2, \ldots$. Since $f(u) \in L^1(\mathbb{R}^N)$, and using (3.15)–(3.17), we then deduce,

$$\begin{aligned} +\infty > \mathcal{L}^N\big(\big\{f(u) > \lambda\big\}\big) &= \sum_{i=1}^{\infty} \mathcal{L}^N\big(\{a_i < u < b_i\}\big) \\ &= \sum_{i=1}^{\infty} \mathcal{L}^N\big(\{a_i < Tu < b_i\}\big) = \mathcal{L}^N\big(\big\{f(Tu) > \lambda\big\}\big) \quad \forall \lambda > 0, \end{aligned} \tag{3.29}$$

and similarly,

$$+\infty > \mathcal{L}^N\big(\big\{f(u) \leqslant \lambda\big\}\big) = \mathcal{L}^N\big(\big\{f(Tu) \leqslant \lambda\big\}\big) \quad \forall \lambda < 0. \tag{3.30}$$

Now the assertion follows from (3.29), (3.30) and Lemma 3.1. □

Theorem 3.2 below has many applications, too. It has been shown by Crowe, Rosenbloom and Zweibel [48] for the Schwarz symmetrization of nonnegative functions. However, it has been observed in [34] and in [120] that their proof carries over to arbitrary rearrangements without difficulties.

THEOREM 3.2. *Let* $F \in C(\mathbb{R}^2)$, $F(0,0) = 0$, *and*

$$F(A,B) - F(a,B) - F(A,b) + F(a,b) \geqslant 0$$
$$\forall a, b, A, B \in \mathbb{R} \text{ with } a \leqslant A,\ b \leqslant B, \tag{3.31}$$

and let T *be a rearrangement. Furthermore, let* u, v *measurable such that* $F(u,0)$, $F(0,v)$, $F(u,v) \in L^1(R^N)$. *Finally, let either*

(i) $u, v \in \mathcal{S}_+$, *or*

(ii) T *satisfies* (3.6) *and* $\mathcal{L}^N(\{|u| > t\}) < \infty$, $\mathcal{L}^N(\{|v| > t\}) < \infty$, $\forall t > 0$. *Then*

$$\int_{\mathbb{R}^N} F(u,v)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} F(Tu, Tv)\,\mathrm{d}x. \tag{3.32}$$

REMARK 3.3. If $F \in C^2$ then (3.31) is equivalent to

$$\frac{\partial^2 F(\lambda,\mu)}{\partial\lambda\partial\mu} \geqslant 0 \quad \forall \lambda, \mu \in \mathbb{R}. \tag{3.33}$$

Furthermore, the conditions on u, v in (ii) are satisfied if, for instance, $u, v \in L^p(\mathbb{R}^N)$ for some $p \in [1, \infty)$.

PROOF OF THEOREM 3.2. We prove the assertion in the case (ii). The proof in the case (i) is similar, and it will be omitted. In view of (3.31), there exists a nonnegative measure, denoted by $\mathrm{d}F_{\lambda\mu}$, such that (see [100, pp. 64–68]),

$$\begin{aligned} F(s,t) - F(s,0) - F(0,t) &= \int_0^t \int_0^s \mathrm{d}F_{\lambda\mu} \\ &= \iint_{\mathbb{R}^2} k(s,t;\lambda,\mu)\,\mathrm{d}F_{\lambda\mu} \quad \forall s,t \in \mathbb{R}, \end{aligned} \tag{3.34}$$

where

$$k(s,t;\lambda,\mu) = \begin{cases} \chi(0,s)(\lambda)\chi(0,t)(\mu) & \text{if } s \geqslant 0, t \geqslant 0, \\ -\chi(0,s)(\lambda)\chi(t,0)(\mu) & \text{if } s \geqslant 0, t \leqslant 0, \\ -\chi(s,0)(\lambda)\chi(0,t)(\mu) & \text{if } s \leqslant 0, t \geqslant 0, \\ \chi(s,0)(\lambda)\chi(t,0)(\mu) & \text{if } s \leqslant 0, t \leqslant 0. \end{cases} \tag{3.35}$$

Notice that in the case $F \in C^2$ we have

$$\mathrm{d}F_{\lambda\mu} = \frac{\partial^2 F(\lambda,\mu)}{\partial\lambda\partial\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu,$$

which is obviously nonnegative by (3.33).

Choosing $s = u(x)$ and $t = v(x)$ in (3.34) and then integrating we find

$$\int_{\mathbb{R}^N} F(u,v)\,\mathrm{d}x = \int_{\mathbb{R}^N} F(u,0)\,\mathrm{d}x + \int_{\mathbb{R}^N} F(0,v)\,\mathrm{d}x + \iint\limits_{\mathbb{R}^2} K(\lambda,\mu)\,\mathrm{d}F_{\lambda,\mu}, \tag{3.36}$$

where

$$K(\lambda,\mu) = \begin{cases} \mathcal{L}^N(\{u>\lambda\}\cap\{v>\mu\}) & \text{if } \lambda>0,\ \mu>0,\\ -\mathcal{L}^N(\{u>\lambda\}\cap\{v<\mu\}) & \text{if } \lambda>0,\ \mu<0,\\ -\mathcal{L}^N(\{u<\lambda\}\cap\{v>\mu\}) & \text{if } \lambda<0,\ \mu>0,\\ \mathcal{L}^N(\{u<\lambda\}\cap\{v<\mu\}) & \text{if } \lambda<0,\ \mu<0 \end{cases} \tag{3.37}$$

and an analogous expression for $\int_{\mathbb{R}^N} F(Tu,Tv)\,\mathrm{d}x$. Now one obtains (3.32) from these representations and from the inequalities below, which follow from (3.6), (3.15) and (3.17),

$$\begin{aligned} &\mathcal{L}^N\big(\{u>\lambda\}\cap\{v>\mu\}\big) \leqslant \mathcal{L}^N\big(\{Tu>\lambda\}\cap\{Tv>\mu\}\big),\\ &\mathcal{L}^N\big(\{u>\lambda\}\cap\{v<-\mu\}\big) \geqslant \mathcal{L}^N\big(\{Tu>\lambda\}\cap\{Tv<-\mu\}\big),\\ &\mathcal{L}^N\big(\{u<-\lambda\}\cap\{v>\mu\}\big) \geqslant \mathcal{L}^N\big(\{Tu<-\lambda\}\cap\{Tv>\mu\}\big), \quad \text{and}\\ &\mathcal{L}^N\big(\{u<-\lambda\}\cap\{v<-\mu\}\big) \leqslant \mathcal{L}^N\big(\{Tu<-\lambda\}\cap\{Tv<-\mu\}\big)\\ &\quad \forall \lambda,\mu>0. \end{aligned}$$

□

Choosing $F(s,t) := -G(|s-t|)$ with a Young function G in (3.32) we obtain

COROLLARY 3.1 (Nonexpansivity of the rearrangement). *Let G be a Young function, T a rearrangement, and let u, v be measurable such that $G(|u|), G(|v|), G(|u-v|) \in L^1(\mathbb{R}^N)$. Furthermore, assume that one of the conditions* (i) *or* (ii) *from Theorem* 3.2 *is satisfied. Then*

$$\int_{\mathbb{R}^N} G\big(|Tu-Tv|\big)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} G\big(|u-v|\big)\,\mathrm{d}x. \tag{3.38}$$

Furthermore, (3.32) with $F(u,v) = uv$ gives the following inequality which is attributed to Hardy and Littlewood, see [68].

COROLLARY 3.2. *Let $u, v \in L^2(\mathbb{R}^N)$, and let T be a rearrangement. Furthermore, let either u and v be nonnegative, or T satisfies* (3.6). *Then*

$$\int_{\mathbb{R}^N} uv\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} TuTv\,\mathrm{d}x. \tag{3.39}$$

Rearrangements are nonexpansive in $L^\infty(\mathbb{R}^N)$, too.

COROLLARY 3.3. *Let $u, v \in L^\infty(\mathbb{R}^N)$, and let T be a rearrangement. Furthermore, let either $u \in \mathcal{S}$ or T satisfies* (3.6). *Then*

$$\|Tu - Tv\|_\infty \leqslant \|u - v\|_\infty. \tag{3.40}$$

PROOF. Let $C := \|u - v\|_\infty$. Then we have $-C \leqslant u - v \leqslant C$ a.e. on $\mathbb{R}^N$. By the monotonicity (3.18) and (3.26) this means that

$$Tv - C = T(v - C) \leqslant Tu \leqslant T(v + C) = Tv + C, \quad \text{a.e. on } \mathbb{R}^N,$$

which implies (3.40). □

REMARK 3.4. For applications it is useful to define rearrangements of functions which are merely defined on a set $M \in \mathcal{M}$. This can be done as follows:

Let T a rearrangement, and let $u : M \to \mathbb{R}$ measurable. Choosing any number $c \in \mathbb{R} \cup \{-\infty\}$, with $c \leqslant \inf_M u$, we extend u onto $\mathbb{R}^N$ by setting

$$u(x) := c \quad \text{if } x \in \mathbb{R}^N \setminus M, \tag{3.41}$$

and define Tu by (3.14). (Notice that in fact Tu does not depend on the particular choice of c. Notice also that if $\mathcal{L}^N(M) < \infty$, and if $c > -\infty$ then this implies $u, Tu \in \mathcal{S}$.) Since

$$\{Tu > \lambda\} \subset TM \quad \forall \lambda > c, \quad \text{and} \tag{3.42}$$

$$Tu(x) = c \quad \forall x \in \mathbb{R}^N \setminus TM, \tag{3.43}$$

we will henceforth not distinguish between the function Tu defined on $\mathbb{R}^N$ and its restriction $Tu|_{TM}$.

The following lemma can be proved analogously as Theorems 3.1 and 3.2, and we leave the details to the reader.

LEMMA 3.3. *Let $M \in \mathcal{M}$ with $\mathcal{L}^N(M) < \infty$, let u, v be measurable functions on M, and let T be a rearrangement. If $f : \mathbb{R} \to \mathbb{R}$ is continuous or nondecreasing and $f(u) \in L^1(M)$ then*

$$\int_M f(u)\,\mathrm{d}x = \int_{TM} f(Tu)\,\mathrm{d}x. \tag{3.44}$$

Furthermore, if $F \in C(\mathbb{R})$ satisfies condition (3.31) *of Theorem* 3.2, *and if $F(u, 0)$, $F(0, v)$, $F(u, v) \in L^1(M)$ then*

$$\int_M F(u, v)\,\mathrm{d}x \leqslant \int_{TM} F(Tu, Tv)\,\mathrm{d}x. \tag{3.45}$$

3.2. *Two-point rearrangement*

We now study the two-point rearrangement. This simple rearrangement can be used to provide simple proofs of many geometric and functional inequalities which are related to symmetrizations (see [68,123,22,57,58,19,18] and [37]).

Let Σ be some $(N-1)$-dimensional affine hyperplane in $\mathbb{R}^N$ and assume that H is one of the two open halfspaces into which $\mathbb{R}^N$ is subdivided by Σ. Let σ_H denote reflection in $\Sigma = \partial H$, that is if $H = \{y\colon\ y \cdot \nu < \lambda\}$, ($\nu$ exterior unit normal to ∂H, $\lambda \in \mathbb{R}$), and if $x \in \mathbb{R}^N$ then $\sigma_H x := x + 2(\lambda - x \cdot \nu)\nu$. Furthermore, if $u : \mathbb{R}^N \to \mathbb{R}$ is measurable, then we define its reflexion $\sigma_H u$ by

$$\sigma_H u(x) := u(\sigma_H x), \quad x \in \mathbb{R}^N,$$

and its *two-point rearrangement* $T^H u$ (with respect to H) by

$$T^H u(x) := \begin{cases} \max\{u(x);\ u(\sigma_H x)\} & \text{if } x \in H, \\ \min\{u(x);\ u(\sigma_H x)\} & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases} \tag{3.46}$$

We then define the two-point rearrangement of a set $M \in \mathcal{M}$ via the two-point rearrangement of its characteristic function, i.e.

$$\chi\big(T^H M\big) := T^H\big(\chi(M)\big). \tag{3.47}$$

With these definitions, it is easy to see that the mapping T^H is indeed a rearrangement in the sense of Section 3.1, and it satisfies (3.6). For the sake of simplicity, we will often use the subscript "$_H$" to denote one of the rearranged objects, i.e. we write u_H, M_H and x_H for $T^H u$, $T^H M$ and $\sigma_H x$, respectively.

REMARK 3.5. In the case that u is continuous and M is open or closed, equations (3.46) and (3.47) have to be understood in the pointwise sense. It then follows that if u is continuous on $\mathbb{R}^N$ then so does $T^H u$, and if M is an open or closed subset of $\mathbb{R}^N$, then $T^H M$ is open, respectively closed, too. The following, more precise statement was proved in [19].

LEMMA 3.4. *Let H be some halfspace and let Ω be a domain in $\mathbb{R}^N$ with $\Omega = \sigma_H \Omega$. If $u \in C(\Omega) \cap L^\infty(\Omega)$, or $u \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, then so does u_H and*

$$\omega_{u_H,\Omega} \leqslant \omega_{u,\Omega}. \tag{3.48}$$

PROOF. For any $t > 0$ and $\varepsilon \in (0, \omega_{u_H,\Omega}(t))$, we find $x, y \in \Omega$ such that $|x - y| \leqslant t$ and $|u_H(x) - u_H(y)| \leqslant \omega_{u_H,\Omega}(t) - \varepsilon$. Now consider two cases.

(i) Let $x, y \in H \cap \Omega$ or $x, y \in \Omega \setminus H$. Then one verifies easily the inequality

$$\begin{aligned} &\max\big\{\big|u(x) - u(y)\big|;\ \big|u(x_H) - u(y_H)\big|\big\} \\ &\quad \geqslant \max\big\{\big|u_H(x) - u_H(y)\big|;\ \big|u_H(x_H) - u_H(y_H)\big|\big\}. \end{aligned}$$

Since $|x-y| = |x_H - y_H|$ this implies

$$\omega_{u_H,\Omega}(t) - \varepsilon \leqslant \omega_{u,\Omega}(t). \tag{3.49}$$

(ii) Let $x, y_H \in H \cap \Omega$ or $y, x_H \in H \cap \Omega$. Then $|x-y| = |x_H - y_H| \geqslant |x - y_H| = |x_H - y|$. Since

$$\begin{aligned}
&\max\big\{\big|u(x)-u(y)\big|; \big|u(x_H)-u(y_H)\big|; \big|u(x)-u(y_H)\big|; \big|u(x_H)-u(y)\big|\big\} \\
&\quad = \max\big\{\big|u_H(x)-u_H(y)\big|; \big|u_H(x_H)-u_H(y_H)\big|; \\
&\qquad \big|u_H(x)-u_H(y_H)\big|; \big|u_H(x_H)-u_H(y)\big|\big\},
\end{aligned}$$

we again deduce (3.49). Since ε was arbitrary, the assertion follows. □

The following lemma shows that the two-point rearrangement depends continuously on its defining halfspace, see [37].

LEMMA 3.5. *Let $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, +\infty)$, and let $\{H_n\}$ be a sequence of halfspaces.*

(1) *If H is a halfspace such that*

$$\lim_{n\to\infty} \mathcal{L}^N\big((H_n \Delta H) \cap B_R\big) = 0 \quad \forall R > 0, \tag{3.50}$$

then

$$u_{H_n} \longrightarrow u_H \quad \text{in } L^p\big(\mathbb{R}^N\big). \tag{3.51}$$

(2) *If $u \geqslant 0$, and if $B_{R_n} \subset H_n$, $n = 1, 2, \ldots$, for some sequence $R_n \nearrow +\infty$, then*

$$u_{H_n} \longrightarrow u \quad \text{in } L^p\big(\mathbb{R}^N\big). \tag{3.52}$$

PROOF. (1) Let H, H_n, $n = 1, 2, \ldots$, satisfy (3.50). Then

$$\lim_{n\to\infty} \sigma_{H_n} x = \sigma_H x, \quad \text{uniformly in compact subsets of } \mathbb{R}^N.$$

This leads to (3.51) in case that u is continuous with compact support.

In the general case let $\varepsilon > 0$. We choose a continuous function v with compact support such that $\|u - v\|_p < \varepsilon/3$, and then n_0 large enough such that

$$\|v_{H_n} - v_H\|_p < \frac{\varepsilon}{3} \quad \forall n \geqslant n_0.$$

By nonexpansivity we then obtain

$$\begin{aligned}
\|u_{H_n} - u_H\|_p &\leqslant \|u_{H_n} - v_{H_n}\|_p + \|v_{H_n} - v_H\|_p + \|v_H - u_H\|_p \\
&\leqslant 2\|u - v\|_p + \|v_{H_n} - v_H\|_p < \varepsilon \quad \forall n \geqslant n_0,
\end{aligned}$$

and (3.51) follows.

(2) Let the sequence $\{H_n\}$ as in case (2) and let $u \geqslant 0$. If u has bounded support then there is some $n_0 \in \mathbb{N}$ such that

$$u_{H_n} = u \quad \forall n \geqslant n_0.$$

In the general case we argue similarly as in part (1) to derive (3.52). □

We now state a convolution-type inequality (see [19] and [18]).

LEMMA 3.6. *Let H be a halfspace, $w : \mathbb{R}^+ \to \mathbb{R}_0^+$ measurable and nonincreasing, G a Young function, and let u, v be measurable functions on $\mathbb{R}^N$. Then*

$$\iint_{\mathbb{R}^{2N}} G\big(|u(x) - v(y)|\big) w\big(|x - y|\big)\,\mathrm{d}x\,\mathrm{d}y \geqslant \iint_{\mathbb{R}^{2N}} G\big(|u_H(x) - v_H(y)|\big) w\big(|x - y|\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.53}$$

PROOF. An elementary calculation shows that $\forall x, y \in H$,

$$\begin{aligned} &G\big(|u(x_H) - v(y)|\big) + G\big(|u(x) - v(y_H)|\big) \\ &\quad \geqslant G\big(|u_H(x_H) - v_H(y)|\big) + G\big(|u_H(x) - v_H(y_H)|\big). \end{aligned} \tag{3.54}$$

In view of the assumption on w we have

$$w\big(|x - y|\big) = w\big(|x_H - y_H|\big) \geqslant w\big(|x_H - y|\big) = w\big(|x - y_H|\big).$$

Together with (3.54) this leads to the inequality

$$\begin{aligned} &G\big(|u(x) - v(y)|\big) w\big(|x - y|\big) + G\big(|u(x_H) - v(y)|\big) w\big(|x_H - y|\big) \\ &\qquad + G\big(|u(x) - v(y_H)|\big) w\big(|x - y_H|\big) + G\big(|u(x_H) - v(y_H)|\big) w\big(|x_H - y_H|\big) \\ &\quad \geqslant G\big(|u_H(x) - v_H(y)|\big) w\big(|x - y|\big) + G\big(|u_H(x_H) - v_H(y)|\big) w\big(|x_H - y|\big) \\ &\qquad + G\big(|u_H(x) - v_H(y_H)|\big) w\big(|x - y_H|\big) \\ &\qquad + G\big(|u_H(x_H) - v_H(y_H)|\big) w\big(|x_H - y_H|\big). \end{aligned}$$

Then an integration over $H \times H$ gives (3.53). □

It is easy to see that if u and v are nonnegative then we may replace the function $G(s,t)$ by $(-st)$ in the above proof.

COROLLARY 3.4. *Let H be a halfspace, $w:\mathbb{R}^+ \to \mathbb{R}_0^+$ measurable and nonincreasing, and let u, v be nonnegative measurable functions on $\mathbb{R}^N$. Then*

$$\iint\limits_{\mathbb{R}^{2N}} u(x)v(y)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant \iint\limits_{\mathbb{R}^{2N}} u_H(x)v_H(y)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.55}$$

Some reflexion invariant integrals do not change under two-point rearrangement as well (see [57,58,37]).

LEMMA 3.7. *Let H be a halfspace, Ω be a domain with $\Omega = \sigma_H\Omega$, and let $u \in W^{1,p}(\Omega)$ for some $p \in [1,+\infty]$, then $u_H \in W^{1,p}(\Omega)$. More precisely, if $H = \{x\colon\ x\cdot\nu < \lambda\}$ (ν: exterior unit normal to ∂H, and $\lambda \in \mathbb{R}$), and if $\partial u/\partial\nu$ denotes the directional derivative $\nu\cdot\nabla u$ and $\nabla_{\nu^\perp}u$ the gradient w.r.t. the $(N-1)$-dimensional subspace orthogonal to ν (that is, $\nabla_{\nu^\perp}u = \nabla u - (\nu\cdot\nabla u)\nu$), then $\|\nabla u\|_p = \|\nabla u_H\|_p$, $\|\partial u/\partial\nu\|_p = \|\partial u_H/\partial\nu\|_p$, and $\|\nabla_{\nu^\perp}u\|_p = \|\nabla_{\nu^\perp}u_H\|_p$. Finally, if $u \in W^{1,\infty}(\Omega)$, and if G is a function satisfying*

$$\begin{aligned}
&G \in L^\infty\big(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}_0^+\times\mathbb{R}^N\big), \quad G = G(x,v,t,y),\\
&G \text{ is continuous in } (v,t,y), \text{ nondecreasing in } t \text{ and convex in } (t,y),\\
&G(x,v,t,y) = G(\sigma_H x,v,t,y) \quad \forall (x,v,t,y)\in\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}_0^+\times\mathbb{R}^N,
\end{aligned}$$

then

$$G\left(x,u,\left|\frac{\partial u}{\partial\nu}\right|,\nabla_{\nu^\perp}u\right) \sim G\left(x,u_H,\left|\frac{\partial u_H}{\partial\nu}\right|,\nabla_{\nu^\perp}u_H\right), \tag{3.56}$$

provided that the left integral in (3.56) *converges.*

PROOF. We set

$$v(x) := u(x_H), \quad w(x) := u_H(x_H), \quad x \in H\cap\Omega.$$

Since $u_H(x) = \max\{u(x);v(x)\} = v(x) + (u(x)-v(x))_+$ and $w(x) = \min\{u(x);v(x)\} = u(x) - (u(x)-v(x))_+$ for all $x\in H\cap\Omega$, we have that $u_H, w \in W^{1,1}_{\mathrm{loc}}(H\cap\Omega)$ and

$$\nabla u_H(x) = \begin{cases} \nabla u(x) & \text{a.e. on } \{u>v\}\cap H\cap\Omega,\\ \nabla v(x) & \text{a.e. on } \{u\leqslant v\}\cap H\cap\Omega,\end{cases}$$

$$\nabla w(x) = \begin{cases} \nabla v(x) & \text{a.e. on } \{u>v\}\cap H\cap\Omega,\\ \nabla u(x) & \text{a.e. on } \{u\leqslant v\}\cap H\cap\Omega.\end{cases}$$

From these formulas the assertions follow immediately. □

COROLLARY 3.5. *Let $\Omega\subset\mathbb{R}^N$ be an open set, let H be a halfspace and $u\in W^{1,p}_{0+}(\Omega)$ ($1\leqslant p<+\infty$), or $u\in C^{0,1}_{0+}(\Omega)$. Then $u_H\in W^{1,p}_{0+}(\Omega_H)$, respectively $u_H\in C^{0,1}_{0+}(\Omega_H)$.*

PROOF. Extending u by zero outside Ω we find that $u_H \in W^{1,p}_+(\Omega_H)$.

Notice first that if $u \in C^{0,1}_{0+}(\Omega)$, then $u_H \in C^{0,1}_{0+}(\Omega_H) \subset W^{1,p}_{0+}(\Omega_H)$. If $u \in W^{1,p}_{0+}(\Omega)$, then we choose a sequence $\{u_n\} \subset C^{0,1}_{0+}(\Omega)$ which converges to u in $W^{1,p}_0(\Omega)$. Since $(u_n)_H \to u_H$ in $L^p(\Omega_H)$, and since the functions $(u_n)_H$ are equibounded in $W^{1,p}_0(\Omega_H)$, we find a function $v \in W^{1,p}_0(\Omega_H)$ and a subsequence $(u_{n'})_H$ which converges to v weakly in $W^{1,p}(\Omega_H)$. This means that for every $\varphi \in C^\infty_0(\Omega_H)$ and $i \in \{1, \dots, N\}$,

$$\int_{\Omega_H} \varphi v_{x_i}\,\mathrm{d}x \longleftarrow \int_{\Omega_H} \varphi \frac{\partial((u_{n'})_H)}{\partial x_i}\,\mathrm{d}x = -\int_{\Omega_H} \varphi_{x_i}(u_{n'})_H\,\mathrm{d}x$$
$$\longrightarrow -\int_{\Omega_H} \varphi_{x_i} u_H\,\mathrm{d}x,$$

that is $v = u_H$. The corollary is proved. □

3.3. *Symmetrizations*

Symmetrizations have been used to prove functional inequalities of isoperimetric type in different measure spaces since many years (see e.g. [106,68,96,45,39,44,20,72,69,112,37, 120] and the references cited therein). In this study we restrict ourselves to three types of symmetrizations on $\mathbb{R}^N$ which are defined below.

(1) Let $M \in \mathcal{M}$. If $\mathcal{L}^N(M) < \infty$, then let $M^\star$ the ball B_R with $\mathcal{L}^N(B_R) = \mathcal{L}^N(M)$, and if $\mathcal{L}^N(M) = \infty$ then let $M^\star = \mathbb{R}^N$. Correspondingly, for any function $u \in \mathcal{S}$ we introduce $u^\star$ by formula (3.1) with $Tu = u^\star$ and $T\{u > \lambda\} = \{u > \lambda\}^\star$. An equivalent definition is

$$u^\star(x) := u^\sharp\big(\kappa_N |x|^N\big), \quad x \in \mathbb{R}^N. \tag{3.57}$$

(Here $\kappa_N = \mathcal{L}^N(B_1)$.)

The objects $M^\star$ and $u^\star$ are called the *Schwarz symmetrizations* of M and u, respectively. The mappings $M \to M^\star$ and $u \to u^\star$ are rearrangements. $u^\star$ is 'radially symmetric and radially nonincreasing', that is, $u^\star$ depends on the radial distance $|x|$ only, and is nonincreasing in $|x|$. The superlevel sets $\{u^\star > \lambda\}$ are balls centered at zero, and they have the same measure as $\{u > \lambda\}$, $\lambda \in \mathbb{R}$. Notice that (3.57) implies that two functions $u, v \in \mathcal{S}$ are equidistributed, $u \sim v$, iff $u^\star = v^\star$.

(2) Next let $l(x')$ denote the line $\{x = (x_1, x')\colon\ x_1 \in \mathbb{R}\}$ $(x' \in \mathbb{R}^{N-1})$. If $M \in \mathcal{M}$, its *Steiner symmetrization* M^* is given by

$$M^* := \big\{x = (x_1, x')\colon\ 2|x_1| < \mathcal{L}^1\big(M \cap l(x')\big),\ x' \in \mathbb{R}^{N-1}\big\}. \tag{3.58}$$

If $u \in \mathcal{S}$, its *Steiner symmetrization* u^* is given by formula (3.1) with $Tu = u^*$ and $T\{u > \lambda\} = \{u > \lambda\}^*$. An equivalent definition is

$$u^*(x_1, x') := \sup\big\{\lambda \in \mathbb{R}\colon\ 2|x_1| < \mathcal{L}^1\big(\{u(\cdot, x') > \lambda\}\big)\big\}, \quad x \in \mathbb{R}^N. \tag{3.59}$$

(Here $\{u(\cdot, x') > \lambda\}$ is a short-hand for $\{x_1\colon\ u(x_1, x') > \lambda\}$.) The function u^* is even in the variable x_1 and nonincreasing in x_1 for $x_1 \geqslant 0$, and

$$\mathcal{L}^1\big(\{u(\cdot, x') > \lambda\}\big) = \mathcal{L}^1\big(\{u^*(\cdot, x') > \lambda\}\big) \quad \forall \lambda \in \mathbb{R} \text{ and for a.e. } x' \in \mathbb{R}^{N-1} \tag{3.60}$$

the set on the right-hand side of (3.60) being an interval centered at zero. Hence the Steiner symmetrization is a rearrangement, too.

(3) Let $P := (1, 0, \ldots, 0)$ – the 'north pole'. If $M \in \mathcal{M}$ then there is for a.e. $r > 0$ a unique value $\rho \geqslant 0$ such that the spherical cap $B_\rho(P) \cap \partial B_r$ has the same $(N-1)$-Lebesgue measure as $M \cap \partial B_r$. We denote this spherical cap by $CM(r)$. The set

$$CM := \big\{x \in \mathbb{R}^N\colon\ x \in CM(r),\ r > 0\big\} \tag{3.61}$$

is called the *cap symmetrization* of M. Furthermore, if $u : \mathbb{R}^N \to \mathbb{R}$ is measurable, we define its *cap symmetrization* Cu by

$$Cu(x) := \sup\big\{\lambda \in \mathbb{R}\colon\ x \in C\{u > \lambda\}\big\}. \tag{3.62}$$

The mapping C is a rearrangement which satisfies (3.6). The superlevel sets $\{Cu > \lambda\} \cap \partial B_r$ are spherical caps centered at P and have the same measure as $\{u > \lambda\} \cap \partial B_r$ ($r > 0$, $\lambda \in \mathbb{R}$). Hence Cu depends only on the radial distance $r = |x|$ and on the geographical latitude $\theta_1 := \arccos(x_1/|x|)$, and is nonincreasing in $\theta_1 \in [0, \pi]$.

(4) Now we introduce pointwise representatives of the symmetrization of open and closed sets and of continuous functions.

If M is an open set, then we agree that the definitions of $M^\star$ and M^* given above are taken in pointwise sense. If M is closed, then let $M^\star$ the closed ball $\overline{B_R}$ having the same measure as M if $\mathcal{L}^N(M) < \infty$, and $M^\star = \mathbb{R}^N$ if $\mathcal{L}^N(M) = \infty$, and let

$$M^* := \big\{x = (x_1, x')\colon\ 2|x_1| \leqslant \mathcal{L}^1\big(M \cap l(x')\big),\ x' \in \mathbb{R}^{N-1}\big\}. \tag{3.63}$$

Notice that the function

$$f(x', \lambda) := \mathcal{L}^1\big(\{u(\cdot, x') > \lambda\}\big), \quad \lambda \in \mathbb{R},\ x' \in \mathbb{R}^{N-1}, \tag{3.64}$$

is lower, respectively upper semicontinuous in x', if M is open, respectively closed, so that M^* given by the above definition is open, respectively closed, too.

Furthermore, if M is open, respectively closed, then let $CM(r)$ the spherical cap $B_\rho \cap \partial B_r$, respectively $\overline{B_\rho} \cap \partial B_r$, such that $\mathcal{L}^{N-1}(M \cap \partial B_r) = \mathcal{L}^{N-1}(B_\rho \cap \partial B_r)$, when $r > 0$. We also let $CM(0) = \{0\}$ iff $0 \in M$ in both cases. The pointwise representative CM is then given by formula (3.61). Notice that the function

$$g(r, \lambda) := \mathcal{L}^{N-1}\big(\{u > \lambda\} \cap \partial B_r\big), \quad \lambda \in \mathbb{R},\ r \geqslant 0, \tag{3.65}$$

is lower, respectively upper semicontinuous, if M is open, respectively closed, so that CM is then open, respectively closed, too.

Finally, if u is a continuous function, then the superlevel sets $\{u > \lambda\}$ ($\lambda \in \mathbb{R}$), are open. We then define pointwise representatives of $u^\star$, u^* and Cu by formula (3.14) choosing pointwise representatives for $\{u > \lambda\}$ ($\lambda \in \mathbb{R}$).

REMARK 3.6. It can be seen from simple examples that the Schwarz and Steiner symmetrizations do not satisfy property (3.6). This can lead to pathologies if one applies formula (3.14) to measurable functions which are not in $\mathcal{S}$:

For instance, if $u(x) = \sin|x|$ and $v(x) = w(x')\sin x_1$, for some positive measurable function w, then (3.1) gives $u^\star(x) \equiv 1$ and $v^*(x) = w(x')$, so that the negative parts of u and v do not show up at their symmetrizations.

We now introduce the following notation: Let $\mathcal{H}$ be the set of all affine halfspaces of $\mathbb{R}^N$, and

$$\mathcal{H}_0 := \{H \in \mathcal{H}\colon\ 0 \in H\},$$
$$\mathcal{H}^* := \big\{H \in \mathcal{H}\colon\ H = \{x\colon\ x_1 > \lambda\} \text{ or } H = \{x\colon\ x_1 < \lambda\},\ \text{for some } \lambda \in \mathbb{R}\big\},$$
$$\mathcal{H}_0^* := \{H \in \mathcal{H}^*\colon\ 0 \in H\},$$
$$C\mathcal{H} := \{H \in \mathcal{H}\colon\ 0 \in \partial H\}, \quad \text{and}$$
$$C\mathcal{H}_P := \{H \in C\mathcal{H}\colon\ P \in H\}.$$

REMARK 3.7.

(1) It is easy to see that, if $u \in \mathcal{S}$, then $(u_H)^\star = u\ \forall H \in \mathcal{H}$ and $(u_H)^* = u^*\ \forall H \in \mathcal{H}^*$. Furthermore, if u is measurable on $\mathbb{R}^N$, then $C(u_H) = Cu\ \forall H \in C\mathcal{H}$.

(2) The two-point rearrangement can be used in order to identify symmetric situations. First observe that if $u \in \mathcal{S}$ and $u = u^\star$, then $u = u_H\ \forall H \in \mathcal{H}_0$. Furthermore, if $u \in \mathcal{S}$, $u = u^*$, then $u = u_H\ \forall H \in \mathcal{H}_0^*$. Finally, if $u : \mathbb{R}^N \to \mathbb{R}$ is measurable and $u = Cu$, then $u = u_H\ \forall H \in C\mathcal{H}_P$.

A more interesting result is the following

LEMMA 3.8. *Let $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, \infty)$.*

(1) *If $\mathcal{L}^N(\{u > 0)\}) > 0$, and if for every $H \in \mathcal{H}^*$ we have either $u = u_H$ or $\sigma_H u = u_H$, then $u \in \mathcal{S}_+$, and there is a point $\xi = (\lambda_0, 0, \dots, 0) \in \mathbb{R}^N$ ($\lambda_0 \in \mathbb{R}$), such that $u(\cdot - \xi) = u^*(\cdot)$. In particular, if $u = u_H\ \forall H \in \mathcal{H}_0^*$ then $u \in \mathcal{S}_+$ and $u = u^*$.*

(2) *If $\mathcal{L}^N(\{u > 0\}) > 0$, and if for every $H \in \mathcal{H}$ we have either $u = u_H$ or $\sigma_H u = u_H$, then there is a point $\xi \in \mathbb{R}^N$ such that $u \in \mathcal{S}_+$ and $u(\cdot - \xi) = u^\star(\cdot)$. In particular, if $u = u_H\ \forall H \in \mathcal{H}_0$ then $u \in \mathcal{S}_+$ and $u = u^\star$.*

(3) *If for every $H \in C\mathcal{H}$ we have either $u = u_H$ or $\sigma_H u = u_H$ then there is a rotation ρ about zero such that $u(\rho\cdot) = Cu(\cdot)$. In particular, if $u = u_H\ \forall H \in C\mathcal{H}_P$ then $u = Cu$.*

PROOF. We only prove the first parts of the assertions (1)–(3) and leave the second parts to the reader.

(1) For simplicity, we write $H_\lambda = \{x\colon\ x_1 > \lambda\}$, $u_{H_\lambda} = u_\lambda$ and $\sigma_{H_\lambda} u = \sigma_\lambda u$ ($\lambda \in \mathbb{R}$). Since $\mathcal{L}^N(\{u > 0\})$, and since u is decaying at infinity, we have that $u = u_\lambda$ when λ is small enough and $\sigma_\lambda u = u_\lambda$ when λ is large enough. By the continuity of the mapping $\lambda \mapsto u_\lambda$ (Lemma 3.5) we find an intermediate value $\lambda_0 \in \mathbb{R}$ such that $u = \sigma_{\lambda_0} u = u_{\lambda_0}$. Assume that there is another value $\lambda_0' \neq \lambda_0$ with the same property. Then $u(x_1, x') = u(2\lambda_0 - x_1, x') = u(2\lambda_0' - x_1, x') = u(x_1 + 2(\lambda_0' - \lambda_0), x')\ \forall (x_1, x') \in \mathbb{R}^N$, that is, u is periodic in x_1 with period $2|\lambda_0' - \lambda_0|$. But this is not possible since $u \in L^p(\mathbb{R}^N)$. Hence we have that $u = u_\lambda$ for $\lambda \leqslant \lambda_0$ and $\sigma_\lambda u = u_\lambda$ for $\lambda \geqslant \lambda_0$, which means that u is nonincreasing in x_1 for $x_1 \geqslant \lambda_0$. Since u decays at infinity this in particular implies that $u \geqslant 0$, and the first part of (1) follows.

(2) Proceeding as in part (1) of the proof for any of the x_i-directions, we find that $u \in \mathcal{S}_+$, and that there is point $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ such that the function $v(\cdot) := u(\cdot - \xi)$ is even in the variables x_i and nonincreasing in x_i for $x_i \geqslant 0$ ($i = 1, \dots, N$).

This implies that if $H \in \mathcal{H}_0$ then $v = v_H$, and if $H \in \mathcal{H} \setminus \mathcal{H}_0$ then $\sigma_H v = v_H$. By continuity this means that if e a unit vector, and $H = \{x\colon\ x \cdot e > 0\}$, then one has $v = \sigma_H v$. Hence v depends on the radial distance $|x|$ only, and is nonincreasing in $|x|$, and the first part of (3) follows.

(3) Fix N mutually orthogonal unit vectors e_i ($i = 1, \dots, N$), and let $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$. Introducing polar coordinates $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ ($\varphi \in [-\pi, \pi]$, $r \geqslant 0$), the function $w(\varphi, r, x'') := u(x)$ ($x'' = (x_3, \dots, x_N)$), is (2π)-periodic in φ. From the assumption it follows that, if $\lambda \in [-\pi, \pi]$, then either $w(\varphi, \cdot) \geqslant w(2\lambda - \varphi, \cdot)\ \forall \varphi \in [\lambda, \lambda + \pi]$ or $w(\varphi, \cdot) \leqslant w(2\lambda - \varphi, \cdot)\ \forall \varphi \in [\lambda, \lambda + \pi]$, where "$\cdot$" stands for (r, x''). We claim that there is a value $\lambda^* \in [-\pi, \pi]$ such that $w(\varphi, \cdot) = w(2\lambda^* - \varphi, \cdot)$ $\forall \varphi \in [-\pi, \pi]$ and such that w is nonincreasing in φ for $\varphi \in [\lambda^*, \lambda^* + \pi]$.

First observe that by periodicity and continuity there exists a value $\lambda_0 \in [0, \pi]$ such that $w(\varphi, \cdot) = w(2\lambda_0 - \varphi, \cdot)\ \forall \varphi \in [-\pi, \pi]$. Assume that there is a sequence of values $\{\lambda_n\}$ with the same symmetry property, and such that $\lambda_n \neq \lambda_0\ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \lambda_n = \lambda_0$. Then we have that $w(\varphi, \cdot) = w(2(\lambda_0 - \lambda_n) + \varphi, \cdot)\ \forall n \in \mathbb{N}$ and $\forall \varphi \in [-\pi, \pi]$, which means that $w(\varphi, \cdot) = w(0, \cdot)\ \forall \varphi \in [-\pi, \pi]$, that is w is independent of φ.

Next assume that a sequence with the above property does not exist. By periodicity and continuity, we then find two numbers $\lambda_i \in [-\pi, \pi]$ with $\lambda_2 \in (\lambda_1, \pi + \lambda_1]$ such that $w(\varphi, \cdot) = w(2\lambda_i - \varphi, \cdot)\ \forall \varphi \in [-\pi, \pi]$ ($i = 1, 2$), and such that $w(\varphi, \cdot) \geqslant w(2\lambda - \varphi, \cdot)$ $\forall \varphi \in [\lambda, \lambda + \pi]$ and $\forall \lambda \in [\lambda_1, \lambda_2]$. Assume that $\lambda_2 < \lambda_1 + \pi$. Then we have that $w(\varphi, \cdot) \geqslant w(2(\lambda_1 - \lambda) + \varphi, \cdot) = w(2(\lambda_2 - \lambda) + \varphi, \cdot)\ \forall \varphi \in [\lambda_2, \lambda_1 + \pi]$ and $\forall \lambda \in [\lambda_1, \lambda_2]$. This again implies that w is independent of φ. It remains to consider the case that $\lambda_2 = \lambda_1 + \pi$. It is then easy to see that w is nonincreasing in φ for $\varphi \in [\lambda_1, \lambda_1 + \pi]$.

We have thus proved the claim in any of the above cases.

Let $l_1 = (\cos\lambda^*, \sin\lambda^*, 0, \dots, 0)$. Notice that $l_1 \in \operatorname{span}\{e_1, e_2\}$. It follows that if $l \in \operatorname{span}\{e_1, e_2\}$ and $H = \{x\colon\ x \cdot l > 0\}$ such that $l_1 \in H$, then $u = u_H$. Analogously we find a vector $l_2 \in \operatorname{span}\{l_1, e_3\}$ such that the following holds: If $l \in \operatorname{span}\{e_1, e_2, e_3\}$ and $H = \{x\colon\ x \cdot l > 0\}$ such that $l_3 \in H$, then $u = u_H \dots$.

Continuing in this manner we find unit vectors $l_k \in \operatorname{span}\{e_1, \dots, e_{k+1}\}$ $(k = 1, \dots, N-1)$, such that the following holds: If $l \in \operatorname{span}\{e_1, \dots, e_{k+1}\}$ and $H = \{x\colon\ x \cdot l > 0\}$ such that $l_k \in H$, then $u = u_H$ $(k = 1, \dots, N-1)$.

By continuity and periodicity this also implies that if $H \in C\mathcal{H}$ such that $l_{N-1} \in \partial H$, then $u = \sigma_H u$.

Finally, let ρ a rotation of the coordinate system about the origin such that $\rho l_{N-1} = P = (1, 0, \dots, 0)$, and introduce a function v by $v(\rho\cdot) := u(\cdot)$. Then the above properties imply that v depends on the variables $|x|$ and $\theta_1 := \arccos(x_1/|x|)$ only and is nonincreasing in $\theta_1 \in [0, \pi]$. This means that $v = Cv$, and the first part of (3) follows. □

The next separation property will be crucial in the proof of inequalities between the Sobolev norm of a function and its symmetrization. It has been proved in [19] for the cap symmetrization and in [18] for the Schwarz and Steiner symmetrization.

LEMMA 3.9. *Let $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, \infty)$.*

(1) *If $u \geqslant 0$, and if $u \neq u^\star$ then there is an $H \in \mathcal{H}_0$ such that*

$$\|u_H - u^\star\|_p < \|u - u^\star\|_p. \tag{3.66}$$

(2) *If $u \geqslant 0$, and if $u \neq u^*$ then there is an $H \in \mathcal{H}_0^*$ such that*

$$\|u_H - u^*\|_p < \|u - u^*\|_p. \tag{3.67}$$

(3) *If $u \neq Cu$ then there is some halfspace $H \in C\mathcal{H}_P$ such that*

$$\|u_H - Cu\|_p < \|u - Cu\|_p. \tag{3.68}$$

PROOF. (1) If $H \in \mathcal{H}_0$ then we have that $u = (u^\star)_H$, and an elementary analysis shows that

$$\begin{aligned} &\big|u_H(x) - u^\star(x)\big|^p + \big|u_H(x_H) - u^\star(x_H)\big|^p \\ &\quad \leqslant \big|u(x) - u^\star(x)\big|^p + \big|u(x_H) - u^\star(x_H)\big|^p \quad \forall x \in H. \end{aligned} \tag{3.69}$$

An integration of (3.69) over H then leads to $\|u_H - u^\star\|_p \leqslant \|u - u^\star\|_p$. Therefore to prove (3.66) it suffices to show that for a suitable choice of H the inequality (3.69) becomes strict on a subset of H of *positive* measure.

Since $u \neq u^\star$, we find some number $c > 0$ such that $\mathcal{L}^N(\{u > c\} \Delta \{u^\star > c\}) > 0$.

Let x^1 and x^2 density points of the sets $\{u > c\} \setminus \{u^\star > c\}$ and $\{u^\star > c\} \setminus \{u > c\}$, respectively. We choose a halfspace H such that $x^1 = \sigma_H x^2$ and $x^2 \in H$. (Note that from $u^\star(x^1) \leqslant c < u^\star(x^2)$ it follows that $0 \in H$.) Hence there is a subset K of H of positive measure containing x^2 such that

$$u^\star(x) > c \geqslant u(x), \quad u^\star(x_H) \leqslant c < u(x_H) \quad \forall x \in K. \tag{3.70}$$

But this means that the inequality (3.69) becomes strict on the set K.

(2) First observe that if $H \in \mathcal{H}_0^*$ then $(u^*)_H = u^*$ and (3.69) holds with $u^\star$ replaced by u^*. Setting

$$M := \{x' \in \mathbb{R}^{N-1}\colon\ \mathcal{L}^1(\{u(\cdot, x') > c\} \Delta \{u^*(\cdot, x') > c\}) > 0\},$$

we see that $\mathcal{L}^{N-1}(M) > 0$. Let x_0' be a density point of M . Then

$$\mathcal{L}^1(\{u(\cdot, x_0') > c\} \setminus \{u^*(\cdot, x_0') > c\}) = \mathcal{L}^1(\{u^*(\cdot, x_0') > c\} \setminus \{u(\cdot, x_0') > c\}) > 0.$$

Furthermore, let y_1 and y_2 be density points of the sets $\{u(\cdot, x_0') > c\} \setminus \{u^*(\cdot, x_0') > c\}$ and $\{u^*(\cdot, x_0') > c\} \setminus \{u(\cdot, x_0') > c\}$, respectively. Setting $H = \{x\colon\ (2x_1 - y_1 - y_2)(y_2 - y_1) > 0\}$ we see that $0, (y_2, x_0') \in H$ and $\sigma_H(y_1, x_0') = (y_2, x_0')$. Hence there is some subset K of H with positive measure containing (y_2, x_0') such that the relations (3.70) are satisfied, with $u^\star$ replaced by u^*, and we conclude analogously as before.

(3) If $H \in C\mathcal{H}_P$ then $(Cu)_H = Cu$, and (3.69) holds with $u^\star$ replaced by Cu. Setting

$$M := \{r \in \mathbb{R}_0^+\colon\ \mathcal{L}^{N-1}([\{u > c\} \Delta \{Cu > c\}] \cap \partial B_r) > 0\},$$

we have that $\mathcal{L}^1(M) > 0$. Let r_0 a density point of M. Then

$$\mathcal{L}^{N-1}([\{u > c\} \setminus \{Cu > c\}] \cap \partial B_{r_0}) = \mathcal{L}^{N-1}([\{Cu > c\} \setminus \{u > c\}] \cap \partial B_{r_0}) > 0.$$

Furthermore, let x^1, x^2 be density points of the sets $[\{u > c\} \setminus \{Cu > c\}] \cap \partial B_{r_0}$ and $[\{Cu > c\} \setminus \{u > c\}] \cap \partial B_{r_0}$, respectively. Choosing $H = \{x\colon\ x \cdot (x^2 - x^1) > 0\}$ we find that $0 \in \partial H$, $x^2 = \sigma_H x^1$ and $P, x_2 \in H$. Hence there is some subset K of H with positive measure containing x^2 such that the relations (3.69) are satisfied, with $u^\star$ replaced by Cu, and we conclude as before. □

4. Inequalities for symmetrizations

In this section we use the results of Section 3.1 to show various inequalities that compare an integral functional of some given functions with the same functional of their symmetrizations. This idea is due to Baernstein and Taylor (see [19]), and it is related to the fact that symmetrizations can be approximated in the L^p-norm via appropriate sequences of two-point rearrangements (see [37] and [117]), although such a sequence does not explicitly occur in our proofs. Instead we use the compactness of certain rearrangement invariant sets and convergence properties of the functionals. Our first result has been proved in [19] for the cap symmetrization, and in [18] for the Schwarz and Steiner symmetrization.

THEOREM 4.1.

(1) *Let $u \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathcal{S}_+$. Then $u^\star, u^* \in C(\mathbb{R}^N)$ and*

$$\omega_{u^\star} \leqslant \omega_{u^*} \leqslant \omega_u. \tag{4.1}$$

(2) *Let* $u \in C(\Omega) \cap L^\infty(\Omega)$ *or* $u \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, *where either* $\Omega = B_{R_2} \setminus \overline{B_{R_1}}$, $\Omega = B_{R_2}$, $\Omega = \mathbb{R}^N$, *or* $\Omega = \mathbb{R}^N \setminus \overline{B_{R_1}}$, *for some* $R_2 > R_1 \geqslant 0$. *Then* $Cu \in C(\Omega)$, *respectively* $Cu \in C(\overline{\Omega})$, *and*

$$\omega_{Cu,\Omega} \leqslant \omega_{u,\Omega}. \tag{4.2}$$

PROOF. (1) Assume first that $u \in C^{0,1}_{0+}(B_R)$ for some $R > 0$. Let

$$A^*(u) := \left\{ v \in C^{0,1}_{0+}(B_R) \colon\ v^* = u^*,\ \omega_{v,B_R} \leqslant \omega_{u,B_R} \right\},$$

and $\delta := \inf\{\|v - u^*\|_2 \colon\ v \in A^*(u)\}$. By Arzelá's Theorem and by the nonexpansivity of the Schwarz symmetrization in L^2, δ is attained for some $U \in A(u)$. Assume that $\delta > 0$. Then we find by Lemma 3.9 a halfspace $H \in \mathcal{H}_0^*$ such that $\|U_H - u^*\|_2 < \delta$. Since $(B_R)_H = B_R$, we have that $U_H \in C^{0,1}_{0+}(B_R)$ by Corollary 3.4. Furthermore, $(U_H)^* = U^* = u^*$ by Remark 3.7(1), and $\omega_{U_H,B_R} \leqslant \omega_{U,B_R} \leqslant \omega_{u,B_R}$ by Lemma 3.4, so that $U_H \in A^*(u)$, contradicting to the minimality of U. Hence $\delta = 0$ and thus $U = u^\star$, which implies the right inequality in (4.1) in this case.

In the general case we choose a sequence $\{u_n\} \subset C^{0,1}_{0+}(\mathbb{R}^N)$ converging to u in $C(\mathbb{R}^N)$. Then we have that $(u_n)^* \to u$ in $C(\mathbb{R}^N)$ by Lemma 3.4, and the right-hand side of (4.1) follows.

Next observe that $(u^*)^\star = u^\star$, since $u^* \sim u$. Using Lemmas 3.4 and 3.9 – with halfspaces in $\mathcal{H}_0$, the left-hand side of (4.1) then follows analogously as above by minimizing $\|v - u^\star\|_2$ over the set $A^\star(u) = \{v \in C^{0,1}_{0+}(B_R) \colon\ v^\star = u^\star,\ \omega_v \leqslant \omega_{u^*}\}$ in case that $u \in C^{0,1}_{0+}(B_R)$, and by a limit procedure in the general case.

(2) First observe that $C\Omega = \Omega_H = \Omega\ \forall H \in C\mathcal{H}$. Hence, if $u \in C(\overline{\Omega})$, and if Ω is bounded, then (4.2) follows analogously as in part (1), by using Lemmas 4 and 9 – with halfspaces in $C\mathcal{H}_P$, and by minimizing $\|v - Cu\|_{2,\Omega}$ over the set $CA(u) = \{v \in C(\overline{\Omega}) \colon\ Cv = Cu,\ \omega_{v,\Omega} \leqslant \omega_{u,\Omega}\}$.

Next let $u \in C(\Omega)$. Since $CK = K$ for any compact annulus $K \subset \Omega$ of the form

$$K = \overline{B_{R'}} \setminus B_{r'}, \quad R' > r' \geqslant 0, \tag{4.3}$$

the above argument shows that $\omega_{Cu,K} \leqslant \omega_{u,K}$. Then (4.2) follows by exhausting Ω from inside with annuli of the form (4.3).

The proof in the case that Ω is unbounded and $u \in C(\overline{\Omega})$ is similar and will be omitted. □

It has been known since many years that a convolution-type integral $\iint u(x)v(y)w(x-y)\,\mathrm{d}x\,\mathrm{d}y$ increases under Schwarz symmetrization of the three functions u, v and w (see [99,68]). Meanwhile many generalizations of this result have been found, see [28,2,92,41], and [56]. Below we show one of those generalizations using two-point rearrangements (see [21] and [18], see also [19] for the special case $G(t) = t^2$).

THEOREM 4.2. *Let* w *and* G *be functions satisfying the conditions of Lemma* 3.6.

(1) *If* $u, v \in \mathcal{S}_+$, *then*

$$\iint\limits_{\mathbb{R}^{2N}} G\big(|u(x)-v(y)|\big)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant \iint\limits_{\mathbb{R}^{2N}} G\big(|u^*(x)-v^*(y)|\big)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant \iint\limits_{\mathbb{R}^{2N}} G\big(|u^\star(x)-v^\star(y)|\big)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.4}$$

(2) *If* $u, v : \Omega \to \mathbb{R}$ *are measurable, where either* $\Omega = B_{R_2} \setminus \overline{B_{R_1}}$, $\Omega = B_{R_2}$, $\Omega = \mathbb{R}^N$, *or* $\Omega = \mathbb{R}^N \setminus \overline{B_{R_1}}$ $(R_2 > R_1 \geqslant 0)$, *then*

$$\iint\limits_{\Omega^2} G\big(|u(x)-v(y)|\big)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant \iint\limits_{\Omega^2} G\big(|Cu(x)-Cv(y)|\big)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.5}$$

PROOF. We proceed similar as in the proof of the previous theorem.

(1) Let $Q(u, v)$ the integral on the left-hand side of (4.4), and let $u, v \in C_{0+}^{0,1}(B_R)$, for some $R > 0$. Setting

$$D^*(u,v) := \big\{(f,g) \in \big(C_0^{0,1}(B_R)\big)^2 \colon\ u^* = f^*,\ \omega_u \leqslant \omega_f,$$
$$v^* = g^*,\ \omega_v \leqslant \omega_g,\ Q(f,g) \leqslant Q(u,v)\big\},$$

we let $\delta := \inf\{\|(f-u^*)^2 + (g-v^*)^2\|_1 \colon\ (f,g) \in D^*(u,v)\}$. Since $D^*(u,v)$ is weakly closed in $C_0^{0,1}(B_R)$, δ is attained for some $(U,V) \in D^*(u,v)$. If $(U,V) \neq (u^*, v^*)$, we find a halfspace $H \in \mathcal{H}_0^*$ such that $\|(U_H - u^*)^2 + (V_H - v^*)^2\|_1 < \delta$, by Lemma 3.9. But then we have that $Q(U_H, V_H) \leqslant Q(U,V)$, by Lemma 3.6, and thus $(U_H, V_H) \in D^*(u,v)$ which contradicts to the minimality of δ. This shows the left inequality of (4.4) in this case.

Next assume that u, v are nonnegative and bounded, with support in an open set M of finite measure. Then we choose sequences $\{u_n\}, \{v_n\}$ of nonnegative Lipschitz functions with compact support K $(K \subset M)$, such that

$$\lim_{n\to\infty} \|u_n - u\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|v_n - v\|_1 = 0,$$

and the left inequality of (4.4) follows by dominated convergence.

In the general case we set $u_n := \min\{(u - (1/n))_+; n\}$ and $v_n := \min\{(v - (1/n))_+; n\}$ $(n \in \mathbb{N})$. Since $u, v \in \mathcal{S}_+$, we have that $u_n, v_n \in L_+^\infty(M_n)$ where M_n is an open set of finite measure. Furthermore, the sequence $\{G(|u_n(x) - v_n(y)|)\}$ is nondecreasing $\forall (x,y) \in \mathbb{R}^{2N}$.

Since also $u_n^\star = \min\{(u^\star - (1/n))_+; n\}$, and analogously for $v_n^\star$ ($n \in \mathbb{N}$), the left inequality of (4.4) follows by monotone convergence.

The right inequality of (4.4) follows the same way by replacing '$*$' by '$\star$' in the above proof and by taking into account that $(u^*)^\star = u^\star$ and $(v^*)^\star = v^\star$.

(2) Extending u and v by zero outside Ω we also have that $Cu = Cv = 0$ in $\mathbb{R}^N \setminus \Omega$. Therefore we may replace the domain of integration in (4.5) by $\mathbb{R}^N$.

Assume first that Ω is bounded. If $u, v \in C(\overline{\Omega})$, then (4.5) follows analogously as in part (1) replacing $C_{0+}^{0,1}(B_R)$ by $C(\overline{\Omega})$, Steiner symmetrization by cap symmetrization, and by working with halfspaces in $C\mathcal{H}_P$ instead of $\mathcal{H}_0^*$. If $u, v \in L^\infty(\Omega)$, then (4.5) follows similarly as in part (1) through an approximation with functions in $C(\overline{\Omega})$. In the general case we set $u_n := \min\{(u+n)_+ - n; n\}$, $v_n := \min\{(v+n)_+ - n; n\}$ ($n \in \mathbb{N}$). Then the sequence $\{G(|u_n(x) - v_n(y)|)\}$ is nondecreasing $\forall (x, y) \in \mathbb{R}^{2N}$. Since also $Cu_n := \min\{(Cu+n)_+ - n; n\}$, and analogously for Cv_n, (4.5) follows in this case, too, by monotone convergence.

Let finally Ω be unbounded. Then we introduce functions U_R, V_R by setting $U_R = u$, $V_R = v$ in $\Omega \cap B_R$ and $U_R = V_R = 0$ in $\mathbb{R}^N \setminus (\Omega \cap B_R)$ ($R > 0$). Since $C(U_R) = Cu$, $C(V_R) = Cv$ in $\Omega \cap B_R$, (4.5) follows from the above considerations by approximation. □

Replacing the function $G(s,t)$ by $(-st)$ in the above proof while using Corollary 3.4 we obtain the following

COROLLARY 4.1. *Let w be as in Lemma* 3.6.

(1) *If $u, v \in \mathcal{S}_+$, then*

$$\iint\limits_{\mathbb{R}^{2N}} u(x)v(y)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant \iint\limits_{\mathbb{R}^{2N}} u^*(x)v^*(y)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y$$

$$\leqslant \iint\limits_{\mathbb{R}^{2N}} u^\star(x)v^\star(y)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.6}$$

(2) *If u, v are nonnegative measurable functions on $\mathbb{R}^N$, then*

$$\iint\limits_{\mathbb{R}^{2N}} u(x)v(y)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant \iint\limits_{\mathbb{R}^{2N}} Cu(x)Cv(y)w\big(|x-y|\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{4.7}$$

Next we deal with integral inequalities involving gradients of functions. Again we will use the two-point rearrangement approach, using now heavily the weak lower semicontinuity of the corresponding functionals. This leads to very general results.

We emphasize that most proofs of the inequalities below that appeared in literature are based on the so-called *method of level sets*. Its idea consists in using the *co-area formula* and taking into account that the perimeter of the superlevel sets $\{u > y\}$ ($t \in \mathbb{R}$), decrease under symmetrization (compare Remark 4.1(2)).

Our fist result is well-known (see e.g. [96,109,93,72,20,38,59] and [18]).

THEOREM 4.3. *Let $u \in C_{0+}^{0,1}(\mathbb{R}^N)$, let $G \in C((\mathbb{R}_0^+)^2)$, $G = G(v,z)$, and let G be convex and nondecreasing in z, with $G(0,0) = 0$. Then*

$$\int_{\mathbb{R}^N} G\big(u, |\nabla u|\big)\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} G\big(u^\star, |\nabla u^\star|\big)\,\mathrm{d}x. \tag{4.8}$$

Furthermore, if $u \in W^{1,p}(\mathbb{R}^N) \cap \mathcal{S}_+$ for some $p \in [1, +\infty]$, then $u^\star \in W^{1,p}(\mathbb{R}^N) \cap \mathcal{S}_+$ and

$$\|\nabla u\|_p \geqslant \|\nabla u^\star\|_p. \tag{4.9}$$

PROOF. (1) Assume $u \in C_{0+}^{0,1}(B_R)$ for some $R > 0$, and

$$B^\star(u) := \big\{v \in C_{0+}^{0,1}(B_R):\ v \sim u,\ \omega_v \leqslant \omega_u,\ J(v) \leqslant J(u)\big\},$$

where

$$J(v) := \int_{\mathbb{R}^N} G\big(v, |\nabla v|\big)\,\mathrm{d}x.$$

Furthermore, let $\delta = \inf\{\|v - u^\star\|_2:\ v \in B^\star(u)\}$. In view of the weak lower semicontinuity of the functional J and the nonexpansivity of the Schwarz symmetrization, $B^\star(u)$ is weakly closed in $C_{0+}^{0,1}(B_R)$. Hence there exists some $U \in B^\star(u)$ with $\delta = \|U - u^\star\|_2$. Since $J(U_H) = J(U)\ \forall H \in \mathcal{H}^\star$, we may then argue as in the proof of Theorem 4.1 to obtain that $\delta = 0$, and thus $U = u^\star$. This shows (4.8).

(2) Let $u \in W_+^{1,p}(\mathbb{R}^N)$ for some $p \in (1, \infty)$. We choose a sequence $\{u_n\} \subset C_{0+}^{0,1}(\mathbb{R}^N)$ which converges to u in $W^{1,p}(\mathbb{R}^N)$. By (4.8) with $G(v,z) = z^p$ we then have that $\|\nabla u_n\|_p \geqslant \|\nabla (u_n)^\star\|_p$, $n = 1, 2, \ldots$. Hence we find a subsequence $\{(u_n')^\star\}$ and a function $v \in W^{1,p}(\mathbb{R}^N)$ such that $(u_n')^\star \rightharpoonup v$ weakly in $W^{1,p}(\mathbb{R}^N)$. But by the nonexpansivity of the Schwarz symmetrization in L^p we have that $(u_n)^\star \to u^\star$ in $L^p(\mathbb{R}^N)$, so that $v = u^\star$. Finally, the weak lower semicontinuity of the norm gives $\|\nabla u^\star\|_p \leqslant \liminf \|\nabla (u_n')^\star\|_p \leqslant \lim \|\nabla u_n\|_p = \|\nabla u\|_p$.

(3) Let $u \in W^{1,\infty}(\mathbb{R}^N) \cap \mathcal{S}_+$. By Rademacher's Theorem, there is a version $u \in C^{0,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap \mathcal{S}_+$. By Theorem 4.1 this implies $\omega_{u^\star} \leqslant \omega_u$ and $u^\star \in C^{0,1}(\mathbb{R}^N)$. Since also $\|\nabla u\|_\infty = \lim_{t \searrow} \omega_u(t)/t$, (4.9) follows for $p = \infty$.

(4) Let $u \in W_+^{1,1}(\mathbb{R}^N)$. We choose a sequence $\{u_n\} \subset C_{0+}^{0,1}(\mathbb{R}^N)$ with $u_n \to u$ in $W^{1,1}(\Omega)$. Then have that for every Young function G,

$$\int_{\mathbb{R}^N} G\big(|\nabla u_n|\big)\,\mathrm{d}x \geqslant \int_{\mathbb{R}^N} G\big(\big|\nabla (u_n)^\star\big|\big)\,\mathrm{d}x, \quad n = 1, 2, \ldots.$$

By a result of [5] this implies that

$$\mathcal{L}^N\big\{|\nabla u_n| > t\big\} \geqslant \mathcal{L}^N\big\{\big|\nabla (u_n)^\star\big| > t\big\}, \quad n = 1, 2, \ldots.$$

Due to a well-known weak compactness criterion in L^1 this implies that there is a function $v \in L^1(\Omega)$ and a subsequence $\{(u_{n'})^\star\}$ such that $|\nabla(u_{n'})^\star| \rightharpoonup v$ weakly in $L^1(\mathbb{R}^N)$. Since $(u_n)^\star \to u$ in $L^1(\mathbb{R}^N)$ this implies that $|\nabla u^\star| = v$ and $u^\star \in W^{1,1}(\mathbb{R}^N)$. Finally, the inequality (4.9) for $p = 1$ follows from the weak lower semicontinuity of the norm. □

The inequalities (4.10) and (4.11) in Theorem 4.4 below are well-known (see [72,101, 18] and [37]), while experts seem to be familiar with (4.10), although we could not find a reference.

THEOREM 4.4. *Let $u \in C^{0,1}_{0+}(\mathbb{R}^N)$, and let G be a function satisfying*

$$G \in L^\infty\big(\mathbb{R}^{N-1} \times \mathbb{R}^+_0 \times \mathbb{R}^+_0 \times \mathbb{R}^{N-1}\big), \quad G = G(x', v, z_1, z'),$$
$$G \text{ is continuous in } (v, z_1, z'), \text{ nondecreasing in } z_1 \text{ and convex in } (z_1, z'),$$
$$G(x', 0, 0, 0) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

Then

$$\int_{\mathbb{R}^N} G\left(x', u, \left|\frac{\partial u}{\partial x_1}\right|, \nabla' u\right) \mathrm{d}x \geqslant \int_{\mathbb{R}^N} G\left(x', u^*, \left|\frac{\partial u^*}{\partial x_1}\right|, \nabla' u^*\right) \mathrm{d}x. \tag{4.10}$$

(*Here* $\nabla' u = (u_{x_2}, \ldots, u_{x_N})$.) *Furthermore, if* $u \in W^{1,p}(\mathbb{R}^N) \cap \mathcal{S}_+$ *for some* $p \in [1, +\infty]$, *then* $u^* \in W^{1,p}(\mathbb{R}^N) \cap \mathcal{S}_+$,

$$\|\nabla u\|_p \geqslant \|\nabla u^*\|_p, \quad \text{and} \tag{4.11}$$

$$\left\| \frac{\partial u}{\partial x_i} \right\|_p \geqslant \left\| \frac{\partial u^*}{\partial x_i} \right\|_p, \quad i = 1, \ldots, N. \tag{4.12}$$

PROOF. (1) First notice that the functional

$$J(v) := \int_{\mathbb{R}^N} G\big(x', v, |v_{x_1}|, \nabla' v\big) \mathrm{d}x$$

is weakly lower semicontinuous on $C^{0,1}_{0+}(\mathbb{R}^N)$, and we have that $J(v) = J(v_H)$ $\forall H \in \mathcal{H}^*$, by Lemma 3.7. Assuming $u \in C^{0,1}_{0+}(B_R)$ $(R > 0)$, (4.11) can then be proved analogously as (4.8) by minimizing $\|v - u^*\|_2$ over the set $B^*(u) = \{v \in C^{0,1}_{0+}(B_R)\colon\ v^* = u^*,\ \omega_v \leqslant \omega_u,\ J(v) \leqslant J(u)\}$, and by working with halfspaces in $\mathcal{H}^*_0$. We leave the details to the reader.

(2) The inequalities (4.11) and (4.12) for $p \in [1, \infty)$ can be shown analogously as in the previous proof choosing $G(x', v, z_1, z') = (\sum_{j=1}^N z_j^2)^{(p/2)}$, respectively $G(x', v, z_1, z') = |z_i|^p$, and by using the weak lower semicontinuity of the functionals $\|\nabla \cdot\|_p$ and $\|(\partial/\partial x_i) \cdot\|_p$, $i = 1, \ldots, N$.

(3) Let $u \in W^{1,\infty}(\mathbb{R}^N) \cap \mathcal{S}_+$. We then show (4.11) as in the previous proof.

Furthermore, let $t > 0$ and $u_t := \max\{u - t; 0\}$. Then $\mathcal{L}^N(\{u_t > 0\}) < \infty$ and $u_t \in W^{1,p}(\mathbb{R}_N)$ $\forall p \in [1, \infty]$. Since $(u_t)^* = (u^*)_t$ by Lemma 3.2, we find that

$$\left\| \frac{\partial u_t}{\partial x_i} \right\|_\infty = \lim_{p\to\infty} \left\| \frac{\partial u_t}{\partial x_i} \right\|_p \geqslant \lim_{p\to\infty} \left\| \frac{\partial (u^*)_t}{\partial x_i} \right\|_p = \left\| \frac{\partial (u^*)_t}{\partial x_i} \right\|_\infty, \quad i = 1, \ldots, N.$$

Passing to the limit $t \searrow 0$, we deduce from this the inequalities (4.12) for $p = \infty$. □

The inequalities (4.16) and (4.17) below are well-known (see e.g. [69,21] and [18]). Furthermore, (4.13) has been proved in the special case that $G = f(|x|)|\nabla u|^p$, where f is nonnegative and continuous, in [104].

THEOREM 4.5. *Let $u \in C^{0,1}(\Omega)$ where either $\Omega = B_{R_2} \setminus \overline{B_{R_1}}$, $\Omega = B_{R_2}$, $\Omega = \mathbb{R}^N$, or $\Omega = \mathbb{R}^N \setminus \overline{B_{R_1}}$, for some $R_2 > R_1 \geqslant 0$, and let G be a function satisfying*

$$G \in L^\infty\big(\mathbb{R}_0^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}_0^+\big), \quad G = G(r, v, s, t),$$

G continuous in (v, s, t), nondecreasing in t and convex in (s, t),

$$G(r, 0, 0, 0) = 0 \quad \forall r \geqslant 0.$$

Then

$$\int_\Omega G\left(|x|, u, \frac{\partial u}{\partial r}, |\nabla_{r^\perp} u|\right) \mathrm{d}x \geqslant \int_\Omega G\left(|x|, Cu, \frac{\partial Cu}{\partial r}, |\nabla_{r^\perp} Cu|\right) \mathrm{d}x, \tag{4.13}$$

provided that the left integral in (4.13) *converges.* (*Here $\partial u/\partial r$ is the radial derivative $(x \cdot \nabla u)/|x|$ and $\nabla_{r^\perp} u$ denotes the tangential gradient on $\partial B_{|x|}$, that is $\nabla_{r^\perp} u = \nabla u - (x/|x|)(\partial u/\partial r)$.*)

Furthermore, if $u \in W^{1,p}(\Omega)$ for some $p \in [1, +\infty]$, then so does Cu,

$$\|\nabla u\|_p \geqslant \|\nabla Cu\|_p, \tag{4.14}$$

$$\left\| \frac{\partial u}{\partial r} \right\|_p \geqslant \left\| \frac{\partial Cu}{\partial r} \right\|_p, \quad \textit{and} \tag{4.15}$$

$$\|\nabla_{r^\perp} u\|_p \geqslant \|\nabla_{r^\perp} Cu\|_p. \tag{4.16}$$

PROOF. (1) Notice that the functional

$$J(v) := \int_\Omega G\left(|x|, v, \frac{\partial v}{\partial r}, |\nabla_{r^\perp} v|\right) \mathrm{d}x \tag{4.17}$$

is weakly lower semicontinuous on $C^{0,1}(\Omega)$, and we have that $J(v) = J(v_H)$ $\forall H \in C\mathcal{H}$ by Lemma 3.7. If then Ω is bounded, one shows (4.13) analogously as (4.10) in the previous theorem by minimizing $\|v - Cu\|_{2,\Omega}$ over the set $CB(u) = \{v \in C^{0,1}(\Omega)\colon\ Cv = Cu,\ \omega_{v,B_R} \leqslant \omega_{u,B_R},\ J(v) \leqslant J(u)\}$, and by working with halfspaces in $C\mathcal{H}_P$.

Taking into account that the values of Cu on the sphere ∂B_r depend only on the values of u on the same sphere, $(r > 0)$, one then finds $J_R(u) \geqslant J_R(Cu)$, where $J_R(v)$ denotes the integral (4.17) with Ω replaced by $\Omega \cap B_R$ $(R > 0)$. Passing to the limit $R \to \infty$, (4.13) follows from this in the general case, too.

(2) The inequalities (4.14)–(4.16) follow analogously as the inequalities (4.11) and (4.12) in the previous Theorem by specializing (4.13). The details are left to the reader. □

COROLLARY 4.2. *Let Ω be a domain in $\mathbb{R}^N$ and $u \in W_{0+}^{1,p}(\Omega)$ for some $p \in [1, \infty)$. Then $u^\star \in W_0^{1,p}(\Omega^\star)$ and $u^* \in W^{1,p}(\Omega^*)$.*

PROOF. Choose a sequence $\{u_n\} \subset C_{0+}^{0,1}(\Omega)$ such that $u_n \to u$ in $W^{1,p}(\Omega)$. Since $\operatorname{supp} u_n \subset \Omega$, it is easy to see that also $\operatorname{supp} u_n^* = (\operatorname{supp} u_n)^* \subset \Omega^*$, so that $u_n^* \in W_0^{1,p}(\Omega)$, $n = 1, 2, \ldots$. Since $u_{n'}^* \rightharpoonup u^*$ in $W^{1,p}(\Omega)$, for a subsequence, we find that $u^* \in W_0^{1,p}(\Omega^*)$.

The proof for the Schwarz symmetrization is analogous and will be omitted. □

REMARK 4.1. The Sobolev norm inequalities in Theorems 4.3–4.5 have many applications.

(1) In particular they are helpful in calculating the best constants in embedding theorems (see e.g. [109,21,113]). Here is an example (see [109]):

Let $p \in (1, N)$. Then there exists a constant $C > 0$ such that

$$\|\nabla v\|_p \geqslant C\|v\|_{Np/(N-p)} \quad \forall v \in W^{1,p}\big(\mathbb{R}^N\big). \tag{4.18}$$

Suppose that there exists a function u such that (4.13) becomes an equality. Then Theorems 3.1 and 4.3 tell us that $|u|^\star$ gives an equality, too, in (4.18).

By standard variational calculus we are thus lead to the problem of finding *radially symmetric and radially decreasing* solutions u of

$$\begin{aligned} &-\Delta_p u \equiv -\nabla\big(|\nabla u|^{p-2}\nabla u\big) = du^{[Np/(N-p)]-1}, \quad u > 0 \quad \text{in } \mathbb{R}^N, \\ &\lim_{|x|\to\infty} u(x) = 0, \end{aligned} \tag{4.19}$$

for some $d > 0$. The solutions of this problem are given by

$$u(x) = \alpha\big(\beta + |x|^{p/(p-1)}\big)^{1-(N/p)}, \quad \text{for some } \alpha, \beta > 0, \tag{4.20}$$

which means that

$$C = \sqrt{\pi}N^{1/p}\left(\frac{N-p}{p-1}\right)^{1-(1/p)}\left[\frac{\Gamma(N/p)\Gamma(1+N-(N/p))}{\Gamma(N)\Gamma(1+(N/2))}\right]^{1/N} \tag{4.21}$$

is the largest possible constant such that (4.18) holds true.

(2) It is well known that the $W^{1,1}$-case in Theorems 4.3–4.5 can be used to obtain some classical isoperimetric inequalities (see e.g. [44,66,113,18,37]). More precisely, (4.9) for $p = 1$ implies the

Isoperimetric inequality in $\mathbb{R}^N$:

Among all smooth sets $\Omega \subset \mathbb{R}^N$ *with given (fixed)* $\mathcal{L}^N$*-measure, the ball yields the smallest perimeter* $(= (N-1)$*-surface measure*).

Furthermore, (4.11) for $p = 1$ implies that the Steiner symmetrization decreases the perimeter of smooth open sets in $\mathbb{R}^N$. Finally, from (4.16) one obtains the

Isoperimetric inequality on the sphere:

Among all smooth open subsets of ∂B_1 *with given* $\mathcal{L}^{N-1}$*-measure a spherical cap* (*which is a set of the form* $\partial B_1 \cap B_\rho(P)$*, for some* $\rho > 0$) *has the smallest perimeter* $(= (N-2)$*-boundary measure*).

(3) Using the inequalities from Theorems 4.3–4.5 one can show isoperimetric inequalities for eigenvalues in PDE.

The following result is well-known, and it was proved simultaneously by Faber and Krahn (see [62] and [77]):

Among all open sets in $\mathbb{R}^N$ *of given* (*fixed*) *measure the ball gives the smallest first eigenvalue of the Dirichlet Laplacian.*

Here is a simple proof: Let Ω be a domain in $\mathbb{R}^N$. The Laplacian on Ω with Dirichlet boundary conditions is a self-adjoint operator with compact inverse, so there exists a sequence of positive eigenvalues $0 < \lambda_1(\Omega) \leqslant \lambda_2(\Omega) \leqslant \cdots$ with $\lim_{n\to\infty} \lambda_n(\Omega) = +\infty$ and a sequence of corresponding eigenfunctions $\{u_n\}$, such that

$$\begin{aligned} &-\Delta u_n = \lambda_n(\Omega) u_n \quad \text{in } \Omega, \\ &u_n = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.22}$$

The eigenvalues are given through the following minimization problems,

$$\lambda_n(\Omega) = \min\left\{ \max\left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega v^2 \,\mathrm{d}x} \colon\ v \in E_k,\ v \neq 0 \right\} \colon \right. \\ \left. E_n \subset W_0^{1,2}(\Omega),\ \text{ linear subspace of dimension } n \right\}. \tag{4.23}$$

For the first eigenvalue, it reads,

$$\lambda_1(\Omega) = \min\left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega v^2 \,\mathrm{d}x} \colon\ v \in W_0^{1,2}(\Omega),\ v \neq 0 \right\}, \tag{4.24}$$

the minimum being achieved by u_1. Notice that the first eigenvalue is simple, and u_1 *does not change sign*, by Krein–Rutman's Theorem. (Actually, u_1 never vanishes in Ω, by the Strong Maximum Principle.) Assuming $u_1 \geqslant 0$, we then have by Theorems 3.2 and 4.3,

$$\lambda_1(\Omega) = \frac{\int_\Omega |\nabla u_1|^2 \,\mathrm{d}x}{\int_\Omega u_1^2 \,\mathrm{d}x} \geqslant \frac{\int_{\Omega^\star} |\nabla u_1^\star|^2 \,\mathrm{d}x}{\int_{\Omega^\star} (u_1^\star)^2 \,\mathrm{d}x} \geqslant \lambda_1\big(\Omega^\star\big),$$

proving the result.

M. Ashbaugh and R. Benguria [14,15] showed that the following result:

Among all domains Ω with given volume, the ball maximizes the quotient $\lambda_2(\Omega)/\lambda_1(\Omega)$.

Their beautiful proof relies on many arguments including inequality (4.9).

Notice, that there have also been obtained some results related to Faber and Krahn's inequality for the bi-Laplacian operator $\Delta^2 = \Delta(\Delta)$, see [94,16].

Finally we emphasize that there are still many open isoperimetric problems for eigenvalues in PDE. Overviews are given, for instance, in [95,13,74] and [70].

(4) Any rearrangement is a continuous mapping on $L^p_+(\mathbb{R}^N)$ $(1 \leqslant p < \infty)$, by Corollary 1. One might ask whether the symmetrizations are continuous in $W^{1,p}_+(\mathbb{R}^N)$, too. Indeed, it has been shown by A. Burchard that the answer is yes for the Steiner symmetrization, (see [40], see also [46] for the one-dimensional case $N = 1$). On the other hand, F. Almgren and E. Lieb proved that the answer is no for the Schwarz symmetrization when $N \geqslant 2$, that is, there are sequences $\{u_n\} \subset W^{1,p}_+(\mathbb{R}^N)$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$ but $(u_n)^\star \not\to u^\star$ in $W^{1,p}(\mathbb{R}^N)$ (see [2]).

There is a vast literature on a-priori estimates for elliptic and parabolic equations using Schwarz symmetrization (see e.g. [8–11,110,111,93,20,6,7,18,4] and the references cited therein). We also emphasize the survey [115] where fundamental ideas and results are given.

Since we are mainly interested in symmetry questions here, we restrict ourselves to a simple, but typical result, which is due to Talenti [110]:

THEOREM 4.6. *Let Ω be a bounded domain in $\mathbb{R}^N$, $p \in (1, N)$, and let $f \in L^q_+(\Omega)$, where $q \in [Np/(Np - N + p), \infty]$. Furthermore, let u, v be weak solutions of the following boundary value problems*:

$$u \in W^{1,p}_0(\Omega), \quad -\Delta_p u = -\nabla\big(|\nabla u|^{p-2}\nabla u\big) = f \quad \text{in } \Omega, \tag{4.25}$$

$$v \in W^{1,p}_0\big(\Omega^\star\big), \quad -\Delta_p v = -\nabla\big(|\nabla v|^{p-2}\nabla v\big) = |f|^\star \quad \text{in } \Omega^\star. \tag{4.26}$$

Then

$$0 \leqslant |u|^\star \leqslant v \quad \text{in } \Omega^\star. \tag{4.27}$$

PROOF. (1) First observe that if $w \in W^{1,p}_0(\Omega)$ solves $-\Delta_p w = |f|$ in Ω, then the Maximum Principle tells us that $|u| \leqslant w$ in Ω so that there is no restriction in assuming that f is nonnegative in (4.25).

(2) Assume that $f \in L^\infty_+(\Omega)$, and that $\partial\Omega \in C^\infty$. This implies (see e.g. [107,86]) that $u \in C^{1,\alpha}(\overline{\Omega})$, and $v \in C^{1,\alpha}(\overline{\Omega^\star})$, for some $\alpha \in (0, 1)$. Hence it follows from Theorem 4.1 that $u^\star \in C^{0,1}_{0+}(\Omega^\star)$. In terms of the symmetric decreasing rearrangement $u^\sharp = u^\sharp(z)$ $(z \in [0, |\Omega|])$ (see formula (3.57)), this reads as $u^\sharp \in C^{0,1}([0, |\Omega|])$.

We will call $s \in [0, \max u)$ a *regular value* if $L_s := \{u = s\}$ is a smooth $(N-1)$-hypersurface, if $\nabla u \neq 0$ on L_s, if $z = \mu_u(s) = \mathcal{L}^N(\{u > s\})$ is a density point of $\partial u^\sharp/\partial z$,

and if $u^\sharp$ is differentiable there. By Sard's Theorem (see [66]), almost all $s \in [0, \max u]$ are regular values.

Let s be regular. Then Green's Theorem together with Corollary 3.2 gives

$$\begin{aligned}\int_{L_s} |\nabla u|^{p-1}\,\mathrm{d}S &= \int_{\{u>s\}} f(x)\,\mathrm{d}x = \int_\Omega f(x)\chi\{u>s\}(x)\,\mathrm{d}x \\ &\leqslant \int_{\Omega^\star} f^\star(x)\chi\{u^\star>s\}(x)\,\mathrm{d}x = \int_{\{u^\star>s\}} f^\star(x)\,\mathrm{d}x. \end{aligned} \tag{4.28}$$

Furthermore, letting $s < s+h < \max u$ and $u_{s,h} = \max\{(u-s)_+; h\}$, we have that $(u_{s,h})^\star = (u^\star)_{s,h}$, and thus

$$\int_{\{s<u<s+h\}} |\nabla u|^p\,\mathrm{d}x = \|\nabla u_{s,h}\|_p^p \geqslant \big\|\nabla (u_{s,h})^\star\big\|_p^p = \int_{\{s<u^\star<s+h\}} |\nabla u^\star|^p\,\mathrm{d}x.$$

By Fleming–Rishel's formula (see e.g. [66,113]), this means

$$\begin{aligned}\int_{L_s} |\nabla u|^{p-1}\,\mathrm{d}S &= \lim_{h\searrow 0} \frac{1}{h}\int_{\{s<u<s+h\}} |\nabla u|^p\,\mathrm{d}x \\ &\geqslant \lim_{h\searrow 0} \frac{1}{h}\int_{\{s<u^\star<s+h\}} |\nabla u^\star|^p\,\mathrm{d}x = \int_{\{u^\star=s\}} |\nabla u^\star|^{p-1}\,\mathrm{d}S.\end{aligned}$$

In view of (4.28) we then obtain

$$\int_{\{u^\star=s\}} |\nabla u^\star|^{p-1}\,\mathrm{d}S \leqslant \int_{\{u^\star>s\}} f^\star(x)\,\mathrm{d}x. \tag{4.29}$$

On the other hand, it is easy to see that $v = v^\star$, and thus

$$\int_{\{u^\star=s\}} |\nabla v|^{p-1}\,\mathrm{d}S = \int_{\{u^\star>s\}} f^\star(x)\,\mathrm{d}x. \tag{4.30}$$

Now (4.29) and (4.30) imply

$$\frac{\partial u^\sharp}{\partial z}\big(\mu_u(s)\big) \geqslant \frac{\partial v^\sharp}{\partial z}\big(\mu_u(s)\big) \quad \text{for all regular values } s. \tag{4.31}$$

Since $u^\sharp(z)$ is constant on intervals $[\liminf_{t\to s}\mu_u(t), \limsup_{t\to s}\mu_u(t)]$ ($s \in [0, \max u]$), and since $u^\sharp(|\Omega|) = v^\sharp(|\Omega|) = 0$, an integration of (4.31) yields

$$u^\sharp(z) \leqslant v^\sharp(z) \quad \forall z \in \big[0, |\Omega|\big],$$

which implies (4.27) in this case.

(3) In the general case we choose a sequence $\{\Omega_n\}$ of C^∞-domains with $\Omega_n \subset\subset \Omega_{n+1}$, $n = 1, 2, \ldots$, such that $\Omega = \bigcup_{n=1}^\infty \Omega_n$, and a sequence $\{f_n\} \subset L^\infty(\Omega)$ with $f_n \leqslant f_{n+1}$,

$n = 1, 2, \ldots$, and such that $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in Ω. If then u_n, v_n denote the solutions of the boundary problems (4.25), respectively (4.26), with Ω, f, respectively $\Omega^\star$, $f^\star$ replaced by Ω_n, f_n, respectively $(\Omega_n)^\star$, $(f_n)^\star$, $n = 1, 2, \ldots$, we find that $u_n \nearrow u$ a.e. in Ω, and $v_n \nearrow v$ a.e. in $\Omega^\star$, and the assertion follows. □

REMARK 4.2. The usefulness of Theorem 4.6 lies in the fact that the solution v of (4.26) can be explicitly computed,

$$v^\sharp(z) = N^{-p/(p-1)}(\kappa_N)^{-p/(N(p-1))} \int_z^{|\Omega|} \left(\int_0^\zeta f^\sharp(\sigma)\,\mathrm{d}\sigma \right)^{1/(p-1)} \times \zeta^{-p(N-1)/(N(p-1))}\,\mathrm{d}\zeta, \quad z \in \big[0, |\Omega|\big]. \tag{4.32}$$

From (4.32) one sees that v is bounded if $f^\star \in L^q(\Omega^\star)$ for some $q > N/p$. By Theorem 4.6 this means that if $f \in L^q(\Omega)$ with $q > N/p$, then $u \in L^\infty(\Omega)$. On the other hand, if

$$f^\star(x) = |x|^{-p'}, \quad \text{where } p' > p,$$

then $f^\star \in L^{(N/p')-\varepsilon}(\Omega^\star)$ $\forall \varepsilon \in (0, (N/p') - 1]$, but v is unbounded. Moreover, if $N > p/(p-1)$, $\Omega^\star = B_R$ for some $R > 0$, and if

$$f^\star(x) = |x|^{-p}\big(\log\big(eR/|x|\big)\big)^{1-p},$$

then $f^\star \in L^{N/p}(\Omega^\star)$, and again $v \notin L^\infty(\Omega^\star)$.

REMARK 4.3. We finally comment on some extensions and other common rearrangements.

(1) The Steiner symmetrization can be defined for a more general class of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ which is characterized by the condition

$$\mathcal{L}^1\big(\big\{u(\cdot, x') > \lambda\big\}\big) < \infty \quad \forall \lambda > \inf u(\cdot, x') \text{ and for a.e. } x' \in \mathbb{R}^{N-1},$$

in the same manner as for the class $\mathcal{S}$. Then u^* are again equidistributed in the sense that (3.60) holds.

(2) *Nondecreasing rearrangement w.r.t.* x_1:
Let $u : \mathbb{R}^N \to \mathbb{R}$ measurable. Furthermore, assume that

$$\mathcal{L}^1\big(\big\{u(\cdot, x') < \lambda\big\} \cap \mathbb{R}_0^+\big) < \infty \quad \text{and} \quad \mathcal{L}^1\big(\big\{u(\cdot, x') > \lambda\big\} \setminus \mathbb{R}_0^+\big) < \infty$$
$$\forall \lambda \in \big(\inf u(\cdot, x'), \sup u(\cdot, x')\big), \text{ and for a.e. } x' \in \mathbb{R}^{N-1}.$$

Notice that if $u \in C(\Omega)$ then the above condition is equivalent to,

$$\lim_{x_1\to\infty} u(x_1, x') = \sup u(\cdot, x') \quad \text{and} \quad \lim_{x_1\to-\infty} u(x_1, x') = \inf u(\cdot, x')$$
$$\forall x' \in \mathbb{R}^{N-1}.$$

Then one may define the *nondecreasing rearrangement of u w.r.t. x_1*, Nu (see [72,23]), by

$$Nu(x_1,x') = \sup\big\{\lambda\in\mathbb{R}\colon\ x_1 > \mathcal{L}^1\big(\big\{u(\cdot,x')<\lambda\big\}\cap\mathbb{R}_0^+\big) - \mathcal{L}^1\big(\big\{u(\cdot,x')>\lambda\big\}\setminus\mathbb{R}_0^+\big)\big\},\quad x\in\mathbb{R}^N. \tag{4.33}$$

The function Nu is nondecreasing in x_1 and the functions u and Nu are equidistributed in the sense that

$$\mathcal{L}^1\big(\big\{\alpha<u(\cdot,x')<\beta\big\}\big)=\mathcal{L}^1\big(\big\{\alpha<u(\cdot,x')<\beta\big\}\big)\quad\forall\alpha,\beta\in\mathbb{R}\text{ with}$$
$$\inf u(\cdot,x')<\alpha<\beta<\sup u(\cdot,x'),\ \text{and for a.e. } x'\in\mathbb{R}^{N-1}. \tag{4.34}$$

In many applications u is only defined on a halfspace. Let, for instance, u be measurable on $H_-:=(-\infty,0)\times\mathbb{R}^{N-1}$, and such that

$$\mathcal{L}^1\big(\big\{u(\cdot,x')>\lambda\big\}\cap(-\infty,0)\big)<\infty\quad\forall\lambda>\inf u(\cdot,x')\text{ and for a.e. } x'\in\mathbb{R}^{N-1}.$$

Extending u onto $\mathbb{R}^N$ by reflexion, $u(x_{H_-}):=u(x)$, $(x\in H_-)$, we then may define the nondecreasing rearrangement of Nu in H_- as the restriction of u^* to H_-. This definition is justified by the fact that Nu is nondecreasing w.r.t. x_1 in H_-, and satisfies (4.34). This clearly shows the similarity of the Steiner symmetrization and the nondecreasing rearrangement, and it is therefore not surprising that the two rearrangements share a lot of properties (see [72] and [23]).

(3) The Steiner symmetrization of a function $u\in\mathcal{S}$ can also be visualized as a Schwarz symmetrization of the function $u(\cdot,x')$ – with $x'\in\mathbb{R}^{N-1}$ fixed – on the real line. Similarly, by decomposing $x=(y,x'')$ $(y\in\mathbb{R}^k, x''\in\mathbb{R}^{N-k})$, one can construct the so-called (k,N)-Steiner symmetrizations as Schwarz symmetrizations of the function $u(\cdot,x'')$ on $\mathbb{R}^k$ $(1\leqslant k\leqslant N-1)$, see [101,18,37].

Furthermore, the cap symmetrization admits lower-dimensional generalizations by 'cap symmetrizing' the function $u(\cdot,x'')$, on $\mathbb{R}^k$ for each $x''\in\mathbb{R}^{N-k}$, see [18].

(4) The two-point rearrangement approach is also applicable to symmetrizations on the hyperbolic space $\mathbf{H}^N$ (see [18,114]).

(5) Some results which are similar to those given in this section can be obtained for a symmetrization in Gauss space (see e.g. [27,79]).

(6) The *anisotropic symmetrization* is a rearrangement adapted to anisotropic variational problems. In those problems, the function of the gradient in the functional does not depend on the Euclidean norm, but on a positively homogeneous function. Integral inequalities including Polya–Szegö-type principles for these rearrangements have been analyzed in [3] and [119].

5. Symmetry results

In this section we use two-point rearrangements in order to prove symmetry results for minimizers of some elliptic variational problems (see [35,36]). We emphasize that this part does not require any knowledge about the symmetrization inequalities of Section 4.

We will study the uniformly elliptic and the nonuniformly elliptic case separately. The proofs in the first case are much simpler, since we may employ the Principle of Unique Continuation, while using the fact that the minimizers are strong $W^{2,2}$-solutions of the corresponding Euler equation. In the second case we deal with weak C^1-solutions of the corresponding Euler equation. Here the symmetry relies on the use of the Strong Maximum Principle.

5.1. *Uniformly elliptic case*

The results of this section are based on the

PRINCIPLE OF UNIQUE CONTINUATION (PUC) (see [71]). *Let Ω be a domain in $\mathbb{R}^N$, and let $u \in W^{2,2}(\Omega)$ satisfy*

$$-a_{ij}u_{x_ix_j} + b_i u_{x_i} + cu = 0 \quad \text{in } \Omega, \tag{5.1}$$

where $a_{ij} \in C^{0,1}(\Omega)$, and $b_i, c \in L^\infty(\Omega)$. Furthermore, suppose that there is a nonempty open subset U of Ω such that $u \equiv 0$ in U. Then $u \equiv 0$ in Ω.

We emphasize however, that some of the results below could be obtained as well by using the more complicated method of Section 5.2. (In particular, Theorem 5.2 is a special case of Theorem 5.5.).

First we consider a problem in a domain with rotational symmetry (compare [35]). Let $\Omega_1 = B_{R_2} \setminus \overline{B_{R_1}}$, or $\Omega_1 = B_{R_2}$, for some $R_2 > R_1 \geqslant 0$, and let F, G, H functions satisfying

$$\begin{aligned}
&F, G, H \in C^2(\mathbb{R}), \quad F' =: f, \quad G' =: g, \quad H' =: h,\\
&\big|f(t)\big|, \big|g(t)\big| \leqslant c\big(1 + |t|^p\big), \qquad \big|h(t)\big| \leqslant c\big(1 + |t|^q\big),\\
&\text{with } c > 0,\ 1 \leqslant p < (N+2)/(N-2),\\
&1 \leqslant q < N/(N-2), \text{ if } N \geqslant 3, \text{ and } p, q \text{ finite if } N = 2, \qquad (5.2)\\
&g \text{ does not vanish on intervals of } \mathbb{R}, \qquad (5.3)
\end{aligned}$$

and

$$K_1 = \left\{ v \in W^{1,2}(\Omega_1) \colon \int_{\Omega_1} G(v)\,\mathrm{d}x = 1 \right\}.$$

We consider the following variational problem

$$(\mathrm{P}_1) \quad J_1(v) \equiv \int_{\Omega_1} \left(\frac{1}{2} |\nabla v|^2 - F(v) \right) \mathrm{d}x + \int_{\partial\Omega_1} H(v)\, \mathrm{d}S \longrightarrow \mathrm{Inf!}, \quad v \in K_1.$$

If u is a minimizer of (P_1) then standard variational calculus (see e.g. [124]) shows that

$$-\Delta u = f(u) + \alpha g(u) \quad \text{in } \Omega_1, \tag{5.4}$$

$$\frac{\partial u}{\partial \nu} + h(u) = 0 \quad \text{on } \partial\Omega_1 \quad (\nu\text{: exterior normal}),$$

$$\int_{\Omega_1} \big(\nabla u \nabla \varphi - f(u)\varphi \big)\, \mathrm{d}x + \int_{\partial\Omega_1} h(u)\varphi\, \mathrm{d}S = 0, \quad \text{and}$$

$$\int_{\Omega_1} \big(|\nabla\varphi|^2 - \big(f'(u) + \alpha g'(u) \big)\varphi^2 \big)\, \mathrm{d}x + \int_{\partial\Omega_1} h'(u)\varphi^2\, \mathrm{d}S \geqslant 0 \tag{5.5}$$

for every $\varphi \in W^{1,2}(\Omega_1)$ satisfying $\int_{\Omega_1} g(u)\varphi\, \mathrm{d}x = 0$, and

$u \in W^{2,2}(\Omega_1) \cap C^1(\overline{\Omega_1})$,

with $\alpha \in \mathbb{R}$. Notice that problem (P_1) with $F(v) = -(1/2)v^2$, $G(v) = |v|^{p+1}$ and $H \equiv 0$, has been extensively studied by M. Esteban and other authors (see [47,60,61]).

We first recall the well-known result that in case of the unconstrained problem, that is $K_1 = W^{1,2}(\Omega_1)$, minimizers u of (P_1) are radial. This statement has been shown in [108] for the case of Dirichlet boundary conditions. But the argument used in [108] is applicable in our case, too. For the convenience of the reader we give a proof:

Setting

$$I(\varphi) := \left(\int_{\Omega_1} \big(|\nabla\varphi|^2 - f'(u)\varphi^2 \big)\, \mathrm{d}x + \int_{\partial\Omega_1} h'(u)\varphi^2\, \mathrm{d}S \right) \left(\int_{\Omega_1} \varphi^2\, \mathrm{d}x \right)^{-1},$$

we find that $I(\varphi) \geqslant 0\ \forall \varphi \in W^{1,2}(\Omega_1)$ with $\varphi \not\equiv 0$, by (5.5). Now fix any coordinate system $x = (x_1, x_2, x'')$ ($x \in \mathbb{R}^N$, $x'' \in \mathbb{R}^{N-2}$), and let $v := x_1 u_{x_2} - x_2 u_{x_1}$ the angular derivative of u in the (x_1, x_2)- plane. We have that $-\Delta v = f'(u)v$ in Ω_1, and $(\partial v/\partial \nu) + h'(u)v = 0$ on $\partial\Omega_1$. Assuming $v \not\equiv 0$ we then find $I(v) = 0$. But this means that v can have one sign only, which is impossible since $\int_{\Omega_1} v\, \mathrm{d}x = 0$. Hence u is radial.

In this connection we also mention the well-known result of Casten and Holland [43] that in case of Neumann boundary conditions, that is $H(v) \equiv 0$, any minimizer of the unconstrained problem is constant.

THEOREM 5.1. *Let u be a minimizer of* (P_1). *Then there is a rotation about zero ρ such that $u(\rho\,\cdot) = Cu(\cdot)$.*

PROOF. We will assume that u is not constant in Ω_1 – since otherwise the assertion is trivially true.

First we claim that u cannot be constant on open subsets of Ω_1. Indeed, if $u \equiv c$ on an open set $U \subset \Omega_1$, then $f(c) + \alpha g(c) = 0$. Setting $v := u - c$ we have that $-\Delta u = c(x)v$ in Ω_1, with some bounded function c, while $v \equiv 0$ in U. The (PUC) then tells us that $v \equiv 0$ throughout Ω_1, a contradiction.

Let $H \in C\mathcal{H}$. Then the results of Section 3.1 show that $J(u_H) = J(u)$ and $u_H \in K_1$, so that u_H is a minimizer, too. This means that u_H satisfies (5.4), (5.5) with the Lagrangian multiplier α possibly replaced by some other number α'.

We next claim that actually $\alpha = \alpha'$. Indeed, there is an open set $V \subset \Omega_1$ with $u = u_H$ or $\sigma_H u = u_H$ on V. Then $\alpha g(u) = \alpha' g(u_H)$, respectively $\alpha g(\sigma_H u) = \alpha' g(u)$, in V. Since u, respectively $\sigma_H u$, is not constant on V this means that $\alpha = \alpha'$, by the nondegeneracy condition (5.3).

Finally, setting $v := u - u_H$, respectively $v := \sigma_H u - u_H$, we find that v satisfies $-\Delta v = c(x)v$ in Ω_1, with some bounded function c, while $v \equiv 0$ in V. By the (PUC) this means that $v \equiv 0$, or equivalently $u \equiv u_H$, respectively $\sigma_H u \equiv u_H$ throughout Ω_1. Since H was arbitrary, the assertion follows by Lemma 3.8. □

REMARK 5.1. Theorem 5.1 remains valid in an unbounded domain $\Omega_1 = \mathbb{R}^N \setminus \overline{B_{R_1}}$ and/or if the functions F, G and H depend additionally on $|x|$ and satisfy appropriate growth conditions. Recently the author obtained symmetry results for global and local minimizers, and also for stationary solutions of (P_1), including situations with more than one integral constraint and/or with Dirichlet boundary conditions. Using the same method, he also proved related symmetry and monotonicity results for variational problems in cylindrical domains, see [35].

Next we study a problem in the entire space (compare [36]). Let F, G as in (5.2) and (5.3) except that the growth conditions are replaced by

$$\big|F(t)\big|, \big|G(t)\big| \leqslant c\big(|t|^p + |t|^q\big), \quad \text{with } c > 0 \text{ and } p, q \geqslant 1, \tag{5.6}$$

and let

$$K_2 := \left\{ v \in L^p\big(\mathbb{R}^N\big) \cap L^q\big(\mathbb{R}^N\big) \colon \nabla v \in L^2\big(\mathbb{R}^N\big),\ \int_{\mathbb{R}^N} G(v)\,\mathrm{d}x = d \right\} \quad (d \in \mathbb{R}).$$

We consider the following variational problem:

$$(P_2) \quad J(v) \equiv \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla v|^2 - F(v) \right) \mathrm{d}x \longrightarrow \text{Inf!}, \quad v \in K_2.$$

Assume that u is a solution of (P_2). Then u satisfies

$$-\Delta u = f(u) + \alpha g(u) \quad \text{on } \mathbb{R}^N, \tag{5.7}$$

where $\alpha \in \mathbb{R}$. If in addition, u is bounded, then standard elliptic estimates show that

$$u \in W^{2,2}_{\mathrm{loc}}\big(\mathbb{R}^N\big) \cap C^1\big(\mathbb{R}^N\big), \quad \text{and} \tag{5.8}$$

$$\lim_{|x|\to\infty} u(x) = 0. \tag{5.9}$$

Proceeding as in the previous proof and working with halfspaces $H \in \mathcal{H}^\star$, we then obtain the following

THEOREM 5.2. *Let u be a bounded minimizer of* (P_2) *and* $\mathcal{L}^N(\{u > 0\}) > 0$. *Then there is a point* $\xi \in \mathbb{R}^N$ *such that* $u(\cdot - \xi) = u^\star(\cdot)$. *In particular, u does not change sign.*

Our method is also applicable to local minimizers of some variational problems.
We will say that u is a *local* minimizer of (P_2) if there exists a number $\varepsilon > 0$ such that

$$\|u - v\|_p + \|u - v\|_q + \|\nabla(u - v)\|_2 < \varepsilon \quad \forall v \in K_2.$$

The next result is close to those that have been obtained for stationary solutions of (P_2) using the Moving Plane Method (MPM) (see the remarks on problem (P) in the introduction). However, the (MPM) requires either an asymptotic estimate of the solution u at infinity (see [65,82]), or some additional condition on the nonlinearity $f(t) + \alpha g(t)$ near 0 (see [83]). Notice that the proof of Theorem 5.3 below uses arguments which are very similar to those used in the (MPM).

THEOREM 5.3. *Let u be a nonnegative and bounded local minimizer of* (P_2). *Then the assertion of Theorem* 5.2 *holds.*

PROOF. Writing $H_\lambda = \{x\colon\ x_1 > \lambda\}$, $u_{H_\lambda} = u_\lambda$, and $\sigma_{H_\lambda} u = \sigma_\lambda u$, we first observe that $u_\lambda \in K_2$ and $J_2(u_\lambda) = J_2(u)$ $\forall \lambda \in \mathbb{R}$. Since u is nonnegative, we have that $u_\lambda \to u$ in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ as $\lambda \to -\infty$, by Lemma 3.5. Hence we find a number $\lambda_1 \in \mathbb{R}$ such that u_λ is a local minimizer, too, $\forall \lambda \leqslant \lambda_1$. Clearly we may assume that $u_\lambda \not\equiv \sigma_\lambda u$ for these λ. Using the (PUC) as in the proof of Theorem 5.1 we then show that $u = u_\lambda$ for $\lambda \leqslant \lambda_1$. Similarly one proves that there exists a number λ_2, $(\lambda_2 > \lambda_1)$, such that $\sigma_\lambda u = u_\lambda$ $\forall \lambda \geqslant \lambda_2$.

Now let

$$\lambda^* := \max\bigl\{\lambda \in \mathbb{R}\colon\ u = u_\mu\ \forall \mu \in (-\infty, \lambda)\bigr\}.$$

By continuity, we have that $\lambda^* \in [\lambda_1, \lambda_2]$ and $u = u_\lambda$ $\forall \lambda \in (-\infty, \lambda^*]$. Using again Lemma 3.5 we have that $u_\lambda \to u$ in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$ and $\nabla u_\lambda \to \nabla u$ in $L^2(\mathbb{R}^N)$ as $\lambda \to \lambda^*$. Hence there is a number $\delta > 0$ such that u_λ is a local minimizer, too, $\forall \lambda \in [\lambda^*, \lambda^* + \delta]$. Using again the (PUC) we then must have $u = u_\lambda$ or $\sigma_\lambda u = u_\lambda$ for any of these λ. On the other hand, we cannot have $u = u_\lambda$ on an interval $[\lambda^*, \lambda^* + \delta']$ $(0 < \delta' \leqslant \delta)$, in view of the maximality of λ^*. By continuity, it then follows that $u = u_{\lambda^*} = \sigma_{\lambda^*} u$. Hence $u = u_\lambda$ for $\lambda \leqslant \lambda^*$ and $\sigma_\lambda u = u_\lambda$ for $\lambda \geqslant \lambda^*$, which means that $u(x_1 - \lambda^*, x') = u^*(x_1, x')$ on $\mathbb{R}^N$. We may repeat these considerations in any rotated coordinate system, thus obtaining that if $H \in \mathcal{H}$ then either $u = u_H$ or $\sigma_H u = u_H$. The assertion then follows by Lemma 3.8. □

REMARK 5.2.

(1) With some more effort one can prove that Theorems 5.2 and 5.3 still hold for a more general functional J_2,

$$J_2(v) = \int_{\mathbb{R}^N} \big(M\big(|\nabla v|\big) - F\big(|x|, v\big)\big)\,\mathrm{d}x,$$

where $M \in C^2(\mathbb{R}_0^+)$, $M'(0) = 0$, $M''(0) > 0$, M strictly convex (a prototype here is $M(t) = \sqrt{1+t^2}$, the minimal surface operator), and $F = F(s,t)$ $((s,t) \in \mathbb{R}_0^+ \times \mathbb{R})$, F twice differentiable w.r.t. t, $F(\cdot,t), F_t(\cdot,t) \in L^\infty(\mathbb{R}^N)$ $\forall t \in \mathbb{R}$, and $F_t(s,t)$ non-increasing in s, and if K_2 contains several integral constraints. Furthermore, since there holds a variant of the (PUC) for elliptic systems (see [90]) the results can be generalized to variational problems associated to cooperative elliptic systems (compare also Theorem 5.4 of the next section).

(2) O. Lopes [90] studied problem (P_1) under Neumann conditions. He also investigated a variational problem that is related to an elliptic system,

$$E(v_1, \ldots, v_m) := \int_{\Omega_1} \left(\frac{1}{2} \sum_{i=1}^{m} |\nabla v_i|^2 - F\big(|x|, v_1, \ldots, v_m\big) \right) \mathrm{d}x \to \mathrm{Inf!},$$

subject to

$$v_i \in W^{1,2}(\Omega_1) \quad (i = 1, \ldots, m), \quad \text{and} \quad \int_{\Omega_1} G\big(|x|, v_1, \ldots, v_m\big)\,\mathrm{d}x = 1,$$

where the functions F and G satisfy appropriate smoothness and growth conditions. He proved both in the scalar and in the vector case, that local minimizers are not radial.

Furthermore, he showed in the system case that any component of the (vector-valued) global minimizer depends, up to some rotation, only on the radial distance $|x|$ and on the geographical latitude $\arccos(x_1/|x|)$, though he did not prove the stronger symmetry property of Theorem 5.1.

In another paper, [91], O. Lopes investigated the same problem in the entire space – with G independent of $|x|$ – showing that any global minimizer is radially symmetric. However, he did not prove that the solutions are monotone in the radial variable which particularly means that they might change sign.

We emphasize that no cooperativity conditions are imposed on the functions F and G in [90,91]. Therefore these results cannot be obtained by our method. (Though there is an overlap with Theorems 5.1–5.3 in the scalar case.)

The proofs in [90,91] are based on a reflexion technique where the (PUC) plays a crucial role, too. On the other hand, the trial functions used in that method are not rearrangements of the solution. Therefore it seems difficult to generalize O. Lopes' results to cases with more than one integral constraint.

5.2. *Degenerate elliptic case*

In this section we study variational problems associated to the p-Laplacian on the whole space. Generalizations of the results below are contained in [36]. Since the difference of two solutions do not satisfy a linear, uniformly elliptic equation, the (PUC) is not applicable. Instead, we will essentially use a general version of the Strong Maximum Principle which has recently been proved by P. Pucci, J. Serrin and H. Zou (see [97] and [98]). We also emphasize that we will not restrict the admissible class to nonnegative functions. (This, in particular, excludes the applicability of the Moving Plane Method!) Nevertheless, in many cases we are able to show that the minimizers do not change sign, as a byproduct of their symmetry.

Let $n \in \mathbb{N}$, $k_i \in \mathbb{N}$, $i = 1, \dots, n$, and $p \in (1, N)$, fixed numbers, let $p^* := Np/(N-p)$, and let $F = F(s, t_1, \dots, t_n)$, $G_{ij} = G_{ij}(t_i)$, be fixed functions defined $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$, $j = 1, \dots, k_i$, $i = 1, \dots, n$, and satisfying

$$
\begin{aligned}
&F \text{ differentiable w.r.t. } t_1, \dots, t_n,\\
&F, F_{t_i} \text{ measurable in } s \text{ and continuous in } t_1, \dots, t_n,\\
&F(\cdot, t_1, \dots, t_n), F_{t_i}(\cdot, t_1, \dots, t_n) \in L^\infty\big(\mathbb{R}_0^+\big) \quad \forall (t_1, \dots, t_n) \in \mathbb{R}^n,\\
&G_{ij} \in C^1(\mathbb{R}), \quad M' =: m, \quad F_{t_i} =: f_i, \quad G'_{ij} =: g_{ij},\\
&F(s, 0, \dots, 0) = f_i(s, 0, \dots, 0) = G_{ij}(0) = g_{ij}(0) = 0,\\
&f_i(s, t_1, \dots, t_n) \text{ nonincreasing in } s \text{ and nondecreasing in } t_k,\\
&\quad \text{for } k \in \{1, \dots, n\},\ k \neq i, \qquad (5.10)\\
&\big|f_i(s, t_1, \dots, t_n)\big|, \big|g_{ij}(t_i)\big| \leqslant c\big(1 + |t|^r\big) \quad \forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n, \qquad (5.11)\\
&\quad \text{for some } c > 0 \text{ and } r \in (1, p^* - 1),\\
&\quad \forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n,\ j = 1, \dots, k_i;\\
&\text{if, for some constants } \beta_1, \dots, \beta_{k_i}, \text{ the function } \textstyle\sum_{j=1}^{k_i} \beta_j g_{ij}(t) \text{ vanishes}\\
&\text{on some interval } c < t < d, \text{ then it vanishes everywhere on } \mathbb{R};\\
&\quad i = 1, \dots, n. \qquad (5.12)
\end{aligned}
$$

Setting

$$
\begin{aligned}
K_3 = \Bigg\{ &v \equiv (v_1, \dots, v_n) \in \big(L^{p^*}\big(\mathbb{R}^N\big)\big)^n\colon\ |\nabla v_i|^p,\ F\big(|\cdot|, v\big), G_{ij}(v_i) \in L^1\big(\mathbb{R}^N\big),\\
&\int_{\mathbb{R}^N} G_{ij}(v_i)\,\mathrm{d}x = c_{ij},\ j = 1, \dots, k_i,\ i = 1, \dots, n \Bigg\}, \qquad (5.13)
\end{aligned}
$$

where $c_{ij} \in \mathbb{R}$, $j = 1, \dots, k_i$, $i = 1, \dots, n$, we then consider the following variational problem:

$$(\mathrm{P}_3) \quad J_3(v) \equiv \int_{\mathbb{R}^N} \left(\frac{1}{p} \sum_{i=1}^{n} |\nabla v_i|^p - F\big(|x|, v\big) \right) \mathrm{d}x \longrightarrow \mathrm{Inf}!, \quad v \in K_3. \tag{5.14}$$

We call u a *minimizer of* (P_3) if $u \in K_3$, and if $J_3(v) \geqslant J_3(u) \ \ \forall v \in K_3$.

Suppose that u is a minimizer. In view of the manifold condition (5.12), standard arguments in the calculus of variations show (see e.g. [124]) that u is a distributional solution of the following system of elliptic PDE,

$$\begin{aligned} -\Delta_p u_i \equiv -\nabla\big(|\nabla u_i|^{p-2} \nabla u_i\big) &= f_i\big(|x|, u\big) + \sum_{j=1}^{k_i} \alpha_{ij} g_{ij}(u_i) \\ &\equiv h_i\big(|x|, u\big) \quad \text{in } \mathbb{R}^N, \ i = 1, \dots, n, \end{aligned} \tag{5.15}$$

with $\alpha_{ij} \in \mathbb{R}$, $j = 1, \dots, k_i$, $i = 1, \dots, n$, as Lagrange multipliers. Notice that $h_i = h_i(s, t_1, \dots, t_n)$ $((s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n)$, is nonincreasing in s and nondecreasing in t_j, $i, j = 1, \dots, n$, $j \neq i$, i.e. the system (5.15) is *cooperative*.

Furthermore, the growth conditions (5.11) ensure that

$$u_i \in L^\infty\big(\mathbb{R}^N\big) \cap C^{1,\alpha}\big(\mathbb{R}^N\big), \quad i = 1, \dots, n. \tag{5.16}$$

for some $\alpha \in (0, 1]$ (see [122]). Since $u \in (L^{p^*}(\mathbb{R}^N))^n$ this in particular implies that

$$u_i(x) \longrightarrow 0 \quad \text{as } |x| \to \infty, \ i = 1, \dots, n. \tag{5.17}$$

The proofs of our symmetry theorems require some preparations. First we will need an integral inequality related to two-point rearrangement that has been proved in [34]. Here we add a careful analysis of the equality sign.

LEMMA 5.1. *Let* $v = (v_1, \dots, v_n) \in (L^q(\mathbb{R}^N))^n$, *let* $F(|\cdot|, v) \in L^1(\mathbb{R}^N)$, *and let* $0 \in \overline{H}$. *Then*

$$\int_{\mathbb{R}^N} F\big(|x|, v_1, \dots, v_n\big) \mathrm{d}x \leqslant \int_{\mathbb{R}^N} F\big(|x|, T^H v_1, \dots, T^H v_n\big) \mathrm{d}x. \tag{5.18}$$

Furthermore, if, for some $i \in \{1, \dots, n\}$, *the function* $(\partial F / \partial t_i)(r, t_1, \dots, t_n)$ *is strictly decreasing in* r *and* $0 \in H$, *then the equality in* (5.18) *is achieved only if* $v_i = T^H v_i$. *Finally, if, for some numbers* $i, j \in \{1, \dots, n\}$, $i \neq j$, *the function* $(\partial F / \partial t_i)(r, t_1, \dots, t_n)$ *is strictly increasing in* t_j, *then the equality in* (5.18) *is achieved only if*

$$\big(v_i(x) - v_i(\sigma_H x)\big)\big(v_j(x) - v_j(\sigma_H x)\big) \geqslant 0 \quad \forall x \in H. \tag{5.19}$$

The proof of Lemma 5.1 requires the following technical

LEMMA 5.2. *Let $r_+ \geqslant r_- \geqslant 0$, $a_i, b_i, c_i^+, c_i^- \in \mathbb{R}$ with $c_i^+ = \max\{a_i; b_i\}$, $c_i^- = \min\{a_i; b_i\}$, $i = 1, \ldots, n$. Then*

$$F(r_-, a_1, \ldots, a_n) + F(r_+, b_1, \ldots, b_n) \leqslant F(r_-, c_1^+, \ldots, c_n^+) + F(r_+, c_1^-, \ldots, c_n^-). \tag{5.20}$$

Furthermore, if, for some $i \in \{1, \ldots, n\}$, the function $(\partial F/\partial t_i)(r, t_1, \ldots, t_n)$ is strictly decreasing in r and if $r_+ > r_-$, then the equality in (5.20) *is achieved only if $a_i = c_i^+$. Finally, if for some numbers $i, j \in \{1, \ldots, n\}$, $i \neq j$, the function $(\partial F/\partial t_i)(r, t_1, \ldots, t_n)$ is strictly increasing in t_j, then equality in* (5.20) *is achieved only if there holds*

$$(a_i - b_i)(a_j - b_j) \geqslant 0. \tag{5.21}$$

PROOF. By regrouping the variables $t_1, \ldots, t_n$, if necessary, we may assume w.l.o.g. that there is some $k \in \{1, \ldots, n\}$ such that $a_i = c_i^-$ for $1 \leqslant i \leqslant k$, and if $k < n$, then also $a_i = c_i^+$ for $i > k$. Introducing the vectors $v' = (c_1^-, \ldots, c_k^-)$, $v'' = (c_{k+1}^-, \ldots, c_n^-)$, $h' = (h_1, \ldots, h_k)$, $h'' = (h_{k+1}, \ldots, h_n)$, where $h_i := c_i^+ - c_i^-$, $i = 1, \ldots, n$, (5.20) reads as

$$I := F(r_-, v' + h', v'' + h'') + F(r_+, v', v'') - F(r_-, v', v'' + h'') - F(r_+, v' + h', v'') \geqslant 0.$$

We have,

$$I = \int_0^1 \sum_{i=1}^{k} h_i \big(F_{t_i}(r_-, v' + th', v'' + h'') - F_{t_i}(r_+, v' + th', v'')\big)\, \mathrm{d}t.$$

Now each summand in the integrand is nonnegative in view of the assumptions on F, proving the first assertion. Moreover, we have $I = 0$ only if $h_i = 0$ or

$$F_{t_i}(r_-, v' + th', v'' + h'') = F_{t_i}(r_+, v' + th', v'') \quad \forall t \in (0, 1),$$

for any $i \in \{1, \ldots, k\}$. From this the assertions in the equality case of (5.20) follow easily. □

PROOF OF LEMMA 5.1. We have by Lemma 5.2, and since $|x| \leqslant |\sigma x|$ $\forall x \in H$,

$$F\big(|x|, v_1(x), \ldots, v_n(x)\big) + F\big(|\sigma x|, v_1(\sigma x), \ldots, v_n(\sigma x)\big) \leqslant F\big(|x|, T^H v_1(x), \ldots, T^H v_n(x)\big) + F\big(|\sigma x|, T^H v_1(\sigma x), \ldots, T^H v_n(\sigma x)\big) \quad \forall x \in H. \tag{5.22}$$

Integrating this inequality over H, the first assertion of Lemma 5.1 follows. Furthermore, if, for some $i \in \{1, \ldots, n\}$, the function $(\partial F/\partial t_i)(r, t_1, \ldots, t_n)$ is strictly decreasing in r, and if $0 \in H$, then we have that $|x| < |\sigma x|$ $\forall x \in H$, and hence equality in (5.22) is achieved only if $v_i(x) = T^H v_i(x)$, by Lemma 5.2.

Finally, if for some numbers $i, j \in \{1, \dots, n\}$, the function $(\partial F/\partial t_i)(r, t_1, \dots, t_n)$ is strictly increasing in t_j, then Lemma 5.2 tells us that equality in (5.22) is achieved only if (5.19) holds. □

Next we show a purely geometrical result.

LEMMA 5.3 (Reflexion Lemma). *Let U be a nonempty open set in $\mathbb{R}^N$ with C^1-boundary S, and assume that*

$$\nu(y) = \nu(z) - \frac{2(\nu(z), y - z)}{|y - z|^2}(y - z) \quad \forall y, z \in S \text{ with } y \neq z, \tag{5.23}$$

where $\nu(x)$ denotes the exterior normal to U at x. Then U is either a halfspace, a ball or the exterior of a ball in $\mathbb{R}^N$.

PROOF. Assume that U is not a halfspace. Then there exist two points $y_1, y_2 \in S$ such that $\nu(y_1) \neq \nu(y_2)$. Letting

$$z_0 := y_1 + \frac{|y_2 - y_1|^2}{2(\nu(y_1), y_2 - y_1)}\nu(y_1),$$

we may assume w.l.o.g. that $z_0 = 0$. It is then easy to see that $|y_1| = |y_2| = r$ for some $r > 0$, and either (i) $\nu(y_i) = y_i/r$ $(i = 1, 2)$, or (ii) $\nu(y_i) = -y_i/r$ $(i = 1, 2)$.

We claim that (i) implies that U is a ball. Clearly it is enough to show that

$$|x| = r \quad \forall x \in S,\ x \neq \pm y_i\ (i = 1, 2). \tag{5.24}$$

Setting $a_i := (r|x|)^{-1}(x, y_i)$, we have $|a_i| < 1$, and using (5.24) we find,

$$r\nu(x) = y_i - \frac{2r^2 - 2a_i|x|r}{r^2 - 2a_i|x|r + |x|^2}(y_i - x) \quad (i = 1, 2). \tag{5.25}$$

Multiplying (5.25) with $x/(r|x|)$, we have

$$\frac{(\nu(x), x)}{|x|} = \frac{-a_i r^2 - a_i|x|^2 + 2r|x|}{r^2 - 2a_i|x|r + |x|^2} \quad (i = 1, 2). \tag{5.26}$$

Introducing the function

$$f(t) := \frac{-tr^2 - t|x|^2 + 2r|x|}{r^2 - 2t|x|r + |x|^2}, \quad t \in (-1, 1),$$

we find that

$$f'(t) = \frac{-(r^2 - |x|^2)^2}{(r^2 - 2t|x|r + |x|^2)^2} < 0 \quad \forall t \in (-1, +1).$$

By (5.26), this means that we must have $a_1 = a_2$. Going back to (5.25) we finally calculate

$$y_2 - y_1 = \frac{2r^2 - 2a_1|x|r}{r^2 - 2a_1|x|r + |x|^2}(y_2 - y_1),$$

which implies that $r = |x|$. This shows (5.24), and the claim is proved.

Similarly one shows in case (ii) that U is the exterior of a ball in $\mathbb{R}^N$. □

Now we are in a position to prove the first symmetry result.

THEOREM 5.4. *Let $u = (u_1, \dots, u_n)$ be a minimizer of* (P). *Then for any $i \in \{1, \dots, n\}$ the following holds*:

(1) $|\nabla u_i|$ *is constant on the set* $\{u_i = t\}$ $\forall t \in (\inf u_i, \sup u_i)$, *and, in particular* $|\nabla u_i| = 0$ *on the set* $\{u_i = 0\}$. *Furthermore, if* $\sup u_i > 0$, *then the superlevel sets* $\{u_i > t\}$ *are balls* $\forall t \in (0, \sup u_i)$, *and if* $\inf u_i < 0$, *then the sublevel sets* $\{u_i < t\}$ *are balls* $\forall t \in (\inf u_i, 0)$.

(2) (*Radial symmetry*) *If the function $f_i = f_i(r, t_1, \dots, t_n)$ is strictly decreasing in r, then u_i is radially symmetric and radially nonincreasing about* 0 – *and, in particular, nonnegative – that is, there is a function $v \in C^1((0, +\infty))$ such that*

$$u_i(x) = v\big(|x|\big) \quad \text{and} \quad v'(r) \leqslant 0 \quad \text{for } 0 < |x| = r < +\infty. \tag{5.27}$$

(3) *If, for some $j \in \{1, \dots, n\}$, $j \neq i$, the function $f_i = f_i(r, t_1, \dots, t_n)$ is strictly increasing in t_j, then the functions u_i and u_j are equally ordered, that is*

$$\big(u_i(x) - u_i(y)\big)\big(u_j(x) - u_j(y)\big) \geqslant 0 \quad \forall x, y \in \mathbb{R}^N. \tag{5.28}$$

PROOF. (1) First observe that if $H \in \mathcal{H}_0$ then $(T^H u_1, \dots, T^H u_n) \in K_3$ and $J_3(T^H u_1, \dots, T^H u_n) \leqslant J_3(u)$, by Lemma 3.7. Hence, $(T^H u_1, \dots, T^H u_n)$ is a minimizer of (P_3), too, and $J_3(T^H u_1, \dots, T^H u_n) = J_3(u)$. In particular, this implies

$$\int_{\mathbb{R}^N} F\big(|x|, u\big)\,dx = \int_{\mathbb{R}^N} F\big(|x|, T^H u_1, \dots, T^H u_n\big)\,dx \quad \forall H \in \mathcal{H}_0. \tag{5.29}$$

Now fix $i \in \{1, \dots, n\}$. Assume $\sup u_i > 0$, and let $t \in (0, \sup u_i)$. Setting $S_i(t) := \{u_i = t\}$ we choose $x, y \in S_i(t)$, $x \neq y$, and a halfspace $H \subset \mathbb{R}^N$ such that $y = \sigma_H x$, and such that $0 \in \overline{H}$. We claim that this implies

$$\nabla u_i(x) = \nabla \sigma_H\big(u_i(x)\big), \tag{5.30}$$

that is, the gradients of u_i at the points x and y are oppositely directed w.r.t. reflexion in ∂H. Indeed, if $\nabla u_i(x) \neq \nabla \sigma_H(u_i(x))$, then $\nabla T^H u_i$ is discontinuous across some C^1-hypersurface S, while $T^H u_i \in C^1(B_\varepsilon(x) \setminus S)$ ($\varepsilon > 0$, small). But this is impossible, since $(T^H u_1, \dots, T^H u_n)$ is a minimizer, and hence it satisfies a system of the

form (5.15), with the Lagrangian multipliers α_{ij} possibly replaced by some other numbers α'_{ij}, $j = 1, \dots, k_i$, $i = 1, \dots, n$.[1]

Repeating the above considerations for all points $z \in S_i(t)$ we find that

$$\nabla u_i(z) = \nabla u_i(x) - \frac{2(\nabla u_i(x), z - x)}{|z - x|^2}(z - x) \quad \forall z \in S_i(t),\ z \neq x. \tag{5.31}$$

This in particular means that $|\nabla u_i| = \text{const} =: c_i(t)$ on $S_i(t)$. Since $u_i \in C^1(\mathbb{R}^N)$, it follows that $|\nabla u_i| = c_i(t)$ on the set $\{u_i = t\}$, and since u_i decays at infinity, also $|\nabla u_i| = 0$ on the set $\{u_i(x) = 0\}$.

Assume $c_i(t) \neq 0$. Then $S_i(t)$ is locally a C^1-hypersurface and (5.31) shows that

$$\nu(z) = \nu(x) - \frac{2(\nu(x), z - x)}{|z - x|^2}(z - x) \quad \forall z \in S_i(t),\ z \neq x,$$

where $\nu(z)$ denotes the exterior normal to $\{u_i > t\}$ at z. By Lemma 5.3 and (5.17) this means that the superlevel set $\{u_i > t\}$ is a ball in this case.

Next let $t \in (0, \sup u_i)$ and $c_i(t) = 0$. Then we find a strictly decreasing sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = t$ and such that $c_i(t_k) \neq 0$, $k = 1, 2, \dots$. Since $\{u_i > t\} = \bigcup_{k=1}^{\infty} \{u_i > t_k\}$, this means that the superlevel set $\{u_i > t\}$ is a ball in this case, too.

Similarly we can prove that any sublevel set $\{u_i < t\}$ is a ball and $|\nabla u_i| = $ const. on $\{u_i = t\}$ if $\inf u_i < 0$ and if $t \in (\inf u_i, 0)$.

(2) Next assume that, for some $i \in \{1, \dots, n\}$, the function $f_i = f_i(r, t_1, \dots, t_n)$ is strictly decreasing in r. In view of (5.29), Lemma 5.1 tells us that $u_i = T^H u_i$ for any halfspace $H \in \mathcal{H}_0$. It is easy to see that this implies the symmetry property (5.27).

(3) Finally let, for some numbers $i, j \in \{1, \dots, n\}$, $i \neq j$, the function

$$f_i = f_i(r, t_1, \dots, t_n)$$

be strictly increasing in t_j, and assume that (5.28) is not satisfied. Then there exist two density points $x, y \in \mathbb{R}^N$ of u_i and u_j such that $u_i(x) > u_i(y)$ and $u_j(x) < u_j(y)$. We choose a halfspace H with $0 \in \overline{H}$ such that $y = \sigma_H x$. Then, applying Lemma 5.1, we obtain that $\int_{\mathbb{R}^N} F(|x|, u)\,\mathrm{d}x < \int_{\mathbb{R}^N} F(|x|, T^H u_1, \dots, T^H u_n)\,\mathrm{d}x$, a contradiction. The theorem is proved. □

Theorem 5.4(1) implies that the functions $(u_i)_+$ and $(-u_i)_+$ $(i = 1, \dots, n)$, satisfy the symmetry property (LS) from the introduction:

COROLLARY 5.1 (Local symmetry). *Let $u = (u_1, \dots, u_n)$ be a minimizer of* (P_3), *and let A be a connected component of the set $\{x\colon\ \nabla u_i(x) \neq 0\}$, $i \in \{1, \dots, n\}$. Then there are numbers $R_1, R_2 \in [0, +\infty]$ with $R_1 < R_2$, and a point $z \in \mathbb{R}^N$ such that $A = \{x\colon\ R_1 < |x - z| < R_2\}$, and u_i is radially symmetric in A, that is, there is a function $v \in C^1((R_1, R_2))$, such that $u_i(x) = v(|x - z|)$, $x \in A$. Moreover, u_i does not change sign*

[1] With some more effort we may actually prove that $\alpha_{ij} = \alpha'_{ij}$, $j = 1, \dots, k_i$, $i = 1, \dots, n$, but that information is not needed here.

in A. Finally, if $u_i > 0$ (*respectively* $u_i < 0$) *in A then* $v'(r) < 0$ (*respectively* $v'(r) > 0$), $r \in (R_1, R_2)$.

PROOF. We use the notation of the previous proof. In view of part (1) of Theorem 5.4, we find two numbers $a, b \in \mathbb{R}$, $a < b$, such that $A = \{x\colon\ a < u_i(x) < b\}$, and for each $t \in (a, b)$ the level set $\{u_i = t\}$ is a ball with $c_i(t) > 0$. Moreover, since $c(0) = 0$, we have that either $a \geqslant 0$ or $b \leqslant 0$, that is, u_i does not change sign in A. Finally, since $u_i \in C^1(\mathbb{R}^N)$, an easy application of the method of steepest descent (see [12]) shows that the level sets $\{u_i = t\}$, $t \in (a, b)$, are concentric spheres. The last assertion of the corollary then follows immediately. □

Using the local symmetry of minimizers of (P_3), we now intend to show their radial symmetry (and their positivity as well!) provided that the functions h_i, $i = 1, \ldots, n$, in (5.15) satisfy some growth conditions near their zero points. Here the key role in the proof is played by the Strong Maximum Principle for the p-Laplacian which is due to Vazquez [121] (see also [98] and [97] for more general versions).

In the sequel, let $\mathcal{A}$ the set of functions $\alpha \in C(\mathbb{R}_0^+)$ satisfying

$$\alpha(0) = 0, \quad \alpha(t) > 0 \quad \text{for } t > 0, \quad \text{and}$$
$$\int_0^1 \left(\int_0^t \alpha(s)\, \mathrm{d}s \right)^{-1/p} \mathrm{d}t = +\infty.$$

STRONG MAXIMUM PRINCIPLE (SMP). *Let* Ω *be a domain in* $\mathbb{R}^N$, *let* $\partial\Omega$ *be smooth in a neighborhood of* $x_0 \in \partial\Omega$, *and let* $u \in C^1(\Omega \cup \{x_0\})$ *satisfy* $u(x_0) = 0$ *and in the sense of distributions*,

$$-\Delta_p u = -\nabla\big(|\nabla u|^{p-2}\nabla u\big) \geqslant -\alpha(u), \quad u \geqslant 0 \quad \text{in } \Omega,$$

where $\alpha \in \mathcal{A}$. *Then*:

$$\frac{\partial u}{\partial \nu}(x_0) \leqslant 0 \quad (\nu\colon \textit{exterior normal}),$$

where the equality sign is attained only if $u \equiv 0$ *in* Ω.

We notice that any function $\alpha(t) = ct^{p-1}$ $(c > 0)$, belongs to $\mathcal{A}$.

DEFINITION 5.1. Let $h \in C(\mathbb{R}_0^+ \times \mathbb{R}^n)$, $h = h(s, t_1, \ldots, t_n)$, and $i \in \{1, \ldots, n\}$.

(1) We say that h has property $H_+(i, \tau)$, respectively $H_-(i, \tau)$, if there holds:

If $h(\sigma, \tau_1, \ldots, \tau_n) = 0$ for some $(\sigma, \tau_1, \ldots, \tau_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ with $\tau_i = \tau$, then there exists a function $\alpha \in \mathcal{A}$ such that

$$h(s, t_1, \ldots, t_n) \geqslant -\alpha(t_i - \tau) \quad \forall (s, t_1, \ldots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$$
$$\text{with } t_i \in [\tau, +\infty), \text{ respectively} \tag{5.32}$$
$$h(s, t_1, \ldots, t_n) \leqslant \alpha(\tau - t_i) \quad \forall (s, t_1, \ldots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$$
$$\text{with } t_i \in (-\infty, \tau]. \tag{5.33}$$

(2) We say that h has property $H(i,\tau)$, if there holds:
If $h(\sigma,\tau_1,\dots,\tau_n)=0$ for some $(\sigma,\tau_1,\dots,\tau_n)\in\mathbb{R}_0^+\times\mathbb{R}^n$ with $\tau_i=\tau$, then there exists a function $\alpha\in\mathcal{A}_m$ such that h satisfies either one of the conditions (5.32) or (5.33) of (1).
(3) We say that h is nice w.r.t. the variable t_i, if h has property $H(i,\tau)$ for any $\tau\in\mathbb{R}$.

THEOREM 5.5 (Radial symmetry). *Let $u=(u_1,\dots,u_n)$ be a minimizer of* (P_3). *Then for every $i\in\{1,\dots,n\}$ the following holds*:
(1) *If the function h_i in* (5.15) *has property $H(i,\tau)$ for any $\tau\in\mathbb{R}\setminus\{0\}$ then u_i is radially symmetric in the sets $\{u_i>0\}$ and $\{u_i<0\}$. More precisely, if* $\sup u_i>0$, *then*

$$\begin{aligned}&\exists z_+\in\mathbb{R}^N,\ R_{1+},R_{2+}\in[0,+\infty],\ R_{1+}<R_{2+},\ v_+\in C^1\big((R_{1+},R_{2+})\big),\\&\quad\text{such that } \{u_i>0\}=\big\{x\colon |x-z_+|<R_{2+}\big\},\ u_i(x)=v\big(|x-z_+|\big),\\&\quad\text{and } v_+'(r)<0 \text{ for } R_{1+}<|x-z_+|=r<R_{2+},\end{aligned}\tag{5.34}$$

and if $\inf u_i<0$, *then*

$$\begin{aligned}&\exists z_-\in\mathbb{R}^N,\ R_{1-},R_{2-}\in[0,+\infty],\ R_{1+}<R_{2-},\ v_-\in C^1\big((R_{1-},R_{2-})\big),\\&\quad\text{such that } \{u_i<0\}=\big\{x\colon |x-z_-|<R_{2-}\big\},\ u_i(x)=v_-\big(|x-z_-|\big),\\&\quad\text{and } v_-'(r)>0 \text{ for } R_{1-}<|x-z_-|=r<R_{2-}.\end{aligned}\tag{5.35}$$

(2) *If the function h_i in* (5.15) *has property $H(i,0)$, then u_i does not change sign.*
(3) *If the function h_i in* (5.15) *is nice w.r.t. the variable t_i, then u_i is radially symmetric and does not change sign. More precisely, if* $\sup u_i>0$, *then u_i is nonnegative and satisfies condition* (5.34), *and if* $\inf u_i<0$, *then u_i is nonpositive and satisfies condition* (5.35).

PROOF. (1) Let h_i have property $H(i,\tau)$ for any $\tau\in\mathbb{R}\setminus\{0\}$. Assume that $\sup u_i>0$, and let $t\in(0,\sup u_i)$ with $c(t)>0$. We define $t_2:=\inf\{s<t\colon c(\tau)>0\ \forall\tau\in(s,t]\}$, and $t_1:=\sup\{s>t\colon c(\tau)>0\,\forall\tau\in[t,s)\}$. By Theorem 5.4, part (1), we have that $t_2\geqslant 0$, and by Corollary 5.1, we find a point $z_+\in\mathbb{R}^N$, numbers $R_{1+},R_{2+}\in[0,+\infty]$, $R_{1+}<R_{2+}$, and a function $v_+\in C^1((R_{1+},R_{2+}))$ such that $A:=\{t_2<u_i<t_1\}=\{x\colon R_{1+}<|x-z_+|<R_{2+}\}$, and $u_i(x)=v_+(|x-z_+|)$, and $v_+'(r)<0$ for $R_{1+}<|x-z|=r<R_{2+}$. Notice that in view of the equation for u_i, $h_i=h_i(|x|,u(x))$ can be written in A as a function of $|x-z_+|$, too.

Now assume that $t_2>0$. Then $R_{2+}<+\infty$, and since $v_+'(R_{2+})=0$, the (SMP) tells us that we must have $h_i(|x|,u(x))=\text{const}=:k\leqslant 0$ on $\partial B_{R_{2+}}(z_+)$. Assume that $k<0$. Since h_i is continuous, we find some $\varepsilon>0$ such that $h_i(|x|,u(x))<0$ in $B_{R_{2+}+\varepsilon}(z_+)\setminus B_{R_{2+}}(z_+)$. Since $u_i(x)\leqslant t_2$ in $B_{R_{2+}+\varepsilon}(z_+)\setminus B_{R_{2+}}(z_+)$, the SMP gives $v_+'(R_{2+})<0$, a contradiction. Thus we have $k=0$, and since h_i has property $H(i,t_2)$, the (SMP) tells us again that we must have $v_+'(R_{2+})<0$, a contradiction! Hence $h_i(|x|,u(x))=0$ on $\partial B_{R_{2+}}(z_+)$ and $t_2=0$.

Next assume that $t_1 < \sup u_i$. Then $R_{1+} > 0$, $\{u_i \geqslant t_1\} = \overline{B_{R_{1+}}(z_+)}$ and $v'_+(R_{1+}) = 0$. Using the SMP analogously as above, we find that $h_i(|x|, u(x)) = 0$ on $\partial B_{R_{1+}}(z_+)$. Then the fact that h_i has the property $H(i, t_1)$, and that $v'_+(R_{1+}) = 0$, leads again to a contradiction. It follows that $t_1 = \sup u_i$. This proves (5.34). If $\inf u_i < 0$ then one shows analogously as above that (5.35) holds.

(2) Let h_i have property $H(i, 0)$, and assume that both sets $\{u_i > 0\}$ and $\{u_i < 0\}$ are nonempty. It then follows from Theorem 5.4 that each of these sets is a ball or a halfspace. The (SMP) then tells us that $h_i(|x|, u(x)) = 0$ and $u_i(x) = 0$ on $\partial\{u_i > 0\} \cup \partial\{u_i < 0\}$. Assume that h_i satisfies condition (5.32). But then the (SMP) gives $|\nabla u_i| \neq 0$ on $\partial\{u_i > 0\}$, a contradiction. Similarly one obtains a contradiction if h_i satisfies condition (5.33).

(3) This property follows directly from part (1) and part (2). □

We can exclude the possibility of 'plateaus' at height 0, $\sup u_i$ and $\inf u_i$, by slightly sharpening the growth conditions for the function h_i in (5.15) at these levels.

COROLLARY 5.2. *Let $u = (u_1, \ldots, u_n)$ be a minimizer of* (P_3), *and let $i \in \{1, \ldots, n\}$. Then the following holds*:

(1) *If* $\sup u_i > 0$, (*respectively* $\inf u_i < 0$), *and if the function h_i in* (5.15) *has property* $H_-(i, \sup u_i)$ (*respectively* $H_+(i, \inf u_i)$), *then the set* $\{x\colon\ u_i(x) = \sup u_i\}$ (*respectively* $\{x\colon\ u_i(x) = \inf u_i\}$), *is a single point.*

(2) *If the function h_i in* (5.15) *has both properties $H_-(i, 0)$ and $H_+(i, 0)$, then either $u_i(x) > 0$, $u_i(x) < 0$ or $u_i(x) \equiv 0$ on $\mathbb{R}^N$.*

(3) *In particular, if h_i has both properties $H_-(i, \tau)$ and $H_+(i, \tau)$ for any $\tau \in \mathbb{R}$, and if u_i is positive* (*respectively negative*), *then there exists a point $z \in \mathbb{R}^N$ and a function $v \in C^1(\mathbb{R}_0^+)$ such that $u_i(x) = v(|x - z|)$, and $v'(r) < 0$ (respectively $v'(r) > 0$), for $0 < |x - z| = r < +\infty$.*

PROOF. (1) Let $\sup u_i > 0$, and assume that h_i has property $H_-(i, \sup u_i)$. By Theorem 5.4, we find a point $x_0 \in \mathbb{R}^N$, and $R \geqslant 0$ such that $\{x\colon\ u_i(x) = \sup u_i\} = \overline{B_R(x_0)}$. Assume $R > 0$. Then the Maximum Principle shows that $h_i(s, t_1, \ldots, t_n) = 0$ whenever $t_i = \sup u_i$. Hence h_i satisfies condition (5.33). Applying the (SMP) to the set $\{x\colon\ u_i(x) < \sup u_i\} = \mathbb{R}^N \setminus \overline{B_R(x_0)}$ we then find that $|\nabla u_i| \neq 0$ on $\partial B_R(x_0)$, a contradiction. Hence $R = 0$.

Analogously one shows that if $\inf u_i < 0$, then the set $\{x\colon\ u_i(x) = \inf u_i\}$ is a single point.

(2) Assume that h_i satisfies both properties $H_-(i, 0)$ and $H_+(i, 0)$. Then u_i does not change sign by Theorem 5.5(2). Assume that $\sup u_i > 0$ and that $\{u_i = 0\} \neq \emptyset$. Then $\{u_i > 0\}$ is either a ball or a halfspace, which means that $h(|x|, u_i(x)) = 0$ on $\{u_i = 0\}$. Hence h_i satisfies condition (5.32) with $\tau = 0$. Applying the (SMP) then shows that we must have $|\nabla u_i(x)| \neq 0$ on $\partial\{u_i > 0\}$, which is impossible. Hence $u_i(x) > 0$ on $\mathbb{R}^N$.

Analogously one shows that if $\inf u_i < 0$ then $u_i(x) < 0$ on $\mathbb{R}^N$.

The assertion (3) then follows from Theorem 5.5(3), and from the assertions (1) and (2) above. □

REMARK 5.3.

(1) The function f_i is strictly decreasing in r if it is, for instance, of the form

$$f_i(r, t_1, \dots, t_n) = \sum_{k=1}^{m_i} a_{ik}(r) b_{ik}(t_1, \dots, t_n),$$

with continuous and positive functions b_{ik}, and with strictly decreasing functions a_{ik}, $k = 1, \dots, m_i$.

Furthermore, one obtains radial symmetry of minimizers u of (P_3) by combining several of the conditions given in Theorem 5.4. For instance, if the function $f_1 = f_1(r, t_1, \dots, t_n)$ is strictly decreasing in r, and if the functions $f_i = f_i(r, t_1, \dots, t_n)$ are strictly increasing in t_j, $j = 1, \dots, n$, $i = 2, \dots, n$, $j \neq i$, then any component u_i has property (5.27), $i = 1, \dots, n$.

(2) The function h_i has both properties $H_-(i, \tau)$ and $H_+(i, \tau)$ $\forall \tau \in \mathbb{R}$, $\tau \neq 0$, if there exist two numbers $a_i \leqslant 0$, $b_i \geqslant 0$ such that $h_i(s, t_1, \dots, t_m) \geqslant 0$ $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ with $t_i \in [a_i, 0] \cup [b_i, +\infty)$ and $h_i(s, t_1, \dots, t_n) \leqslant 0$ $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ with $t_i \in (-\infty, a_i] \cup [0, b_i]$. Notice that the above inequalities are difficult to check in general, since they also depend on the Lagrangian multipliers in (5.15). Furthermore, h_i has both properties $H_-(i, 0)$ and $H_+(i, 0)$ if f_i has these properties, and if, for some $c > 0$,

$$\left|g_{ij}(t)\right| \leqslant c|t|^{p-1} \quad \forall t \in \mathbb{R},\ j = 1, \dots, k_i.$$

Finally, for certain differential operators, the required growth conditions for h_i in Corollary 5.2(3) are fulfilled if the functions f_i and g_{ij}, $j = 1, \dots, k_i$, $i = 1, \dots, n$, in (5.15) have additional smoothness properties:

Assume for instance that $p \in (1, 2]$. Choosing $\alpha(s) = cs^{p-1}$ $(c > 0)$, in (5.32)–(5.33), we see that h_i has both properties $H_-(i, \tau)$ and $H_+(i, \tau)$ $\forall \tau \in \mathbb{R}$, provided that $f_i = f_i(s, t_1, \dots, t_n)$, $g_{ij} = g_{ij}(t_i)$, $j = 1, \dots, k_i$, satisfy a Hölder condition with exponent $(p - 1)$ w.r.t. t_i, uniformly $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$, $i = 1, \dots, n$.

(3) Our results can be extended to situations where the admissible set K contains further or other constraints that are *invariant under two-point rearrangement* (see [36]).

(4) The flexibility of two-point rearrangements has been demonstrated once more in two interesting papers of J. van Schaftingen [116,118]. He analyzed symmetry properties of critical points of certain variational problems obtained via a minimax procedure. For example, if there is a solution obtained via the Mountain Pass Theorem, then there is a *symmetric* solution with the same energy.

6. Other symmetry results

Our method is also applicable to variational problems with potentials. We restrict ourselves to a simple example in $\mathbb{R}^3$.

Let $\gamma \in (4/3, +\infty)$, $K > 0$, $M > 0$, and $j \in C([0,1])$, such that

$$j(0) = 0, \quad j \text{ nondecreasing and } j^2 \in C^1\big([0,1]\big). \tag{6.1}$$

We consider the following variational problem,

$$\begin{aligned}(\mathrm{P}_4)\quad J_4(g) &\equiv J_4^1(g) + J_4^2(g) + J_4^3(g)\\ &:= -\frac{1}{2}\iint\limits_{\mathbb{R}^6} \frac{g(x)g(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^3} \frac{j^2(m_g(|x'|))}{|x'|^2}\,\mathrm{d}x\\ &\quad + \frac{K}{\gamma-1}\int_{\mathbb{R}^3} g^\gamma(x)\,\mathrm{d}x\\ &\longrightarrow \mathrm{Inf}!, \quad v \in K_4,\end{aligned} \tag{6.2}$$

where m_g is defined by

$$\begin{aligned} m_g(r) &:= \frac{1}{M}\int_{\{|x'|<r\}} g(x)\,\mathrm{d}x \quad \forall r > 0, \quad \text{and}\\ K_4 &:= \left\{ g \in L^1_+\big(\mathbb{R}^3\big) \cap L^\gamma_+\big(\mathbb{R}^3\big)\colon\ g = g\big(x_1, |x'|\big),\ \int_{\mathbb{R}^3} g\,\mathrm{d}x = M \right\}.\end{aligned} \tag{6.3}$$

Problem (P_4) represents a model for rotating, axisymmetric equilibrium figures of a compressible, inviscid and irrotational fluid. It has been studied by many authors (see [63], and the references cited therein). The function g is the density of the fluid having total mass $M = \int_{\mathbb{R}^3} g$, and the fluid rotates about the x_1-axis. We assume a polytropic law, that is the pressure p inside the fluid is

$$p = Kg^\gamma.$$

The functional J_4 is the energy of the fluid with J_4^1, J_4^2, J_4^3 being its gravitational, rotational, and internal components, respectively. The function $j(m_g(r))$ represents the total angular momentum of the fluid lying within a distance r of the axis, and it is assumed to be invariant during the evolution which leads to the equilibrium.

The existence of global minimizers of (P_4) and their 'Steiner symmetry' is well-known (compare [63]). Our aim is to show the symmetry of *local* minimizers as well (see [31]).

We say that f is a local minimizer of (P_4), if $f \in K_4$, and if there is a number $\varepsilon > 0$ such that $J_4(g) \geqslant J_4(f)$ $\forall g \in K_4$ satisfying $\|f-g\|_1 + \|f-g\|_\gamma < \varepsilon$. Notice that any local minimizer f satisfies

$$\begin{aligned} &-\int_{\mathbb{R}^3} \frac{f(y)}{|x-y|}\,\mathrm{d}y + \int_{|x'|}^{+\infty} \frac{j^2(m_f(r))}{r^3}\,\mathrm{d}r + \frac{K\gamma}{\gamma-1} f^{\gamma-1}(x)\\ &\begin{cases} = \alpha & \forall x \in G := \{f > 0\},\\ \geqslant \alpha & \forall x \in \mathbb{R}^3 \setminus G, \end{cases} \qquad \text{where } \alpha \in \mathbb{R}.\end{aligned} \tag{6.4}$$

THEOREM 6.1. *Let f a local minimizer of* (P_4). *Then there is a number* $\lambda^* \in \mathbb{R}$ *such that* $u(\cdot - \xi) = u^*(\cdot)$ *where* $\xi = (\lambda^*, 0, 0)$.

PROOF. Let $H \in \mathcal{H}^*$. Then we have that $m_f(r) = m_{f_H}(r)\ \forall r > 0$, and thus $J_4^2(f) = J_4^2(f_H)$ and $J_4^3(f) = J_4^3(f_H)$, by Theorem 3.1. Furthermore, Lemma 3.6 – with $F(s,t) = st$ and $w(r) = 1/r$ – tells us that $J_4^1(f) \geqslant J_4^1(f_H)$. Moreover, the last inequality can be sharpened by slightly modifying the proof of Lemma 3.6. We have

$$\begin{aligned}&\frac{f(x)f(y) + f(x_H)f(y_H)}{|x-y|} + \frac{f(x_H)f(y) + f(x)f(y_H)}{|x_H - y|}\\ &\quad\leqslant \frac{f_H(x)f_H(y) + f_H(x_H)f_H(y_H)}{|x-y|} + \frac{f_H(x_H)f_H(y) + f_H(x)f_H(y_H)}{|x_H - y|}\\ &\qquad \forall x, y \in H, \end{aligned} \tag{6.5}$$

and an easy calculation shows that (6.5) becomes an equality iff either $f(x) \geqslant f(x_H)$ and $f(y) \geqslant f(y_H)$, or $f(x) \leqslant f(x_H)$ and $f(y) \leqslant f(y_H)$. Using this and integrating (6.5) over $H \times H$ we find that $J_4^1(f) \geqslant J_4^1(f_H)$, with equality iff either $f = f_H$ or $\sigma_H f = f_H$. The assertion then follows by proceeding analogously as in the proof of Theorem 5.3. The details are left to the reader. □

REMARK 6.1. The following modifications of problem (P_4) are also well-known:

(1) One may prescribe the angular velocity $V = V(|x'|)$ instead of the function $j(m)$ $(m \in [0,1])$. In that case, the rotational energy term becomes

$$-\int_{\mathbb{R}^3} g(x) B\big(|x'|\big)\,\mathrm{d}x,$$

where B is defined by

$$B(r) := \int_0^r s V^2(s)\,\mathrm{d}s \quad (r > 0).$$

(2) *Incompressible fluid :* In that case the corresponding variational problem is

$$\begin{aligned}&\widetilde{J}_4(g) \equiv J_4^1(g) + J_4^2(g) \longrightarrow \text{Inf!},\\ &g \in K_4, \quad 0 \leqslant g \leqslant 1. \end{aligned} \tag{6.6}$$

In both cases, the above symmetry result still holds.

We finally mention that the author has obtained symmetry properties for *stationary solutions* of problem (P) from the Introduction. The proofs are based on another rearrangement tool – the *continuous Steiner symmetrization* (CStS) (see [30,33], see also [73] and [105] for other variants). The (CStS) is a semigroup of rearrangements $\{T^t\}$ $(0 \leqslant t \leqslant +\infty)$, connecting a given function $u \in \mathcal{S}$ with its Steiner symmetrization, such that $T^0 u = u$

and $T^{\infty}u = u^*$. The family is continuous in $L^p_+(\mathbb{R}^N)$ and continuous from the right in $W^{1,p}_+(\mathbb{R}^N)$ for any $p \in [1, +\infty)$. Naturally, the rearrangements T^t $(0 \leqslant t \leqslant +\infty)$, share many properties with the Steiner symmetrization which have been obtained in Section 4. On the other hand, these rearrangements cannot be approximated via two-point rearrangements. Therefore the (CStS) falls out of the scope of this presentation.

Below we mention one sample result without proof. Its first part has been obtained in [33] using (CStS). The second part has been shown in [32], using arguments similar as in the proof of Theorem 5.5.

THEOREM 6.2. *Let $p > 1$, $R > 0$, and let $u \in C^1(\overline{B_R})$ be a weak solution of the following problem*,

$$\begin{aligned} &-\Delta_p u \equiv -\nabla\big(|\nabla u|^{p-2}\nabla u\big) = f\big(|x|, u\big), \quad u \geqslant 0 \quad \text{in } B_R, \\ &u = 0 \quad \text{on } \partial B_R, \end{aligned} \tag{6.7}$$

where $f \in C((\mathbb{R}^+_0)^2)$, $f = f(s,t)$, and f is nonincreasing in s. Then the superlevel sets $\{u > t\}$ $(t \geqslant 0)$, are countable unions of mutually disjoint balls, and $|\nabla u| = \text{const}$ on the boundary of each of these balls. Furthermore, if f satisfies the following property:

(i) *If $f(\sigma, \tau) = 0$ for some $(\sigma, \tau) \in (\mathbb{R}^+_0)^2$ with $\tau \neq 0$, then there exists a function $\alpha \in \mathcal{A}$ such that*

$$f(s,t) \geqslant -\alpha(t - \tau) \quad \forall (s,t) \in \big(\mathbb{R}^+_0\big)^2 \text{ with } t \in [\tau, +\infty), \text{ or} \tag{6.8}$$

$$f(s,t) \leqslant \alpha(\tau - t) \quad \forall (s,t) \in \big(\mathbb{R}^+_0\big)^2 \text{ with } t \in (-\infty, \tau]; \tag{6.9}$$

(ii) *If $f(\sigma, 0) = 0$ for some $\sigma \in \mathbb{R}^+_0$, then there exists a function $\alpha \in \mathcal{A}$ such that*

$$f(s,t) \geqslant -\alpha(t) \quad \forall (s,t) \in \big(\mathbb{R}^+_0\big)^2; \tag{6.10}$$

then $u = u^{\star}$.

REMARK 6.2.

(1) The first assertion in Theorem 6.2 implies that u satisfies the symmetry property (LS) from the introduction.

(2) The conditions (i) and (ii) in Theorem 6.2 are satisfied in each one of the following cases:
   (a) There exists a number $b \geqslant 0$ such that $f(s,t) \geqslant 0$ $\forall (s,t) \in \mathbb{R}^+_0 \times [b, +\infty)$ and $f(s,t) \leqslant 0$ $\forall (s,t) \in \mathbb{R}^+_0 \times [0, b]$.
   (b) $p \in (1, 2]$, and $f(s,t)$ satisfies a Hölder condition w.r.t. t, uniformly $\forall s \in \mathbb{R}^+_0$.

List of notations

$\mathbb{R}^+$	$(0, +\infty)$
$\mathbb{R}^+_0$	$[0, +\infty)$

$a_+ = \max\{a; 0\}$
$x = (x_1, \dots, x_N) = (x_1, x')$ points in $\mathbb{R}^N$
$x \cdot y = \sum_{i=1}^N x_i y_i$ Euclidean scalar product
$|x| = \sqrt{x \cdot x}$ Euclidean norm
$M \Delta N = (M \setminus N) \cup (N \setminus M)$ symmetric difference between the sets M and N
$B_r(x)$ ball of radius r centered at x
$B_r = B_r(0)$
$\mathcal{L}^k$ k-dimensional Lebesgue measure, $1 \leqslant k \leqslant N$
$\mathcal{M}$ set of $\mathcal{L}^N$-measurable subsets of $\mathbb{R}^N$
$\omega_{u,\Omega}$ modulus of continuity of u in the domain Ω (p. 6)
$\|\cdot\|_p$ norm in $L^p(\mathbb{R}^N)$
$W^{1,p}(\mathbb{R}^N)$, $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ Sobolev spaces (p. 6)
$W_+^{1,p}(\mathbb{R}^N)$, $L_+^p(\Omega)\dots$ nonnegative functions of the corresponding set (p. 6)
$C^{0,1}(M)$ Lipschitz continuous functions on the set M (p. 6)
$C_0^{0,1}(\Omega)$ Lipschitz continuous functions with compact support in the domain Ω (p. 6)
$\mathcal{S}$ symmetrizable functions (p. 8)
$\mathcal{S}_+ := \{u \in \mathcal{S}\colon \operatorname{ess\,inf} u = 0\}$ (p. 8)
μ_u distribution function of u (p. 8)
$u \sim v$ u and v are equidistributed (p. 8)
$\chi(M)$ characteristic function of a set M
Tu rearrangement of a function u
$u^\sharp$ symmetric decreasing rearrangement of u (p. 8)
$\kappa_N = \pi^{N/2}/\Gamma((N/2)+1)$ measure of the N-dimensional unit ball
$\sigma_H = x_H$ reflexion in ∂H, where H is a halfspace (p. 15)
$T^H u = u_H$ two-point rearrangement of u w.r.t. H (p. 15)
$u^\star$ Schwarz symmetrization of u (p. 19)
u^* Steiner symmetrization of u (p. 19)
Cu cap symmetrization of u (p. 20)
$\mathcal{H}$ set of all affine halfspaces in $\mathbb{R}^N$ (p. 21)
$\mathcal{H}^*$, $C\mathcal{H}$, $\mathcal{H}_0$, $\mathcal{H}_0^*$, $C\mathcal{H}_P$ subsets of $\mathcal{H}$ (p. 21)

Acknowledgement

I would like to thank M. Ashbaugh (Columbia), A. Baernstein II (St. Louis) and B. Kawohl (Cologne) for some useful conversations.

Santiago de Chile, December 2005

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York (1975).
[2] J.A. Almgren and E.H. Lieb, *Symmetric decreasing rearrangement is sometimes continuous*, J. Amer. Math. Soc. **2** (1989), 683–773.

[3] A. Alvino, V. Ferone, P.-L. Lions and G. Trombetti, *Convex symmetrization and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (2) (1997), 275–293.
[4] A. Alvino, V. Ferone and G. Trombetti, *Estimates for the gradient of solutions of nonlinear elliptic equations with L^1 data*, Ann. Mat. Pura Appl. (4) **178** (2000), 129–142.
[5] A. Alvino, P.-L. Lions and G. Trombetti, *On optimization problems with prescribed rearrangements*, Nonlinear Anal. TMA **13** (1989), 185–220.
[6] A. Alvino, P.-L. Lions and G. Trombetti, *Comparison results for elliptic and parabolic equations via Schwarz symmetrization*, Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (2) (1990), 37–65.
[7] A. Alvino, P.-L. Lions and G. Trombetti, *Comparison results for elliptic and parabolic equations via symmetrization: a new approach*, Differential Integral Equations **4** (1) (1991), 25–50.
[8] A. Alvino and G. Trombetti, *Sulle migliori costanti di maggioraz ione per una classe di equazioni ellittiche degeneri*, Ricerche Mat. **27** (1978), 413–428.
[9] A. Alvino and G. Trombetti, *Equazioni ellitiche con termini di ordine inferiore e riordinamenti*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **66** (8) (1979), 1–7.
[10] A. Alvino and G. Trombetti, *Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri e non*, Ricerche Mat. **30** (1981), 15–33.
[11] A. Alvino and G. Trombetti, *Su una classe di equazioni ellitiche non lineari degeneri*, Ricerche Mat. **30** (1981), 193–212.
[12] G. Aronsson and G. Talenti, *Estimating the integral of a function in terms of a distribution function of its gradient*, Boll. Un. Mat. Ital. B (5) **18** (1981), 885–894.
[13] M. Ashbaugh, *Open problems on eigenvalues of the Laplacian*, Analytic and Geometric Inequalities and their Applications, vol. 4787, T.M. Rassias and H.M. Srivastava, eds, Kluwer (1999).
[14] M. Ashbaugh and R. Benguria, *Proof of the Payne–Polya–Weinberger conjecture*, Bull. Amer. Math. Soc. **25** (1) (1991), 19–29.
[15] M. Ashbaugh and R. Benguria, *A sharp bound for the ratio of the first two eigenvalues of the Dirichlet Laplacians and extensions*, Ann. Math. **135** (3) (1992), 601–628.
[16] M. Ashbaugh and R. Benguria, *On Rayleigh's conjecture for the clamped plate and its generalizations to three dimensions*, Duke Math. J. **78** (1995), 1–17.
[17] M. Badiale and E. Nabana, *A note on radiality of solutions of p-Laplacian equation*, Appl. Anal. **52** (1994), 35–43.
[18] A. Baernstein II, *A unified approach to symmetrization*, Partial Differential Equations of Elliptic Type, A. Alvino et al., eds, Symposia Matematica, vol. 35, Cambridge Univ. Press (1995), 47–91.
[19] A. Baernstein II and B.A. Taylor, *Spherical rearrangements, subharmonic functions, and *-functions in n-space*, Duke Math. J. **43** (1976), 245–268.
[20] C. Bandle, *Isoperimetric Inequalities and Applications*, Pitman Research Notes, Boston (1986).
[21] W. Beckner, *Sobolev inequalities, the Poisson semigroup and analysis on the sphere S^n*, Proc. Nat. Acad. Sci. USA **89** (1992), 4816–4819.
[22] Y. Benyamini, *Two-point symmetrization, the isoperimetric inequality on the sphere and some applications*, Texas Functional Analysis Seminar 1983/84, Austin, TX, Longhorne Notes, University of Texas Press, Austin, TX (1984), 53–76.
[23] H. Berestycki and T. Lachand-Robert, *Some properties of the monotone rearrangement with applications to elliptic equations in cylinders*, Math. Nachr., to appear.
[24] H. Berestycki and L. Nirenberg, *Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations*, J. Geom. Phys. **5** (1988), 237–275.
[25] H. Berestycki, L. Nirenberg, *Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains*, Analysis etc., P. Rabinowitz et al., eds, Academic Press, Boston (1990), 115–164.
[26] H. Berestycki and L. Nirenberg, *The method of moving planes and the sliding method*, Bol. Soc. Brasil. Mat. (N.S.) **22** (1991), 1–37.
[27] C. Borell, *Geometric bounds on the Ornstein–Uhlenbeck velocity process*, Z. Wahrsch. Verw. Gebiete **70** (1985), 1–13.
[28] H.J. Brascamp, E.H. Lieb and J.M. Luttinger, *A general rearrangement inequality for multiple integrals*, J. Funct. Anal. **17** (1974), 227–237.
[29] H. Brezis, *Symmetry in nonlinear PDE's*, Proc. Sympos. Pure Math. **65** (1999), 1–12.
[30] F. Brock, *Continuous Steiner-symmetrization*, Math. Nachr. **172** (1995), 25–48.

[31] F. Brock, *Continuous polarization and symmetry of solutions of variational problems with potentials*, Calculus of Variations, Applications and Computations, C. Bandle et al., eds, Pont-a-Mousson, 1994, Pitman Research Notes in Math., vol. 326 (1995), 25–35.

[32] F. Brock, *Radial symmetry for nonnegative solutions of semilinear elliptic problems involving the p-Laplacian*, Progress in Partial Differential Equations, vol. 1, H. Amann et al., eds, Pont-á-Mousson, 1997, Pitman Research Notes in Math., vol. 383, 46–58.

[33] F. Brock, *Continuous rearrangement and symmetry of solutions of elliptic problems*, Proc. Indian Acad. Sci. Math. Sci. **110** (2) (2000), 157–204. Available at: http://www.ias.ac.in/mathsci/vol110/may2000/Pm1707.pdf.

[34] F. Brock, *A general rearrangement inequality á la Hardy–Littlewood*, J. Inequal. Appl. **5** (4) (2000), 309–320.

[35] F. Brock, *Symmetry and monotonicity of solutions to some variational problems in cylinders and annuli*, Electronic J. Differential Equations **2003** (108) (2003), 1–20.

[36] F. Brock, *Positivity and radial symmetry of solutions to some variational problems in* $\mathbb{R}^N$, J. Math. Anal. Appl. **296** (2004), 226–243.

[37] F. Brock and A.Yu. Solynin, *An approach to symmetrization via polarization*, Trans. Amer. Math. Soc. **352** (4) (2000), 1759–1795.

[38] J. Brothers and W.P. Ziemer, *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math. **384** (1988), 153–179.

[39] Yu.D. Burago, V.A. Zalgaller, *Geometric Inequalities*, Grundlehren Math. Wiss., vol. 285, Springer (1980).

[40] A. Burchard, *Steiner symmetrization is continuous in* $W^{1,p}$, GAFA **7** (1997), 823–860.

[41] A. Burchard and M. Schmuckenschläger, *Comparison theorems for exit times*, GAFA **11** (2001), 651–692.

[42] J. Busca and B. Sirakov, *Symmetry results for semilinear elliptic systems in the whole space*, J. Differential Equations **163** (1) (2000), 41–56.

[43] R. Casten and C. Holland, *Instability results for a reaction-diffusion equation with Neumann boundary condition*, J. Differential Equations **27** (1978), 266–273.

[44] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press (1984).

[45] K.M. Chong and N.M. Rice, *Equimeasurable Rearrangements of Functions*, Queen's Papers in Pure and Appl. Math., vol. 28, Queen's Univ., Ontario (1983).

[46] J.-M. Coron, *The continuity of rearrangement in* $W^{1,p}(\mathbb{R})$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **11** (1984), 57–85.

[47] V. Coti-Zelati and M. Esteban, *Symmetry breaking and multiple solutions for a Neumann problem in an exterior domain*, Proc. Roy. Soc. Edinburgh Sect. A **116** (1990), 327–339.

[48] J.A. Crowe, P.C. Rosenbloom and J.A. Zweibel, *Rearrangements of functions*, J. Funct. Anal. **66** (1986), 432–438.

[49] L. Damascelli, *Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results*, Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), 493–516.

[50] L. Damascelli and F. Pacella, *Monotonicity and symmetry of solutions of p-Laplace equations,* $1 < p < 2$*, via the moving plane method*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (4) (1998), 689–707.

[51] L. Damascelli and F. Pacella, *Monotonicity and symmetry results for p-Laplace equations and applications*, Adv. Differential Equations **5** (7–9) (2000), 1179–1200.

[52] L. Damascelli, F. Pacella and M. Ramaswamy, *Symmetry of ground states of p-Laplace equations via the moving plane method*, Arch. Ration. Mech. Anal. **148** (4) (1999), 291–308.

[53] L. Damascelli, F. Pacella and M. Ramaswamy, *Symmetry of* C^1 *solutions of p-Laplace equations in* $\mathbb{R}^N$, Adv. Nonlinear Stud. **1** (1) (2001), 40–64.

[54] L. Damascelli and B. Sciunzi, *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Differential Equations **206** (2004), 483–515.

[55] J.I. Diaz, *Nonlinear PDE and Free Boundaries. Vol. I, Elliptic Equations*, Pitman Research Notes, vol. 106, Boston (1985).

[56] C. Draghici, *A general rearrangement inequality*, Proc. Amer. Math. Soc. **133** (3) (2005), 735–743.

[57] V.N. Dubinin, *Transformation of functions and Dirichlet's principle*, Math. Notes **38** (1985), 539–542.

[58] V.N. Dubinin, *Capacities and geometric transformations in n-space*, GAFA **3** (1993), 342–369.

[59] V.N. Dubinin, *Symmetrization in the geometric theory of functions of a complex variable*, Russian Math. Surveys **49** (1) (1994), 1–79.

[60] M. Esteban, *Nonsymmetric ground states of symmetric variational problems*, Comm. Pure Appl. Math. **XLIV** (1991), 259–274.
[61] M. Esteban and W. Strauss, *Nonlinear bound states outside an insulated sphere*, Comm. Partial Differential Equations **19** (1994), 177–197.
[62] G. Faber, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitz. Ber. Bayer. Akad. Wiss. (1923), 169–172.
[63] A. Friedman, *Variational Principles and Free-Boundary Problems*, Wiley-Interscience, New York (1982).
[64] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
[65] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Math. Anal. Appl., Part A, L. Nachbin, ed., Adv. Math. Suppl. Stud., vol. 7A, Academic Press, New York (1981), 369–402.
[66] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, Basel (1984).
[67] M. Grossi, S. Kesavan, F. Pacella and M. Ramaswamy, *Symmetry of positive solutions of some nonlinear equations*, Topol. Methods Nonlinear Anal. **12** (1) (1998), 47–59.
[68] G. Hardy, J. Littlewood and G. Polya, *Inequalities*, Cambridge University Press, London (1952).
[69] W. Hayman, *Subharmonic Functions, Vol. II*, Academic Press, London (1989).
[70] A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators*, Frontiers in Mathematics, Birkhäuser, Basel (2006), 202 pp.
[71] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Vol. III*, Grundlehren Math. Wiss., vol. 274, Springer-Verlag, Berlin (1985).
[72] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Math., vol. 1150, Springer (1985).
[73] B. Kawohl, *On the simple shape of stable equilibria*, Geometry of Solutions to Partial Differential Equations, G. Talenti, ed., Academic Press (1989), 73–89.
[74] B. Kawohl, *Some nonconvex optimization problems*, Optimal Shape Design, B. Kawohl, O. Pironneau, L. Tartar and J.P. Zolesio, eds, Lecture Notes in Math., vol. 1740, Springer, New York (2000).
[75] S. Kesavan and F. Pacella, *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities*, Appl. Anal. **54** (1994), 7–37.
[76] V.I. Kolyada, *Rearrangements of functions and embedding theorems*, Russian Math. Surveys **44:5** (1989), 73–117.
[77] E. Krahn, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann. **94** (1924), 97–100.
[78] I. Kuzin and S. Pohozaev, *Entire Solutions of Semilinear Elliptic Equations*, Progr. Nonlinear Differential Equations Appl., vol. 33, Birkhäuser, Basel (1997).
[79] M. Ledoux, *Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space*, Bull. Sci. Math. **118** (1994), 485–510.
[80] C.-M. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on bounded domains*, Comm. Partial Differential Equations **16** (1991), 491–526.
[81] C.-M. Li, *Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains*, Comm. Partial Differential Equations **16** (1991), 585–615.
[82] Y. Li and W.-M. Ni, *On the asymptotic behavior and radial symmetry of positive solutions of semilinear elliptic equations in* $\mathbb{R}^n$, Arch. Ration. Mech. Anal. **118** (1992), 195–243.
[83] Y. Li and W.-M. Ni, *Radial symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^n$, Comm. Partial Differential Equations **18** (1993), 1043–1054.
[84] E.H. Lieb, *Inequalities. Selecta of Elliott H. Lieb*, Edited, with a preface and commentaries, by M. Loss and M.B. Ruskai, Springer-Verlag, Berlin (2002).
[85] E.H. Lieb and M. Loss, *Analysis*, second ed., Graduate Studies in Mathematics, vol. 14, Amer. Math. Soc., Providence, RI (2001).
[86] G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (2&3) (1991), 311–361.
[87] P.-L. Lions, *Two geometrical properties of solutions of semilinear problems*, Appl. Anal. **12** (1981), 267–272.

[88] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case*, Ann. Inst. H. Poincaré Anal. Non Linéaire, Part 1 **1** (2) (1984), 109–145; Part 2 **1** (4) (1984), 223–283.

[89] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case*, Rev. Mat. Iberoamericana, Part 1 **1** (1) (1985), 145–201; Part 2 **1** (2) (1985), 45–121.

[90] O. Lopes, *Radial and nonradial minimizers for some radially symmetric functionals*, Electronic J. Differential Equations **3** (1996), 1–14.

[91] O. Lopes, *Radial symmetry of minimizers for some translation and rotation invariant functionals*, J. Differential Equations **124** (1996), 378–388.

[92] C. Morpurgo, *Sharp inequalities for functional integrals and traces of conformally invariant operators*, Duke Math. J., to appear.

[93] J. Mossino, *Inégalités Isopérimetriques et Applications en Physique*, Herman, Paris (1984).

[94] N.S. Nadirashvili, *Rayleigh's conjecture on the principal frequency of the clamped plate*, Arch. Ration. Mech. Anal. **129** (1995), 1–10.

[95] L.E. Payne, *Some comments on the past fifty years of isoperimetric inequalities*, Inequalities (Birmingham, 1987), Lecture Notes in Pure and Appl. Math., vol. 129, Dekker, New York (1991), 143–161.

[96] G. Polya and G. Szegö, *Isoperimetric Inequalities in Mathematical Physics*, Ann. of Math. Stud., vol. 27, Princeton Univ. Press (1952).

[97] P. Pucci and J. Serrin, *A note on the strong maximum principle for singular elliptic inequalities*, J. Math. Pure Appl. **79** (2000), 57–71.

[98] P. Pucci, J. Serrin and H. Zou, *A strong maximum principle and a compact support principle for singular elliptic inequalities*, J. Math. Pures Appl. **78** (1999), 769–789.

[99] F. Riesz, *Sur une inégalité intégrale*, J. London Math. Soc. **5** (1930), 162–168.

[100] S. Saks, *Theory of the Integral*, Monografje Matematyczne, Warszaw/Lvov (1937) (transl. L.C. Young).

[101] J. Sarvas, *Symmetrization of condensers in n-space*, Ann. Acad. Sci. Fenn. Ser. A1 **522** (1972), 1–44.

[102] J. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal. **43** (1971), 304–318.

[103] J. Serrin and H. Zou, *Symmetry of ground states of quasilinear elliptic equations*, Arch. Ration. Mech. Anal. **148** (1999), 265–290.

[104] D. Smets, M. Willem, *Partial symmetry and asymptotic behavior for some elliptic variational problems*, Calc. Var. Partial Differential Equations **18** (1) (2003), 57–75.

[105] A.Yu. Solynin, *Continuous symmetrization of sets*, Zapiski Nauchnykh Seminarov LOMI Akademii Nauk SSSR **185** (1990), 125–139.

[106] J. Steiner, *Einfacher Beweis der isoperimetrischen Hauptsätze*, Crelle J. Reine Angew. Math. **18** (1838), 281–296; Ges. Werke II, Berlin (1882), 77–91 (in German).

[107] M. Struwe, *Variational Methods*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, Berlin (2000).

[108] G. Sweers, *A sign-changing global minimizer on a convex domain*, Progress in PDE's: Elliptic and Parabolic Problems, C. Bandle et al., eds, Pitman Research Notes, vol. 266 (1992), 251–258.

[109] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (4) (1976), 353–372.

[110] G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Pisa Cl. Sci. **3** (4) (1976), 697–718.

[111] G. Talenti, *Linear elliptic PDE's: level sets, rearrangements and a-priori estimates of solutions*, Boll. Un. Mat. Ital. B (6) **4** (1985), 917–949.

[112] G. Talenti, *The Standard Isoperimetric Theorem*, Handbook of Convex Geometry A (1993), 431–455.

[113] G. Talenti, *Inequalities in rearrangement invariant function spaces*, Nonlinear Analysis, Function Spaces and Applications, vol. 5, M. Krbec et al., eds, Proceedings of the Spring School, held in Prague, May 1994, Prometheus Publ., Prague (1994), 177–231.

[114] G. Talenti, *A weighted version of a rearrangement inequality*, Ann. Univ. Ferrara Sez. 7 Sc. Mat. **XLIII** (1997), 121–133.

[115] G. Trombetti, *Symmetrization methods for partial differential equations*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. 8, **3**(8) (3) (2000), 601–643.

[116] J. van Schaftingen, *Symmetrization and minimax principles*, Commun. Contemp. Math. **7** (4) (2005), 463–481.

[117] J. van Schaftingen, *Universal approximation of symmetrizations by polarizations*, Proc. Amer. Math. Soc. **134** (1) (2005), 177–186.

[118] J. van Schaftingen, *Approximation of symmetrizations and symmetry of critical points*, Topol. Methods Nonlinear Anal., to appear.
[119] J. van Schaftingen, *Anisotropic symmetrizations*, Ann. Inst. H. Poincaré Anal. Non Linéare **23** (4) (2006), 539–565.
[120] J. van Schaftingen and M. Willem, *Set transformations, symmetrizations and isoperimetric inequalities*, Nonlinear Analysis and Applications to Physical Sciences, Springer, Italia, Milan (2004), 135–152.
[121] J.L. Vazquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. (1984), 191–202.
[122] L. Veron, *Singularities of solutions of second order quasilinear equations*, Pitman Res. Notes, vol. 353, Longman (1996).
[123] V. Wolontis, *Properties of conformal invariants*, Amer. J. Math. **74** (1952), 587–606.
[124] E. Zeidler, *Nonlinear Functional Analysis and its Applications, Vol. III, Variational Methods and Optimization*, Springer-Verlag, New York (1985).

CHAPTER 2

Liouville-Type Theorems for Elliptic Problems

Alberto Farina

LAMFA, CNRS UMR 6140, Université de Picardie Jules Verne, Faculté de Mathématiques et d'Informatique, 33, rue Saint-Leu, 80039 Amiens, France
E-mail: alberto.farina@u-picardie.fr

Contents

1. Introduction 63
2. Cauchy and Liouville 63
3. Hadamard and Liouville 65
4. Poisson and Liouville 67
5. Bernstein and Liouville 75
6. Jörgens and Liouville 81
7. De Giorgi and Liouville: Part I 83
8. De Giorgi and Liouville: Part II 93
9. Harnack and Liouville 98
10. Moser and Liouville 108
Acknowledgements 113
References 113

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 4
Edited by M. Chipot
© 2007 Elsevier B.V. All rights reserved

This page intentionally left blank

1. Introduction

This work is devoted to Liouville theorems and it is conceived like a walk through the fascinating world of the classification of solutions to linear and non-linear second-order partial differential equations of elliptic type defined on the entire N-dimensional Euclidean space. We have chosen to illustrate the results, ideas and tools, by focusing on problems naturally arising in different areas of mathematics and its applications. The considered topics have been selected among problems in complex analysis, differential geometry, geometric analysis, phase transition and non-linear potential theory, according to the author's taste. The attention is focused on those problems (old and new), whose solution depends on some Liouville-type theorem. Let us mention, in this respect, the Bernstein problem for minimal graphs, a conjecture of De Giorgi and the problem of deciding what is the conformal type of a given Riemann surface. Also, the sharpness of the considered results is discussed by means of examples and counterexamples. The work is composed of ten sections:

1. Introduction
2. Cauchy and Liouville
3. Hadamard and Liouville
4. Poisson and Liouville
5. Bernstein and Liouville
6. Jörgens and Liouville
7. De Giorgi and Liouville: Part I
8. De Giorgi and Liouville: Part II
9. Harnack and Liouville
10. Moser and Liouville

We wish you a good walk!

2. Cauchy and Liouville

One of the most celebrated theorems in complex analysis states that:

THEOREM 2.1. *A bounded complex function $f : \mathbf{C} \to \mathbf{C}$ which is holomorphic on the entire complex plane is a constant function.*

This theorem is attached to the name of the French mathematician Joseph Liouville, who announced it [54,55] in a Note in the Comptes Rendus de l' Académie des Sciences (Paris, December 9, 1844) for the special case of a doubly-periodic function.

Fourteen days later, in another Note in the Comptes Rendus de l' Académie des Sciences (Paris, December 23, 1844), Augustin Cauchy [13,14] published the first proof of the above stated theorem.

A proof of Theorem 2.1, usually based on Cauchy's integral formula, can be found in any textbook of complex analysis. Another way to prove the above theorem is to use the following classical result concerning real-valued harmonic functions defined on the entire Euclidean space $\mathbb{R}^N$.

THEOREM 2.2. *Let $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a bounded solution of $-\Delta u = 0$ in $\mathbb{R}^N$, $N \geqslant 1$. Then u is a constant function.*

To prove Theorem 2.1 we recall that the real and the imaginary parts of a holomorphic function satisfy the Cauchy–Riemann equations, therefore they are bounded real-valued harmonic functions on $\mathbb{R}^2$. The desired conclusion then follows by applying Theorem 2.2.

There also exists a stronger version of the above Theorem 2.2:

THEOREM 2.3 (Strong Liouville property). *Assume $N \geqslant 2$ and let k be a positive integer. Let $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a solution of*:

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^N, \\ u(x) = \mathrm{O}(|x|^k) & \text{as } |x| \to +\infty. \end{cases}$$

Then u is a polynomial of at most degree k.

These results are well known but, for sake of completeness we shall give a quick proof of them. To this end we recall the following two well-known properties of harmonic functions (see for instance [22,36]):

THEOREM 2.4. *Assume $N \geqslant 1$ and let $u \in C^2(\Omega, \mathbb{R})$ be a solution of $-\Delta u = 0$ in an open set $\Omega \subset \mathbb{R}^N$, then*:

(i) Mean value properties: *for any open ball $B = B(x, r) \subset \Omega$ we have*:

$$u(x) = \frac{1}{|\partial B|} \int_{\partial B} u(s)\,\mathrm{d}s = \frac{1}{|B|} \int_B u(y)\,\mathrm{d}y,$$

here, as usual, $|\partial B|$ denotes the surface measure of the boundary of the N-dimensional ball B, while $|B|$ denotes the Lebesgue measure of the N-dimensional ball B.

(ii) Smoothness: *the function u belongs to $C^\infty(\Omega)$. In particular all partial derivatives of u are smooth harmonic functions.*

We are now ready to prove the above stated theorems.

PROOF OF THEOREM 2.2. We present a simple and elegant proof due to E. Nelson [62]. For every $x \in \mathbb{R}^N$ and every $R > 0$ let $B(x, R)$ be the open ball centered at x and with radius R. The mean value formula gives

$$\begin{aligned} \big|u(x) - u(0)\big| &= \frac{1}{|B(0,R)|} \left| \int_{B(x,R)} u(y)\,\mathrm{d}y - \int_{B(0,R)} u(y)\,\mathrm{d}y \right| \\ &= \frac{1}{|B(0,R)|} \left| \int_{D(x,0,R)} u(y)\,\mathrm{d}y \right| \leqslant \sup_{\mathbb{R}^N} |u| \frac{|D(x,0,R)|}{|B(0,R)|}, \end{aligned}$$

where $D(x,0,R)$ is the symmetric difference of $B(x,R)$ and $B(0,R)$. Since $\frac{|D(x,0,R)|}{|B(0,R)|}$ converges to 0 when $R \to +\infty$ we obtain $u(x) = u(0)$. Thus, u is constant. □

PROOF OF THEOREM 2.3. By part (ii) of Theorem 2.4, we have that every partial derivative $\partial u/\partial x_j$ is a smooth harmonic function on $\mathbb{R}^N$. Furthermore, by assumption there are positive constants A and B such that $|u(x)| \leqslant A + B|x|^k$ for every $x \in \mathbb{R}^N$.

For any $x \in \mathbb{R}^N$ let B be the open ball centered at x and with radius $|x| + 1$. Then, by part (i) of Theorem 2.4, every partial derivative $\partial u/\partial x_j$ satisfies:

$$\left|\frac{\partial u}{\partial x_j}(x)\right| = \frac{1}{|B|}\left|\int_B \frac{\partial u}{\partial x_j}(y)\,\mathrm{d}y\right| = \frac{1}{|B|}\left|\int_{\partial B} u(s)\nu_j\,\mathrm{d}s\right|$$
$$\leqslant \frac{N}{|x|+1}\big(A + B\big(2|x|+1\big)^k\big) \leqslant AN + BN2^k\big(|x|+1\big)^{k-1}.$$

By repeating a finite number of times the last procedure we get that any partial derivative of order k is a bounded harmonic function defined on the entire Euclidean space $\mathbb{R}^N$. The desired conclusion then follows by applying Theorem 2.2. □

3. Hadamard and Liouville

Theorem 2.2 admits an extension to super-harmonic functions defined on the Euclidean plane $\mathbb{R}^2$ and which are bounded below. More precisely we have:

THEOREM 3.1. *Let $u \in C^2(\mathbb{R}^2, \mathbb{R})$ be a solution of*:

$$\begin{cases} -\Delta u \geqslant 0 & \text{in } \mathbb{R}^2, \\ u(x) \geqslant -C & \forall x \in \mathbb{R}^2, \end{cases}$$

for some positive constant C. Then u is a constant function.

The above theorem does not hold if we replace the assumption that u is bounded from below by the assumption that u is bounded from above. Indeed, the function $u(x) = -|x|^2$ is a non-constant super-harmonic function which is bounded above.

We prove Theorem 3.1 by making use of the so-called *Hadamard three-circles theorem* for sub-harmonic functions [70], which states:

THEOREM 3.2. *Let Ω be a two-dimensional domain containing the closure of the annulus $A_{r_1,r_2} := \{x \in \mathbb{R}^2 \colon 0 < r_1 < |x| < r_2\}$. Let $v \in C^2(\Omega, \mathbb{R})$ be a sub-harmonic function in Ω, i.e., a function satisfying the differential inequality*:

$$-\Delta v \leqslant 0 \quad \text{in } \Omega.$$

If $M(r)$ denotes the maximum of v on any circle $|x| = r$, then $M(r)$ is a convex function of $\ln(r)$, *i.e., for* $r_1 < r < r_2$:

$$M(r) \leqslant M(r_1)\frac{\ln(r_2/r)}{\ln(r_2/r_1)} + M(r_2)\frac{\ln(r/r_1)}{\ln(r_2/r_1)}. \tag{3.1}$$

The above result is a slightly different version of the original result announced by J. Hadamard [38] in 1896. The original Hadamard three-circles theorem refers to a holomorphic function $f \not\equiv 0$, defined in the annulus $0 < r_1 < |z| < r_2$, and states that the function $r \to \ln(\max_{|z|=r} |f(z)|)$ is a convex function of $\ln(r)$.

PROOF OF THEOREM 3.2. For every $x \in \overline{A_{r_1,r_2}}$ we consider the function

$$w(x) = v(x) - M(r_1)\frac{\ln(r_2/|x|)}{\ln(r_2/r_1)} - M(r_2)\frac{\ln(|x|/r_1)}{\ln(r_2/r_1)}.$$

Then w belongs to $C^2(\overline{A_{r_1,r_2}}, \mathbb{R})$ and satisfies:

$$\begin{cases} -\Delta w \leqslant 0 & \text{in } A_{r_1,r_2}, \\ w \leqslant 0 & \text{on } \partial A_{r_1,r_2}. \end{cases}$$

The desired conclusion is then a consequence of the maximum principle for the Laplace operator. □

PROOF OF THEOREM 3.1. The function $v = -u$ is bounded above and satisfies the assumptions of Theorem 3.2. Hence, letting $r_2 \to +\infty$ in inequality (3.1), we have:

$$\forall r_1 > 0,\ \forall r \geqslant r_1, \quad M(r) \leqslant M(r_1).$$

On the other hand, by the maximum principle, M is a non-decreasing function, therefore:

$$\forall r > 0, \quad M(r) = M(0) = u(0).$$

The claim then follows by applying the strong maximum principle to v. □

In the language of geometry Theorem 3.1 says that $\mathbb{R}^2$, endowed with the standard flat metric, is a *parabolic Riemannian manifold*. We recall that a Riemannian manifold (M, g) is *parabolic* if the only C^2 non-negative super-harmonic functions are the constants, that is, the conditions: $u \in C^2(M)$, $-\Delta_g u \geqslant 0$ and $u \geqslant 0$ on M imply $u = const$. (here Δ_g denotes the Laplace–Beltrami operator, i.e., the Laplace operator associated with the metric g).

An example of Riemannian manifold which is *not parabolic* is given by $\mathbb{R}^N$ (endowed with the standard flat metric) with $N \geqslant 3$. Indeed, in this case, the function

$$u(x) := \left(\frac{\sqrt{N(N-2)}}{1+|x|^2}\right)^{(N-2)/2},$$

is a positive, non-constant solution of the Lane–Emden equation $-\Delta u = u^{(N+2)/(N-2)} \geqslant 0$ on $\mathbb{R}^N$. In particular, this proves that Theorem 3.1 does not hold for $N \geqslant 3$. Nevertheless not all is lost, since we have the following:

THEOREM 3.3. *Let $N \geqslant 1$ and let $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a solution of $-\Delta u = 0$ in $\mathbb{R}^N$. If u is bounded from below* (*above*) *then u is a constant.*

PROOF. To prove the claim it is enough to consider the case of a non-negative u. Pick $x \in \mathbb{R}^N$, $R > 0$ and set $B = B(x, R)$, the open ball centered at x and with radius R. The mean value property, together with the positivity of the solution u, yields:

$$\left| \frac{\partial u}{\partial x_j}(x) \right| = \frac{1}{|B|} \left| \int_{\partial B} u(s) \nu_j \, \mathrm{d}s \right| \leqslant \frac{1}{|B|} \int_{\partial B} u(s) \, \mathrm{d}s.$$

On the other hand we have:

$$\frac{1}{|B|} \int_{\partial B} u(s) \, \mathrm{d}s = \frac{N}{R} \frac{1}{|\partial B|} \int_{\partial B} u(s) \, \mathrm{d}s = \frac{N}{R} u(x)$$

again by the mean value formula. Therefore one obtains:

$$\forall j = 1, \ldots, N, \ \forall x \in \mathbb{R}^N, \quad \left| \frac{\partial u}{\partial x_j}(x) \right| \leqslant \frac{N}{R} u(x)$$

which implies the constancy of u by letting $R \to +\infty$. □

Theorem 3.3 says that an entire non-constant harmonic function must be unbounded both from below and from above. In Section 9, we shall see that this property is not a prerogative of harmonic functions. Indeed, it will be seen that the Laplace operator Δ shares this property with the class of second order uniformly elliptic operators.

4. Poisson and Liouville

In this section we prove some Liouville-type theorems for the semilinear Poisson equation:

$$-\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N, \tag{4.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function.

The first result is a consequence of standard Liouville Theorems 2.3 and/or 3.3.

PROPOSITION 4.1. *Assume $N \geqslant 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, non-decreasing function, $f \not\equiv 0$. Let $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a solution of*:

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \mathbb{R}^N, \\ u(x) = \mathrm{o}(|x|^2) & \text{as } |x| \to +\infty. \end{cases}$$

Then u is a constant function, say $u \equiv c \in \mathbb{R}$, and $f(c) = 0$.

PROOF. We use a slight modification of a procedure described by J. Serrin in the paper [78] (cf. pp. 349–350) which, in its turn, is based on an idea of R. Redheffer [72].

For every $x_0 \in \mathbb{R}^N$ and every $\epsilon > 0$ consider the function:

$$v(x) = u(x) - u(x_0) + 2\epsilon - \epsilon|x - x_0|^2, \quad \forall x \in \mathbb{R}^N.$$

Since $v(x_0) > 0$ and $\lim_{|x|\to+\infty} v(x) = -\infty$, there exists a point $y_0 \in \mathbb{R}^N$ such that $v(y_0) = \max_{x\in\mathbb{R}^N} v(x) > 0$ and thus $\Delta v(y_0) \leqslant 0$. Therefore we have:

$$u(x_0) - 2\epsilon < u(y_0), \qquad \Delta u(y_0) \leqslant 2\epsilon N$$

and the monotonicity of f implies:

$$f\big(u(x_0) - 2\epsilon\big) \leqslant f\big(u(y_0)\big) = \Delta u(y_0) \leqslant 2\epsilon N.$$

By letting $\epsilon \to 0$ in the latter inequality we obtain that $f(u(x_0)) \leqslant 0$. By a similar argument, involving the function $w(x) = u(x) - u(x_0) - 2\epsilon + \epsilon|x - x_0|^2$ instead of v, we also have that $f(u(x_0)) \geqslant 0$.

By summarizing we have proved that:

$$\forall x \in \mathbb{R}^N, \quad -\Delta u(x) = -f\big(u(x)\big) = 0 \tag{4.2}$$

and in particular u is a harmonic function on $\mathbb{R}^N$ which cannot be surjective since otherwise we would have $f \equiv 0$, a contradiction. This yields immediately the desired conclusion by invoking either Theorem 2.3 or Theorem 3.3. □

REMARKS 4.2.

(i) Despite the simplicity of its proof the above result is sharp, as one can easily see by considering the function $u(x) = |x|^2$ which solves equation (4.1) with $f = 2N$.
Also the assumption $f \not\equiv 0$ cannot be removed without affecting the result, since the Laplace equation admits non-constant affine solutions on $\mathbb{R}^N$.

(ii) Under the more restrictive assumption $u(x) = o(|x|)$ as $|x| \to +\infty$, the above Proposition 4.1 (including the case $f \equiv 0$) was proved by J. Serrin [78] in 1972.

If one looks only to non-negative solutions of the semilinear Poisson equation (4.1) a more precise result is available.

THEOREM 4.3. *Assume $N \geqslant 1$ and let $f : [0, +\infty) \to [0, +\infty)$ be a continuous, non-decreasing function such that*

$$\begin{cases} f(0) = 0, \\ f(t) > 0, \quad t > 0. \end{cases} \tag{4.3}$$

Then, the only non-negative solution $u \in C^2(\mathbb{R}^N, \mathbb{R})$ of:

$$-\Delta u + f(u) \leqslant 0 \quad \text{in } \mathbb{R}^N \tag{4.4}$$

is $u \equiv 0$ *if and only if*:

$$\forall s_0 > 0 \quad \int_{s_0}^{+\infty} \left[\int_{s_0}^{s} f(t)\,\mathrm{d}t \right]^{-1/2} \mathrm{d}s < +\infty. \tag{4.5}$$

The above result is essentially due to J.B. Keller [46] and R. Osserman [65] who, working independently, proved it simultaneously in 1957. In the mathematical literature the integral condition (4.5) is known as *Keller–Osserman condition*.

To prove Theorem 4.3 we need the following:

LEMMA 4.4. *Assume* $N \geqslant 1$ *and let* $f : [0, +\infty) \to [0, +\infty)$ *be a continuous, non-decreasing function satisfying* (4.3).

If the differential inequality (4.4) *has a non-trivial, non-negative solution* $u \in C^2(\mathbb{R}^N, \mathbb{R})$, *then equation* (4.1) *admits a positive radially symmetric solution* (*a positive even solution if* $N = 1$) $v = v(|x|)$ *of class* C^2 *such that* $v(0) > 0$ *and* $v'(0) = 0$. *Furthermore*, $v'(r) > 0$ *for* $r > 0$ *and* $\lim_{r \to +\infty} v(r) = +\infty$.

(*Notice that we do not assume that* f *satisfies the Keller–Osserman condition* (4.5).)

PROOF. By the translation invariance of (4.4) we may suppose that $u(0) > 0$. Fix $\alpha \in (0, u(0))$ and consider the solution v of the following initial value problem:

$$\begin{cases} (r^{N-1} v'(r))' = r^{N-1} f(v(r)), \quad r \in (0, R_M), \\ v(0) = \alpha, \quad v'(0) = 0. \end{cases} \tag{4.6}$$

where $(0, R_M)$ is the maximal interval of existence of v. Let us prove that $R_M = +\infty$, so that v will be the required solution. Let assume, to the contrary, that R_M is finite, then:

$$\forall r > 0, \quad v'(r) = r^{1-N} \int_0^r s^{N-1} f\big(v(s)\big)\,\mathrm{d}s \tag{4.7}$$

which, in view of (4.3), immediately implies $v'(r) > 0$ for $r \in (0, R_M)$ and so that $\lim_{r \to R_M} v(r) = +\infty$ (otherwise we would have also $\lim_{r \to R_M} v'(r) \in (0, +\infty)$ and v could be extended beyond R_M, as a solution of (4.6), contradicting thus the maximality of R_M). Therefore we can find $r_1 \in (0, R_M)$ such that $u - v \leqslant 0$ on the boundary of $B(0, r_1)$, the open ball centered at the origin and of radius r_1. Hence:

$$\begin{cases} \Delta(u - v) \geqslant f(u) - f(v) & \text{in } B(0, r_1), \\ u - v \leqslant 0 & \text{on } \partial B(0, r_1). \end{cases}$$

Multiplying the latter differential inequality by $(u - v)^+$, the positive part of $u - v$, and integrating by parts we get $\int_{B(0,r_1)} |\nabla (u - v)^+|^2 \leqslant 0$ which, in particular gives $u(0) \leqslant v(0)$, a contradiction. Therefore, $R_M = +\infty$. To complete the proof we notice that (4.7), (4.3) and the monotonicity of $t \to f(t)$ imply $\lim_{r \to +\infty} v(r) = +\infty$. □

We are now in position to prove Theorem 4.3.

PROOF OF THEOREM 4.3. *Necessity of condition* (4.5). Consider the solution v of the initial value problem (4.6) with $v(0) = s_0 > 0$ and let $(0, R_M)$ be the maximal interval of existence of v. Then R_M is finite (otherwise $v = v(|x|)$ would be a positive solution of (4.4), a contradiction). By proceeding as in the above Lemma 4.4. we find that $\lim_{r\to R_M} v(r) = +\infty$ and $v'(r) > 0$ for $r \in (0, R_M)$. The positivity of v' implies that $v'v'' < f(v)v'$ on $(0, R_M)$ and an integration from 0 to $r < R_M$ yields:

$$0 < \big[v'(r)\big]^2 < 2\int_0^r f\big(v(s)\big)v'(s)\,\mathrm{d}s = 2\int_{v(0)=s_0}^{v(r)} f(t)\,\mathrm{d}t.$$

Hence, for every $r \in (0, R_M)$, we have:

$$\left[\int_{s_0}^{v(r)} f(t)\,\mathrm{d}t\right]^{-1/2} v'(r) < \sqrt{2}$$

and another integration from 0 to R_M gives:

$$\int_{v(0)=s_0}^{v(R_M)=+\infty}\left[\int_{s_0}^{s} f(t)\,\mathrm{d}t\right]^{-1/2}\mathrm{d}s < \sqrt{2}R_M < +\infty,$$

which implies condition (4.5).

Sufficiency of condition (4.5). Suppose to the contrary that the differential inequality (4.4) has a non-negative, non-trivial solution of class C^2. Then, equation (4.1) admits a positive solution v satisfying all the properties listed in Lemma 4.4. Since the function $s \to f(v(s))$ is non-decreasing, it follows from (4.7) that $v'(r) \leqslant \frac{r}{N} f(v(r))$ for $r > 0$. Using the latter inequality in the ordinary differential equation satisfied by v we obtain:

$$\forall r > 0, \quad v''(r) = -\frac{N-1}{r}v'(r) + f\big(v(r)\big) \geqslant \frac{f(v(r))}{N}.$$

We can now proceed as in the previous step to find that $[\int_{v(0)}^{v(r)} f(t)\,\mathrm{d}t]^{-1/2}v'(r) \geqslant \sqrt{2/N}$ for $r > 0$. An integration over $(0, r)$ gives:

$$\int_{v(0)>0}^{v(r)}\left[\int_{v(0)}^{s} f(t)\,\mathrm{d}t\right]^{-1/2}\mathrm{d}s \geqslant \sqrt{\frac{2}{N}}\,r.$$

Letting $r \to +\infty$ in the latter inequality and recalling that $\lim_{r\to+\infty} v(r) = +\infty$ we obtain $\int_{v(0)>0}^{+\infty}[\int_{v(0)}^{s} f(t)\,\mathrm{d}t]^{-1/2}\,\mathrm{d}s = +\infty$, a contradiction. This concludes the proof of Theorem 4.3. □

What happens if the second assumption in (4.3) is replaced by the weaker one: $f(t) \geqslant 0$ for every $t \geqslant 0$? Clearly, if we allow $f(t) = 0$ for some positive values of t, we must take into account the corresponding positive constant solutions of inequality (4.4). Hence, the following natural question arises. Constant functions are the only non-negative solutions of differential inequality (4.4), if and only if, a Keller–Osserman condition is satisfied?

In $N \geqslant 3$ dimensions, the answer to the above question is negative. Indeed, in this case, the function $u(x) := 1 - (1 + |x|^2)^{-(N-2)/2}$, is a non-constant, non-negative and bounded solution of (4.4) with the non-decreasing function

$$f(t) := \begin{cases} 0, & t \in [0, 1], \\ (t-1)^2, & t \geqslant 1. \end{cases}$$

Clearly, the function f satisfies the Keller–Osserman condition (4.5) for any $s_0 > 1$.

In one or two spatial dimensions the answer to the above question is affirmative, providing thus a non-linear Liouville-type theorem for inequality (4.4). The reason of the discrepancy between the case $N \leqslant 2$ and the case $N \geqslant 3$ is the Liouville-type Theorem 3.1, namely, the parabolicity of $\mathbb{R}^2$. More precisely we have:

PROPOSITION 4.5. *Assume $N = 1$ or $N = 2$ and let $f : [0, +\infty) \to [0, +\infty)$ be a continuous, non-decreasing function such that*

$$\begin{cases} f(0) = 0, \\ f \not\equiv 0, \end{cases}$$

and let $\mathcal{M}$ be the largest zero of the function f.

Then, the only non-negative solutions of class C^2 to:

$$-\Delta u + f(u) \leqslant 0 \quad \text{in } \mathbb{R}^N$$

are constants, if and only if:

$$\forall s_0 > \mathcal{M} \quad \int_{s_0}^{+\infty} \left[\int_{s_0}^{s} f(t)\, \mathrm{d}t \right]^{-1/2} \mathrm{d}s < +\infty. \tag{4.8}$$

PROOF. Since any solution for $N = 1$ is also a solution for $N = 2$, it is enough to prove the claim in the two-dimensional case.

Necessity of condition (4.8). By the assumptions the set of zeros of f, denoted by Z_f, must be an interval of the form $[0, \mathcal{M}]$. Consider the solution v of the initial value problem (4.6) with $v(0) = s_0 > \mathcal{M}$ and let $(0, R_M)$ be the maximal interval of existence of v. By the definition of $\mathcal{M}$, the solution v cannot be constant. Hence, by proceeding as in the proof of Theorem 4.3 (cf. proof of *Necessity of condition* (4.5)) we obtain: $R_M < +\infty$, $\lim_{r \to R_M} v(r) = +\infty$ and $v'(r) > 0$ for $r \in (0, R_M)$. These informations immediately imply that f satisfies the integral condition (4.8).

Sufficiency of condition (4.8). Let us suppose, to the contrary, that the differential inequality (4.4) has a non-constant, non-negative solution u of class C^2. Then, u must be unbounded from above, by virtue of the parabolicity of $\mathbb{R}^2$ (cf. Theorem 3.1). Consequently, there is a point $x_0 \in \mathbb{R}^2$ such that $\mathcal{M} < u(x_0)$. By the translation invariance of (4.4) we may suppose $x_0 = 0$. At this point we consider the solution v of the initial value problem (4.6) with $v(0) = \alpha \in (\mathcal{M}, u(x_0))$ and observe that the comparison argument used in the proof of Lemma 4.4. yields: $R_M = +\infty$, $\lim_{r \to +\infty} v(r) = +\infty$ and $v'(r) > 0$ for $r \in (0, +\infty)$.

As in the proof of Theorem 4.3 (cf. proof of *Sufficiency of condition* (4.5)), the latter informations lead to a contradiction. This completes the proof of Proposition 4.5. □

A further step towards a complete classification of solutions to the monotone semilinear Poisson equation (4.1) can be made by refining a little bit more the assumptions on the function f. A typical example in this direction is provided by the following result proved by H. Brezis [11] in 1984.

THEOREM 4.6 (H. Brezis [11]). *Assume $N \geqslant 1$ and $p > 1$.*

(i) *Let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of*:

$$-\Delta u + |u|^{p-1}u \leqslant 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^N\big).$$

Then $u \leqslant 0$ a.e. on $\mathbb{R}^N$.

(ii) *Let $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of*:

$$-\Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^N\big).$$

Then $u = 0$ a.e. on $\mathbb{R}^N$.

The original proof of H. Brezis is based on the *explicit knowledge* of positive, radially symmetric super-solutions of the equation $-\Delta u + |u|^{p-1}u = 0$ on arbitrary open balls, with the further property that these functions blow up at the boundary of the considered balls. This fact is crucially related to the shape of the non-linear function $f(t) = |t|^{p-1}t$ and it does not easily extend to more general functions f. For this reason, we give here a different proof based on Theorem 4.3 and Jensen's inequality. This approach has the advantage to apply to a larger class of problems.

THEOREM 4.7. *Assume $N \geqslant 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\begin{cases} f(0) = 0, \\ f(t) > 0, \quad t > 0, \\ f \text{ is non-decreasing and convex on } [0, +\infty), \\ f \text{ satisfies condition (4.5)}. \end{cases}$$

(i) *Suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is such that $f(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfies*:

$$-\Delta u + f(u) \leqslant 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^N\big).$$

Then $u \leqslant 0$ a.e. on $\mathbb{R}^N$.

(ii) *Assume also that f is an odd function and suppose $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is such that $f(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfies*:

$$-\Delta u + f(u) = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^N\big).$$

Then $u = 0$ a.e. on $\mathbb{R}^N$.

Notice that the function $f(t) = |t|^{p-1}t$, $p > 1$, satisfies all the assumptions of Theorem 4.7.

PROOF OF THEOREM 4.7. As in the original proof of Theorem 4.6 we start by using a variant of Kato's inequality (see [45,11]) to obtain:

$$\Delta u^+ \geqslant f(u)\mathbf{1}_{\{u>0\}} = f(u^+) \quad \text{on } \mathcal{D}'\big(\mathbb{R}^N\big).$$

Let $\{\rho_n\}_{n\in\mathbf{N}^\star}$ be a sequence of standard *mollifiers* and set $u_n = u^+ * \rho_n$, where $*$ denotes the convolution operation. Therefore we have:

$$\begin{aligned}\forall x \in \mathbb{R}^N, \quad \Delta u_n(x) &= \big[(\Delta u^+) * \rho_n\big](x) \geqslant \big(f(u^+) * \rho_n\big)(x) \\ &= \int_{\mathbb{R}^N} f\big(u^+(y)\big)\rho_n(x-y)\,\mathrm{d}y \\ &\geqslant f\bigg(\int_{\mathbb{R}^N} u^+(y)\rho_n(x-y)\,\mathrm{d}y\bigg) = f(u_n)(x),\end{aligned}$$

where in the latter we have used the property: $\int_{\mathbb{R}^N} \rho_n(x)\,\mathrm{d}x = 1$, for every $n \in \mathbf{N}^\star$, which enable us to apply Jensen's inequality.

Hence, for every $n \in \mathbf{N}^\star$, u_n is a smooth, non-negative solution of the differential inequality (4.4), with f satisfying (4.5). Thus, it must vanish everywhere by Theorem 4.3. By letting $n \to +\infty$ we then obtain $u^+ = 0$ a.e. on $\mathbb{R}^N$. This proves the first part of the Theorem. For the second one we observe that $-u$ is also a solution of the equation $-\Delta u + f(u) = 0$, since f is odd. Therefore an application of part (i) yields the desired conclusion. □

We continue this section with an application of the above results to determine the conformal type of a class of Riemann surfaces. This was one of the main motivations leading R. Osserman to study differential inequality (4.4).

THEOREM 4.8 (R. Osserman [65]). *If a simply-connected surface S has Riemannian metric whose Gauss curvature K satisfies $K \leqslant -\epsilon < 0$ everywhere, then S is conformally equivalent to the interior of the unit circle* (*i.e. it is of hyperbolic type*).

PROOF. Considering S as a Riemann surface it is known that it is conformally equivalent either to the interior of the unit circle or to the entire plane $\mathbb{R}^2$. Let us prove that the second case cannot occur. Suppose, to the contrary, that the second case takes place, then there are global isothermal parameters x and y such that the Riemannian metric on S can be expressed in the form $\mathrm{d}s^2 = \lambda^2(\mathrm{d}x^2 + \mathrm{d}y^2)$. Therefore we would have $\Delta\log(\lambda) = -K\lambda^2$ on $\mathbb{R}^2$. The change of variable $u = \log(\lambda)$ and the assumption on K yields then:

$$\Delta u = -K\mathrm{e}^{2u} \geqslant \epsilon\mathrm{e}^{2u} \geqslant 2\epsilon|u|u \quad \text{on } \mathbb{R}^2.$$

The latter implies $u \leqslant 0$ everywhere on $\mathbb{R}^2$, by Theorem 4.7 (or Theorem 4.6). Hence, u must be constant by virtue of Theorem 3.1, an evident contradiction. □

The above result implies that it is impossible to find a conformal Riemannian metric on the entire Euclidean plane with Gaussian curvature K satisfying the condition: $K \leqslant -\epsilon < 0$ everywhere. In this respect we have the following theorem proved by D.H. Sattinger [74] in 1972.

THEOREM 4.9 (D.H. Sattinger [74]). *Let K be a smooth function defined on $\mathbb{R}^2$. If $K \leqslant 0$ everywhere on $\mathbb{R}^2$, and*

$$\exists r_0 > 1,\ \exists C > 0: \quad K(x) \leqslant -\frac{C}{|x|^2} \quad \forall x \in \mathbb{R}^2 \setminus B(0, r_0), \tag{4.9}$$

then the equation:

$$-\Delta u - K\mathrm{e}^{2u} = 0, \tag{4.10}$$

has no entire solutions in $\mathbb{R}^2$. Equivalently, under the above assumptions on K, it is impossible to find a conformal Riemannian metric on the entire Euclidean plane with Gaussian curvature equal to K.

PROOF. The proof is by contradiction. Suppose that equation (4.10) admits a solution u and consider its spherical mean v defined by:

$$v(r) := \frac{1}{2\pi r}\int_{\partial B(0,r)} u(s)\,\mathrm{d}s \quad \forall r > 0. \tag{4.11}$$

By the Gauss–Green formula we have:

$$-\int_{B(0,r)} K\mathrm{e}^{2u} = \int_{B(0,r)} \Delta u = \int_{\partial B(0,r)} \frac{\partial u}{\partial \nu} = 2\pi r v'(r) \quad \forall r > 0. \tag{4.12}$$

Hence $v'(r) > 0$ for all $r \geqslant r_0$, by the assumptions $K \leqslant 0$ everywhere and (4.9). Differentiating the latter, and using the assumption (4.9), we find that

$$\frac{(rv'(r))'}{r} \geqslant \frac{1}{2\pi r}\int_{\partial B(0,r)} (-K)\mathrm{e}^{2u} \geqslant \frac{C}{r^2}\left(\frac{1}{2\pi r}\int_{\partial B(0,r)} \mathrm{e}^{2u}\right) \quad \forall r \geqslant r_0.$$

An application of Jensen's inequality leads to the following ordinary differential inequality:

$$\frac{(rv'(r))'}{r} \geqslant \frac{C}{r^2}\exp\left\{2\left[\frac{1}{2\pi r}\int_{\partial B(0,r)} u(s)\,\mathrm{d}s\right]\right\} = \frac{C}{r^2}\mathrm{e}^{2v(r)} \quad \forall r \geqslant r_0. \tag{4.13}$$

Set $r = \ln t$ and $w(t) = v(\mathrm{e}^t)$. A direct computation gives:

$$\frac{\mathrm{d}w}{\mathrm{d}t}(t) > 0, \quad \frac{\mathrm{d}^2 w}{\mathrm{d}t^2}(t) \geqslant C\mathrm{e}^{2w(t)} \quad \forall t \geqslant t_0 := \ln r_0 > 0. \tag{4.14}$$

Now, the same argument used in the proof of Keller and Osserman's Theorem (cf. proof of *sufficiency of condition* (4.5)) yields:

$$\int_{w(t_0)}^{w(t)} \left[\int_{w(t_0)}^{s} \mathrm{e}^{2\theta}\, \mathrm{d}\theta \right]^{-1/2} \mathrm{d}s \geqslant \sqrt{2C}(t - t_0) \quad \forall t \geqslant t_0 := \ln r_0 > 0,$$

that is:

$$\int_{w(t_0)}^{w(t)} \left[\frac{\mathrm{e}^{2s} - \mathrm{e}^{2w(t_0)}}{2} \right]^{-1/2} \mathrm{d}s \geqslant \sqrt{2C}(t - t_0) \quad \forall t \geqslant t_0 := \ln r_0 > 0,$$

a clear contradiction. This concludes the proof. □

The above theorem is essentially sharp. Indeed, in 1982 W.M. Ni [63] proved that equation (4.10) possesses infinitely many solutions if K is a non-positive Hölder-continuous function, defined on $\mathbb{R}^2$ and satisfying $K(x) \geqslant -C/|x|^l$ at infinity , for some constant $l > 2$. This result was proved by the method of super-solutions and sub-solutions.

5. Bernstein and Liouville

In 1762 J.L. Lagrange considered the problem of determining the graph of a smooth function $u = u(x, y)$ over a two-dimensional bounded open domain Ω, having least area among all graphs that assume given values at the boundary of Ω.

J.L. Lagrange found that the function u is a solution of the quasilinear partial differential equation:

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } \Omega. \tag{5.1}$$

Because of the property of minimizing area, the latter equation has been called *minimal surface equation*. On the other hand, from the viewpoint of geometry, equation (5.1) says that the *mean curvature* of the graph of the function u (view as a non-parametric surface in $\mathbb{R}^3$) vanishes identically. Owing to this fact it has become customary to call *minimal surface* any surface whose *mean curvature* is zero.

In 1915 S.N. Bernstein proved the following beautiful theorem, known as *Bernstein's Theorem*:

THEOREM 5.1 (S.N. Bernstein [8]). *Let $u \in C^2(\mathbb{R}^2, \mathbb{R})$ be a solution of the minimal surface equation* (5.1) *on the entire Euclidean plane $\mathbb{R}^2$. Then u is an affine function, i.e., the graph of u is a plane in $\mathbb{R}^3$.*

Bernstein's Theorem can be seen as a Liouville-type theorem although no assumptions are made on the growth of the solution u. However, there is a deeper relation between Bernstein's result and Liouville-type theorems. Indeed, S.N. Bernstein deduced its celebrated

theorem from the following Liouville-type theorem for solutions of elliptic, not necessarily uniformly elliptic, equations in $\mathbb{R}^2$.

THEOREM 5.2 (S.N. Bernstein [8]). *Let $a, b, c : \mathbb{R}^2 \to \mathbb{R}$ be functions such that the symmetric matrix*:

$$\begin{pmatrix} a(x,y) & b(x,y) \\ b(x,y) & c(x,y) \end{pmatrix} \quad \text{is positive definite for every } (x,y) \in \mathbb{R}^2. \tag{5.2}$$

Let $u \in C^2(\mathbb{R}^2, \mathbb{R})$ be a solution of:

$$\begin{cases} a(x,y)\frac{\partial^2 u}{\partial x^2}(x,y) + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y}(x,y) + c(x,y)\frac{\partial^2 u}{\partial y^2}(x,y) = 0 \quad \text{in } \mathbb{R}^2, \\ u(x,y) = o(\sqrt{x^2+y^2}) \quad \text{as } \sqrt{x^2+y^2} \to +\infty. \end{cases} \tag{5.3}$$

Then u is a constant function.

Note that no continuity or boundedness assumptions are imposed to the functions a, b and c.

PROOF OF THEOREM 5.2. From the ellipticity condition (5.2) and the equation satisfied by u we have:

$$0 \leqslant a\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + 2b\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 u}{\partial x^2} + c\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = -c\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2\right]$$

and

$$0 \leqslant a\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 + 2b\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 u}{\partial y^2} + c\left(\frac{\partial^2 u}{\partial y^2}\right)^2 = -a\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2\right].$$

Hence

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \leqslant 0$$

everywhere in $\mathbb{R}^2$ and the equality holds only at points where

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = 0,$$

that is, only at points where

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} = 0$$

since the equation is elliptic. To conclude the proof Theorem of 5.2 we invoke the following result:

THEOREM 5.3 (S.N. Bernstein [8], E. Hopf [42]). *If u belongs to $C^2(\mathbb{R}^2, \mathbb{R})$ and satisfies*:

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \leqslant 0, \quad \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \not\equiv 0 \quad \text{in } \mathbb{R}^2,$$

then $u = u(x, y)$ cannot be $o(\sqrt{x^2 + y^2})$ as $\sqrt{x^2 + y^2} \to +\infty$.

By applying the above theorem we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 0$$

everywhere in $\mathbb{R}^2$, thus u is an affine function and the desired conclusion follows immediately, in view of the sub-linear growth of u. □

Now we are ready to prove Theorem 5.1.

PROOF OF THEOREM 5.1. Any solution of the minimal surface equation is smooth (actually real-analytic) see for instance [37,61,71]. Then, a direct calculation shows that the smooth functions,

$$\psi_1 = \arctan\left(\frac{\partial u}{\partial x}\right), \qquad \psi_2 = \arctan\left(\frac{\partial u}{\partial y}\right)$$

are bounded solutions of the equation:

$$\left(1 + \left(\frac{\partial u}{\partial y}\right)^2\right)\frac{\partial^2 v}{\partial x^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x \partial y} + \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 v}{\partial y^2} = 0 \quad \text{in } \mathbb{R}^2.$$

An application of Theorem 5.2, with

$$a(x, y) = \left(1 + \left(\frac{\partial u}{\partial y}\right)^2\right), \quad b(x, y) = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \quad \text{and}$$
$$c(x, y) = \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right),$$

then implies $\nabla u = const.$, thus u is an affine function. □

Before proceeding further we wish to make some remarks on the previous results.

REMARKS 5.4. (i) Bernstein's original proof of Theorem 5.3 contained a gap. This gap was discovered and fixed by E. Hopf in 1950 [42]. Almost at the same time, another proof of Theorem 5.3 was provided by E.J. Mickle [58]. We refer to [42] and [58] for a proof of Theorem 5.3.

(ii) Theorem 5.2 and Theorem 5.3 are sharp. Indeed, the linear elliptic equation in (5.3) always admits the linear solution $u(x, y) = y$. On the other hand, the function $u(x, y) = y\tanh(x)$ satisfies

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 < 0$$

everywhere in $\mathbb{R}^2$ and has linear growth at infinity.

(iii) Theorem 5.2 does not extend to $\mathbb{R}^N$, for $N \geqslant 3$. A counterexample of this fact was built by E. Hopf [41] in 1929. Hopf's construction is not only instructive in itself but also provides further important informations related to the topics under consideration. Here is Hopf's construction.

First we observe that the function $v(x, y) = \mathrm{e}^{x-\mathrm{e}^y}$ is a positive solution of the linear *elliptic* (non-uniformly elliptic) equation:

$$L(v) := a(x, y)\frac{\partial^2 v}{\partial x^2}(x, y) + 2b(x, y)\frac{\partial^2 v}{\partial x \partial y}(x, y) + c(x, y)\frac{\partial^2 v}{\partial y^2}(x, y) = 0 \quad \text{in } \mathbb{R}^2, \tag{5.4}$$

with $a(x, y) = 1 + \mathrm{e}^y$, $b(x, y) = 1$ and $c(x, y) = \mathrm{e}^{-y}$.

It follows then the function $w(x, y) = \mathrm{e}^{-v(x,y)}$ satisfies:

$$L(w) = Vw, \quad 0 < w < 1 \text{ in } \mathbb{R}^2, \tag{5.5}$$

where $V = V(x, y)$ is the smooth and positive potential,

$$V = a\left(\frac{\partial v}{\partial x}\right)^2 + 2b\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + c\left(\frac{\partial v}{\partial y}\right)^2 > 0 \quad \text{in } \mathbb{R}^2. \tag{5.6}$$

Next we consider the function $u(x, y, z) = 1 + w(x, y)\sin(z)$. A direct computation, using (5.4) and (5.5), shows that u is a positive and bounded solution of the linear *elliptic* equation:

$$a\frac{\partial^2 u}{\partial x^2}(x, y, z) + 2b\frac{\partial^2 u}{\partial x \partial y}(x, y, z) + c\frac{\partial^2 u}{\partial y^2}(x, y, z) + V\frac{\partial^2 u}{\partial z^2}(x, y, z) = 0 \quad \text{in } \mathbb{R}^3. \tag{5.7}$$

Thus, the function $1 + w(x_1, x_2)\sin(x_3)$ provides a counterexample to Theorem 5.2 in any dimension $N \geqslant 3$, since it also solves the linear *elliptic* equation

$$a\frac{\partial^2 u}{\partial x_1^2} + 2b\frac{\partial^2 u}{\partial x_1 \partial x_2} + c\frac{\partial^2 u}{\partial x_2^2} + V\frac{\partial^2 u}{\partial x_3^2} + \sum_{j=4}^{M}\frac{\partial^2 u}{\partial x_j^2} = 0 \quad \text{in } \mathbb{R}^M,\ M \geqslant 4.$$

(iv) Notice that the function u built by E. Hopf also satisfies the linear *elliptic* equation:

$$\frac{a}{d}\frac{\partial^2 u}{\partial x^2}(x, y, z) + 2\frac{b}{d}\frac{\partial^2 u}{\partial x \partial y}(x, y, z) + \frac{c}{d}\frac{\partial^2 u}{\partial y^2}(x, y, z) + \frac{V}{d}\frac{\partial^2 u}{\partial z^2}(x, y, z) = 0 \quad \text{in } \mathbb{R}^3, \tag{5.8}$$

where $d = a + 2b + c + V > 2$ everywhere in $\mathbb{R}^3$. This proves that Theorem 5.2 cannot be extended to $\mathbb{R}^N$, $N \geqslant 3$, even under the more restrictive assumptions that the coefficients of the elliptic equation are smooth and bounded functions and the solution is supposed to be smooth, positive and bounded.

(v) The examples given by equations (5.4), (5.7) and (5.8) also show that neither Theorem 3.1 nor Theorem 3.3 hold true for second order operators of the form $\sum_{i,j=1}^{N} a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j}$ if one only assumes *ellipticity*. The same remark also applies to Theorem 2.2 for $N \geqslant 3$.

Bernstein's Theorem stimulated a lot of research concerning the study of the higher dimensional minimal graph equation, i.e., the study of C^2 solutions $u = u(x_1, \ldots, x_N)$ of the equation:

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \quad \text{in } \mathbb{R}^N,\ N \geqslant 3. \tag{5.9}$$

One of the main open problems was whether Bernstein's Theorem generalizes to dimension $N \geqslant 3$, i.e., whether every solution of (5.9) is an affine function. E. De Giorgi proved Bernstein's Theorem for $N = 3$ in 1965 [20], F.J. Almgren proved it for $N = 4$ in 1966 [3] while, the case $N \leqslant 7$, was established by J. Simons in 1968 [81]. Their proofs are not based on Liouville-type theorems but on the connection between minimal graphs defined over $\mathbb{R}^N$ and minimal hypercones in $\mathbb{R}^N$ (see for instance [37,80]). In 1969, E. Bombieri, E. De Giorgi and E. Giusti [9], settled Bernstein's problem proving the existence of a non-affine solution of the minimal surface equation (5.9) for any $N \geqslant 8$.

Further results concerning the classification of entire solutions of the minimal surface equation (5.9) will be considered in Section 10.

For more contributions about non-affine solutions of (5.9) we refer to the survey of L. Simon [80] (and the references therein).

We conclude this section by considering the case of minimal graphs of higher codimension. The graph of a C^2 vector-valued function $u : \mathbb{R}^N \to \mathbb{R}^M$ is a minimal submanifold of $\mathbb{R}^{N+M}$, of dimension N and codimension M, if and only if u satisfies the *minimal*

surface system (see [51,67]):

$$-\sum_{i,j=1}^{N} g^{ij}\frac{\partial^2 u_k}{\partial x_i \partial x_j} = 0 \quad \text{for each } k = 1, \dots, M, \tag{5.10}$$

where, as in the classical differential geometry,

$$\left(g^{ij}\right) = (g_{ij})^{-1}, \quad g_{ij} = \delta_{ij} + \sum_{k=1}^{M} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}.$$

Note that, for $M = 1$ system (5.10) reduces to a single equation which is equivalent to the minimal surface equation (5.9) (or (5.1)). See for instance [51] and [67].

In view of the above results it is natural to look for Liouville-type theorems for the case of the *minimal surface system* (5.10). In the higher codimensional case the situation is more involved and new phenomena appear (see [51,67,68]). For instance, the analogue of Bernstein's Theorem is no longer true as soon as $M \geqslant 2$. Indeed, one can construct many "non-affine" solutions of the minimal surface system (5.10). To this end, we note that it is enough to produce such a counterexample only for the case $N = M = 2$.

EXAMPLE 5.5. Let $f : \mathbf{C} \to \mathbf{C}$ be any holomorphic function. Set $u_1 = \Re f$, $u_2 = \Im f$ and consider the vector-valued function $u : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $u := (u_1, u_2)$. This function solves the minimal surface system (5.10), for $N = M = 2$, as one can see by observing that

$$g_{11} = g_{22} > 0, \qquad g_{12} = g_{21} = 0,$$

thanks to Cauchy–Riemann equations. Thus system (5.10) reduces to:

$$\Delta u_1 = \Delta u_2 = 0 \quad \text{in } \mathbb{R}^2.$$

The desired conclusion then follows since u_1 and u_2 are the real part and the imaginary part of the holomorphic function f.

By taking $f(z) = z^k$, $k \in \mathbf{N}$, one obtains infinitely many solutions of (5.10) with polynomial growth (of arbitrary degree) at infinity.

Nevertheless, the following Liouville-type result holds true for the minimal surface system (5.10) of dimension $N = 2$ and codimension M greater than 1.

THEOREM 5.6. *Let $N = 2$, $M \geqslant 2$ and $u \in C^2(\mathbb{R}^2, \mathbb{R}^M)$ be a solution of the system* (5.10) *satisfying*:

$$\left|u(x, y)\right| = o\left(\sqrt{x^2 + y^2}\right) \quad \text{as } \sqrt{x^2 + y^2} \to +\infty.$$

Then u is a constant function.

PROOF OF THEOREM 5.6. Since the symmetric matrix (g^{ij}) is positive definite, the above result follows immediately by applying Theorem 5.2 to every components of the vector-valued solution u. □

The above Theorem 5.6 is sharp. Indeed, for any $M \geqslant 2$, the vector-valued function $u = u(x, y) := (x, y, 0, \ldots, 0)$ is a non-constant solution of (5.10) satisfying $|u(x, y)| = \sqrt{x^2 + y^2}$ everywhere in $\mathbb{R}^2$.

Theorem 5.6 is a very special case of a more general result of R. Osserman [66] concerning the complete description of all entire minimal graphs of dimension 2 and codimension $M \geqslant 2$. It is interesting to note that classical Liouville theorems for harmonic functions (cf. Theorems 2.2 and 3.1) enter in the proof of Osserman's results [66].

For further results about minimal surfaces we refer the interested reader to the beautiful survey of R. Osserman [68].

6. Jörgens and Liouville

The archetype of fully non-linear partial differential equations is the well-known Monge–Ampère equation:

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^2, \tag{6.1}$$

where $D^2 u$ denotes the Hessian matrix of u. Fully non-linear equations of Hessian type arise in many different geometrical problems and a lot of works has been devoted to their understanding [36]. In 1954, K. Jörgens [44], proved the following beautiful classification result for the Monge–Ampère equation (6.1).

THEOREM 6.1 (K. Jörgens [44]). *Let $u \in C^2(\mathbb{R}^2, \mathbb{R})$ be a solution of* (6.1). *Then u is a quadratic polynomial.*

PROOF. Instead of the original Jörgens' argument we present here the elegant proof due to J.C.C. Nitsche [64]. This proof is based on a simple application of the classical Liouville Theorem for entire holomorphic function (cf. Theorem 2.1).

From equation (6.1) we see that the eigenvalues of the Hessian matrix $D^2 u$ are either both positive or both negative. Hence (up to consider $-u$ instead of u, if necessary) we can assume that $D^2 u$ is positive definite everywhere in $\mathbb{R}^2$. Thus, u is a convex function.

Let us consider the mapping $(x, y) \to \xi := (\xi_1, \xi_2)$ defined by:

$$\xi_1(x, y) := x + \frac{\partial u}{\partial x}, \quad \xi_2(x, y) := y + \frac{\partial u}{\partial y}, \quad \forall (x, y) \in \mathbb{R}^2. \tag{6.2}$$

We claim that ξ is a C^1-diffeomorphism of $\mathbb{R}^2$ onto itself.

The convexity of u implies that the map ξ satisfies:

$$|x_1 - x_2|^2 \leqslant (x_1 - x_2) \cdot \big(\xi(x_1) - \xi(x_2)\big) \tag{6.3}$$

for every two points x_1 and x_2 of $\mathbb{R}^2$. Consequently, ξ is a one-to-one transformation.

To prove that ξ maps the entire Euclidean plane onto itself we proceed as follows. For any $\bar{\xi} := (\bar{\xi}_1, \bar{\xi}_2) \in \mathbb{R}^2$, we consider the function $v : \mathbb{R}^2 \to \mathbb{R}$ defined by:

$$v(x_1, x_2) = x_1^2 + x_2^2 + 2u(x_1, x_2) - 2x_1\bar{\xi}_1 - 2x_2\bar{\xi}_2, \quad \forall (x, y) \in \mathbb{R}^2.$$

Since v is a C^1-convex function satisfying $v(x_1, x_2) \to +\infty$ as $\sqrt{x_1^2 + x_2^2} \to +\infty$, there exists a point $\bar{x} := (\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$ such that $v(\bar{x}) = \min_{\mathbb{R}^2} v$. It follows then:

$$0 = \nabla v(\bar{x}) = 2\bar{x} + 2\nabla u(\bar{x}_1, \bar{x}_2) - 2\bar{\xi}$$

that is, $\bar{\xi} = \bar{x} + \nabla u(\bar{x}_1, \bar{x}_2) = \xi(\bar{x})$. Which gives the desired conclusion.

Furthermore, the Jacobian matrix of the map ξ is given by:

$$J := \begin{pmatrix} 1 + \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} & 1 + \frac{\partial^2 u}{\partial y^2} \end{pmatrix} \tag{6.4}$$

and $\det J = \Delta u + 2 > 0$, by (6.1) and the convexity of u.

The above properties immediately imply that ξ is a C^1-diffeomorphism of $\mathbb{R}^2$ onto itself.

We introduce the complex variable $\zeta := \xi_1 + \mathrm{i}\xi_2$ and consider the complex function $f(\zeta) := x - \partial u/\partial x - \mathrm{i}(y - \partial u/\partial y)$. A direct computation, using the inverse of the matrix J, gives:

$$\begin{aligned} \frac{\partial \Re f}{\partial \xi_1} &= \frac{\partial^2 u/\partial y^2 - \partial^2 u/\partial x^2}{\det J} = \frac{\partial \Im f}{\partial \xi_2}, \\ \frac{\partial \Re f}{\partial \xi_2} &= -\frac{2\partial^2 u/\partial x \partial y}{\det J} = -\frac{\partial \Im f}{\partial \xi_1}, \end{aligned} \tag{6.5}$$

hence, f satisfies the Cauchy–Riemann equations and consequently it is an entire holomorphic function of the complex variable ζ. Therefore we have:

$$f'(\zeta) = \frac{\partial^2 u/\partial y^2 - \partial^2 u/\partial x^2}{\det J} + \mathrm{i}\frac{2\partial^2 u/\partial x \partial y}{\det J}, \quad \forall \zeta \in \mathbf{C}, \tag{6.6}$$

and

$$\left| f'(\zeta) \right|^2 = \frac{(\Delta u)^2 - 4}{(\Delta u + 2)^2} < 1, \quad \forall \zeta \in \mathbf{C}. \tag{6.7}$$

By the classical Liouville Theorem 2.1, the holomorphic function f' must be constant. The latter, together with (6.6) and (6.7), implies that the Hessian matrix $D^2 u$ must be constant too and hence u is a polynomial of degree two. □

Both Jörgens' Theorem and Bernstein's Theorem in $\mathbb{R}^2$ are proved without any restriction on the growth of solutions. Hence, the following question naturally arises. Is the analogy between the two problems still true in the higher dimensional case? The answer is negative. Indeed, any smooth convex solution u of the Monge–Ampère equation $\det D^2u = 1$ in $\mathbb{R}^N$ is a quadratic polynomial for any $N \geqslant 2$. This was eventually proved by E. Calabi [12] for $N \leqslant 5$ in 1958 and later, in 1972, by A.V. Pogorelov [69] for any $N \geqslant 2$.

7. De Giorgi and Liouville: Part I

In 1978 E. De Giorgi raised the following question:

DE GIORGI'S CONJECTURE ([21]). *Let us consider a solution $u \in C^2(\mathbb{R}^N, \mathbb{R})$ of*

$$\Delta u = u^3 - u, \tag{7.1}$$

such that

$$|u| \leqslant 1, \qquad \frac{\partial u}{\partial x_N} > 0$$

in the whole $\mathbb{R}^N$. Is it true that, for every $\lambda \in \mathbb{R}$, the sets $\{u = \lambda\}$ are hyperplanes, at least if $N \leqslant 8$?

Equivalently, De Giorgi's conjecture can be reformulated by saying that the considered solution u depends only on one variable.

N. Ghoussoub and C. Gui [34] proved the above conjecture for $N = 2$ in 1998 while L. Ambrosio and X. Cabré [4] proved it for $N = 3$ in 2000. De Giorgi's conjecture is still an open question for all dimensions $N \geqslant 4$. Furthermore, to the best of our knowledge, there are no counterexamples to the above conjecture when $N \geqslant 8$. For a result in this respect see [43].

Under the additional assumption that

$$\forall x' \in \mathbb{R}^{N-1} \qquad \lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1, \tag{7.2}$$

the conjecture was proved by O. Savin [75] in 2003 for $N \leqslant 8$.

If one supposes that the limits in (7.2) are uniform with respect to $x' \in \mathbb{R}^{N-1}$ (but without the assumption about the monotonicity of u) then, for any $N \geqslant 2$, the solution is an increasing function which only depends on the variable x_N. This result was proved, independently and with different methods, by A. Farina [23,24] in 1999 (see also [26]), by H. Berestycki, F. Hamel and R. Monneau [7] in 2000 and by M. Barlow, R. Bass and C. Gui [5] in 2000.

The rest of this section is devoted to the study of De Giorgi's conjecture (and some extensions of it) to the more general context of quasilinear elliptic equations [25] and fully non-linear elliptic equations in dimension two. The fully non-linear case is new and,

although it concerns a special class of fully non-linear equations, it is, to the best of our knowledge, the only one about De Giorgi's conjecture in its full generality.

More precisely we study the extension of the above problem in three main directions:

(i) by only assuming $|\nabla u| > 0$ in $\mathbb{R}^2$ instead of the monotonicity hypothesis $\partial u/\partial x_2 > 0$ in $\mathbb{R}^2$,

(ii) by only assuming $|\nabla u| \in L^\infty(\mathbb{R}^2)$,

(iii) by replacing the Allen–Cahn equation (7.1) by quasilinear equations and fully non-linear equations of the form:

$$\operatorname{div}\big(a\big(|\nabla u|\big)\nabla u\big) + f(u) = 0 \quad \text{in } \mathbb{R}^2, \tag{7.3}$$

and

$$F\big[\operatorname{div}\big(a\big(|\nabla u|\big)\nabla u\big), u\big] = 0 \quad \text{in } \mathbb{R}^2. \tag{7.4}$$

Each one of the above considered directions (i)–(iii) contains a generalization of the assumptions appearing in the classical De Giorgi conjecture. This is clear for (i) and (iii). For (ii) we refer the reader to Remarks 7.5 at the end of this section.

The function a will be assumed to satisfy the (minimal) structural assumptions:

$$a \in C^{1,1}_{\mathrm{loc}}(0, +\infty), \quad \lim_{t\to 0} ta(t) = 0, \quad \big(ta(t)\big)' > 0, \quad \forall t > 0, \tag{S_1}$$

and

$$\exists T' > 0,\ \exists C = C(T') > 0: \quad \big(ta(t)\big)' \leqslant Ct^{-2}, \quad \forall t \in (0, T']. \tag{S_2}$$

These assumptions are satisfied, for instance, by every function of the form:

$$a(t) = t^\alpha\big(\eta + t^2\big)^\beta \tag{7.5}$$

with, $\eta > 0$, $\alpha > -1$ and $\beta \geqslant -(\alpha + 1)/2$. For a suitable choice of the parameters in (7.5), we recover the m-Laplacian operator, with $1 < m < +\infty$, the minimal surfaces operator as well as some more general operators satisfying non-standard growth conditions.

We first treat the quasilinear case. This enables us to explain the main ideas and tools necessary to obtain the desired result. Later on we shall consider the fully nonlinear case. In the sequel we shall denote by $\arg(\nabla u)$ the argument of the gradient of u. This is a well-defined real-valued function since $|\nabla u| > 0$ in $\mathbb{R}^2$.

The main results about the quasilinear case are:

THEOREM 7.1 ([25]). *Let $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ and suppose that the structural assumptions* (S_1), (S_2) *are satisfied. Let u be a $C^1(\mathbb{R}^2)$ solution of*

$$\begin{cases} \operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2), \\ |\nabla u| > 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{7.6}$$

with $\nabla u \in L^\infty(\mathbb{R}^2)$. *Assume that there exists* $\delta < 1$ *such that*

$$\big|\arg(\nabla u)(x)\big| = \mathrm{O}\big(\ln^\delta |x|\big), \quad \text{as } |x| \to +\infty, \tag{7.7}$$

then u *is one-dimensional and monotone*, *i.e.*, *there are* $\nu \in \mathbf{S^1}$ *and a function* $g \in C^2(\mathbb{R})$ *satisfying*

$$u(x) = g(\nu \cdot x), \quad \forall x \in \mathbb{R}^2,$$
$$\big|g'(t)\big| > 0, \quad \forall t \in \mathbb{R}.$$

When we suppose that u satisfies

$$\frac{\partial u}{\partial x_1} > 0 \quad \text{in } \mathbb{R}^2,$$

we have that the argument of the gradient of u can be written as:

$$\arg(\nabla u) = \arctan\left(\frac{u_2}{u_1}\right) \quad \text{in } \mathbb{R}^2,$$

where $u_j := \partial u/\partial x_j$, for $j = 1, 2$. Since in this case condition (7.7) is automatically satisfied, a direct application of the above Theorem 7.1 gives the following extension of De Giorgi's conjecture:

COROLLARY 7.2 ([25]). *Let* $f \in C^{0,1}_{\mathrm{loc}}(\mathbb{R})$ *and suppose that the structural assumptions* (S_1), (S_2) *are satisfied. Let* $u \in C^1(\mathbb{R}^2)$ *be a solution of*

$$\begin{cases} \operatorname{div}(a(|\nabla u|)\nabla u) + f(u) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2), \\ \frac{\partial u}{\partial x_1} > 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{7.8}$$

such that $\nabla u \in L^\infty(\mathbb{R}^2)$. *Then*, u *is one-dimensional*, *i.e.*, *there are* $\nu \in \mathbf{S^1}$ *and a function* $g \in C^2(\mathbb{R})$ *such that*

$$u(x) = g(\nu \cdot x), \quad \forall x \in \mathbb{R}^2.$$

The proofs of the above results is based on the following Liouville-type result proved by D. Gilbarg and J. Serrin [35] in 1955 (see also a prior Note [27] of R. Finn, where a similar result was announced).

THEOREM 7.3 (D. Gilbarg and J. Serrin [35], R. Finn [27]). *Assume* $0 < \delta < 1$ *and let* $B = (b_{hk})$ *be a symmetric real matrix*, *whose entries are bounded locally Lipschitz-continuous functions defined on* $\mathbb{R}^2$ *and satisfying*:

$$\forall x \in \mathbb{R}^2, \ \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \quad \sum_{h,k=1}^{2} b_{hk}(x)\xi_h\xi_k > 0. \tag{7.9}$$

Then every distribution solution $u \in C^1(\mathbb{R}^2, \mathbb{R})$ of

$$\begin{cases} \operatorname{div}(B(x)\nabla u) = 0 & \text{in } \mathbb{R}^2, \\ u(x) = \mathrm{O}(\ln^{\delta}(|x|)) & \text{as } |x| \to +\infty, \end{cases} \tag{7.10}$$

is a constant function.

PROOF. We split the proof into two steps.

Step 1. For every $r > 0$ and every $v \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$ we have:

$$\int_{B(0,r)} (B\nabla u \cdot \nabla v) = \int_{\partial B(0,r)} v(B\nabla u \cdot \nu), \tag{7.11}$$

where $B(0, r)$ denotes the open ball centered at the origin and of radius r.

Observe that $u \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$ by classical regularity results for linear elliptic partial differential equations and therefore the equation in (7.10) is also satisfied almost everywhere in $\mathbb{R}^2$. By a standard regularization procedure one can construct sequences of smooth functions $(u_n)_{n\in\mathbf{N}}$, $(v_n)_{n\in\mathbf{N}}$ and a sequence of smooth matrices $(B_n)_{n\in\mathbf{N}}$ such that

$$u_n \to u \quad \text{in } C^1_{\mathrm{loc}}\big(\mathbb{R}^2\big) \cap H^2_{\mathrm{loc}}\big(\mathbb{R}^2\big), \tag{7.12}$$

$$v_n \to v \quad \text{in } C^0_{\mathrm{loc}}\big(\mathbb{R}^2\big) \cap H^1_{\mathrm{loc}}\big(\mathbb{R}^2\big), \tag{7.13}$$

$$B_n \to B \quad \text{in } C^0_{\mathrm{loc}}\big(\mathbb{R}^2, \mathbb{R}^{2\times 2}\big) \cap H^1_{\mathrm{loc}}\big(\mathbb{R}^2, \mathbb{R}^{2\times 2}\big). \tag{7.14}$$

Hence, for every $n \in \mathbf{N}$, we have:

$$\begin{aligned} \int_{B(0,r)} v_n \operatorname{div}(B_n\nabla u_n) + \int_{B(0,r)} (B_n\nabla u_n \cdot \nabla v_n) &= \int_{B(0,r)} \operatorname{div}(v_n B_n \nabla u_n) \\ &= \int_{\partial B(0,r)} v_n (B_n \nabla u_n \cdot \nu). \end{aligned}$$

The identity (7.11) then follows by letting $n \to +\infty$, and observing that (7.12)–(7.14) imply $v_n \operatorname{div}(B_n\nabla u_n) \to v \operatorname{div}(B\nabla u) = 0$ in L^1_{loc}.

Step 2. End of the proof. Fix a real number γ such that $0 < \gamma < \max\{1, 1/\delta - 1\}$. For any $\epsilon > 0$ we consider the locally Lipschitz-continuous function $v_{\epsilon,\gamma} = (u^+ + \epsilon)^{\gamma}$. Inserting the latter in (7.11) and letting $\epsilon \to 0$ we get

$$\gamma \int_{B(0,r)} u_{\gamma-1}(B\nabla u^+ \cdot \nabla u^+) = \int_{\partial B(0,r)} u_{\gamma}(B\nabla u^+ \cdot \nu), \tag{7.15}$$

where, for $\theta \in \mathbb{R}$, we have set:

$$u_\theta(x) := \begin{cases} 0 & \text{if } u(x) \leqslant 0, \\ (u^+)^{\theta} & \text{if } u(x) > 0. \end{cases}$$

Then,

$$\gamma \int_{B(0,r)} u_{\gamma-1}(B\nabla u^+ \cdot \nabla u^+) \leqslant \int_{\partial B(0,r)} u_\gamma (B\nabla u^+ \cdot \nabla u^+)^{1/2} (B\nu \cdot \nu)^{1/2}$$
$$\leqslant C_1 \int_{\partial B(0,r)} u_{(\gamma+1)/2} u_{(\gamma-1)/2} (B\nabla u^+ \cdot \nabla u^+)^{1/2},$$

where C_1 is a positive constant independent of r. Therefore, there are $r_0 > 1$ and $0 < \beta < 1$ such that, for every $r > r_0$:

$$\left(\int_{B(0,r)} u_{\gamma-1}(B\nabla u^+ \cdot \nabla u^+) \right)^2$$
$$\leqslant C_2 r \ln^\beta(r) \int_{\partial B(0,r)} u_{\gamma-1}(B\nabla u^+ \cdot \nabla u^+) \tag{7.16}$$

for some positive constant C_2 independent of r.

For every $r > 0$ we set $I(r) := \int_{B(0,r)} u_{\gamma-1}(B\nabla u^+ \cdot \nabla u^+)$ and observe that (7.16) becomes the differential inequality:

$$I^2(r) \leqslant C_2 r \ln^\beta(r) I'(r) \quad \text{on } (r_0, +\infty). \tag{7.17}$$

Integrating the latter we find, for some $r_1 > r_0$:

$$\exists A > 0,\ B > 0\colon\ \forall r > r_1 \quad I(r) \leqslant \frac{1}{A + B \ln^{1-\beta}(r)}.$$

The latter implies $I \equiv 0$, which immediately gives $\nabla u^+ = 0$ on $\mathbb{R}^2$. Since also $-u$ is a solution of (7.10) we also obtain that $\nabla u^- = 0$ on $\mathbb{R}^2$. Hence u is constant. □

Now, we are in position to prove Theorem 7.1.

PROOF OF THEOREM 7.1. We split the proof into three steps.

Step 1. The solution u belongs to $C^2(\mathbb{R}^2, \mathbb{R})$. Since u has no critical points, one can use classical interior regularity results for quasilinear equations, due to P. Tolksdorf [83] and Ladyzhenskaya and Uraltseva [50]. These results immediately imply that any C^1 distribution-solution u of (7.6) is actually of class C^2, since f is locally Lipschitz-continuous and $a \in C^{1,1}_{\text{loc}}(0, +\infty)$ satisfies the structural assumption (S_1).

Step 2. For any solution u of (7.6) *we have*

$$\begin{cases} \operatorname{div}(\rho^2 A\nabla\theta) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2), \\ \operatorname{div}(A\nabla\rho) = \rho(-f'(u) + (A\nabla\theta)\nabla\theta) & \text{in } \mathcal{D}'(\mathbb{R}^2), \end{cases} \tag{7.18}$$

where $\theta := \arg(\nabla u)$, $\rho := |\nabla u|$ *and* $A = (a_{hk})$ *is the real symmetric matrix whose entries are* $C^{0,1}_{\mathrm{loc}}$ *functions given by*:

$$a_{hk} := \frac{a'(|\nabla u|)}{|\nabla u|} u_h u_k + a\big(|\nabla u|\big)\delta_{hk}. \tag{7.19}$$

By the assumptions on ∇u and step 1 we have that the vector field $\nabla u/|\nabla u|$ is well-defined and belongs to $C^1(\mathbb{R}^2, \mathbf{S^1})$. Hence, there exists a $C^1(\mathbb{R}^2, \mathbb{R})$ function θ such that

$$\nabla u(x) = \big|\nabla u(x)\big| e^{i\theta(x)} := \rho(x) e^{i\theta(x)} \quad \text{in } \mathbb{R}^2. \tag{7.20}$$

For $s = 1, 2$ recall that $u_s := \partial u/\partial x_s$. Differentiating the equation in (7.6) yields:

$$\operatorname{div}\Big(\big[a\big(|\nabla u|\big)\nabla u\big]_s\Big) + f'(u)u_s = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big). \tag{7.21}$$

The vector field $a(|\nabla u|)\nabla u$ belongs to C^1, then a direct calculation gives

$$\operatorname{div}\big(A(x)\nabla u_s\big) + f'(u)u_s = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big), \tag{7.22}$$

where A is the matrix whose entries are given by (7.19).

Define the complex function $z = u_1 + iu_2$ then, $z \in C^1$ and satisfies the following complex Schrödinger equation:

$$\operatorname{div}(A\nabla z) + f'(u)z = 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big). \tag{7.23}$$

Inserting (7.20) into (7.23) we find:

$$\begin{aligned} -f'(u)\rho e^{i\theta} = -f'(u)z &= \operatorname{div}(A\nabla z) = \operatorname{div}\big(e^{i\theta} A\nabla\rho\big) + i\operatorname{div}\big(\rho e^{i\theta} A\nabla\theta\big) \\ &= e^{i\theta}\operatorname{div}(A\nabla\rho) + ie^{i\theta}(A\nabla\rho)\nabla\theta + i\rho e^{i\theta}\operatorname{div}(A\nabla\theta) \\ &\quad + ie^{i\theta}(A\nabla\theta)\nabla\rho - e^{i\theta}\rho(A\nabla\theta)\nabla\theta \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big), \end{aligned} \tag{7.24}$$

hence

$$\begin{aligned} -f'(u)\rho &= \operatorname{div}(A\nabla\rho) - \rho(A\nabla\theta)\nabla\theta + 2i(A\nabla\rho)\nabla\theta + i\rho\operatorname{div}(A\nabla\theta) \\ &\quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big), \end{aligned}$$

where in the last identity we used the symmetry of A.

Separating the imaginary and the real parts we obtain:

$$\begin{cases} \rho\operatorname{div}(A\nabla\theta) + 2(A\nabla\rho)\nabla\theta = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2), \\ \operatorname{div}(A\nabla\rho) - \rho(A\nabla\theta)\nabla\theta + \rho f'(u) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^2). \end{cases} \tag{7.25}$$

In particular, the second equation in (7.18) is established. To prove the first one we notice that

$$\begin{aligned}\operatorname{div}\big(\rho^2 A\nabla\theta\big) &= \rho^2 \operatorname{div}(A\nabla\theta) + 2\rho(A\nabla\theta)\nabla\rho \\ &= \rho\big(\rho \operatorname{div}(A\nabla\theta) + 2(A\nabla\rho)\nabla\theta\big) \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big)\end{aligned}$$

thus, the claim follows from the first equation in (7.25).

Step 3. End of the proof. We observe that the structural assumption (S_2) yields the boundedness of the coefficients of the matrix $\rho^2 A$ (the matrix A being defined by (7.19)). By virtue of this fact, an application of Liouville-type Theorem 7.3 to the first equation in (7.18) yields that θ is a constant function. Thus $\nabla u(x) = |\nabla u(x)|e^{i\theta_0}$ in $\mathbb{R}^2$, for a real constant θ_0. Setting $\tau = (-\sin(\theta_0), \cos(\theta_0))$ we have $\nabla u \cdot \tau = 0$ everywhere. The latter implies the desired conclusion with $\nu = (\cos(\theta_0), \sin(\theta_0))$. □

Next we consider the fully non-linear equation (7.4). In this case we assume that the function $F = F(s,t)$ satisfies the following (ellipticity) assumptions:

$$F \in C^1\big(\mathbb{R}^2\big),\ \forall (s,t) \in \mathbb{R}^2 \quad \frac{\partial F}{\partial s}(s,t) > 0. \tag{7.26}$$

Under the above conditions we can state and prove the following:

THEOREM 7.4. *Let suppose that structural assumptions* (S_1), (S_2) *and* (7.26) *are satisfied and let* u *be a* $C^3(\mathbb{R}^2)$ *solution of*:

$$F\big[\operatorname{div}\big(a\big(|\nabla u|\big)\nabla u\big), u\big] = 0 \quad \text{in } \mathbb{R}^2,$$

such that $\nabla u \in L^\infty(\mathbb{R}^2)$.

(1) *If* $|\nabla u| > 0$ *in* $\mathbb{R}^2$ *and the argument of* ∇u *satisfies the growth condition* (7.7). *Then* u *is one-dimensional and monotone, i.e., there are* $\nu \in \mathbf{S^1}$ *and a function* $g \in C^3(\mathbb{R})$ *such that*

$$u(x) = g(\nu \cdot x), \quad \forall x \in \mathbb{R}^2, \tag{7.27}$$

$$\big|g'(t)\big| > 0, \quad \forall t \in \mathbb{R}. \tag{7.28}$$

(2) *If* $\partial u/\partial x_1 > 0$ *in* $\mathbb{R}^2$, *then* u *is one-dimensional. More precisely, it satisfies* (7.27) *and* (7.28) *for some* $\nu \in \mathbf{S^1}$ *and* $g \in C^3(\mathbb{R})$.

PROOF. Differentiating the equation in (7.4) yields, for $k = 1, 2$,

$$F_s \operatorname{div}\big(A(x)\nabla u_k\big) + F_t u_k = 0 \quad \text{in } \mathbb{R}^2, \tag{7.29}$$

where

$$F_s = \frac{\partial F}{\partial s}\big(\operatorname{div}\big(a\big(|\nabla u|\big)\nabla u\big), u\big) \quad \text{and} \quad F_t = \frac{\partial F}{\partial t}\big(\operatorname{div}\big(a\big(|\nabla u|\big)\nabla u\big), u\big).$$

By the assumption (7.26) we have $F_s > 0$ everywhere in $\mathbb{R}^2$, therefore the partial derivatives u_k, $k = 1, 2$, satisfy equation (7.22) with $f'(u)$ replaced by F_t/F_s. Now we can go on as in the proof of Theorem 7.1 and conclude by invoking the Liouville-type Theorem 7.3. □

We wish to conclude this section with several remarks concerning the above results.

REMARKS 7.5.

(i) In Theorems 7.1, 7.4 and theirs corollaries we assumed $\nabla u \in L^\infty$ instead of the more restrictive hypothesis $u \in L^\infty$ appearing in the original formulation of De Giorgi's conjecture. Indeed, by standard elliptic estimates we know that any *bounded* C^1 distribution solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$ has bounded gradient. Moreover, this property also holds true for a large class of quasilinear equations of the form (7.3). Indeed, by making use of regularity results of Tolksdorf [83], it is easy to see that the property: $u \in L^\infty \Rightarrow \nabla u \in L^\infty$ is still true for the m-Laplacian operator or, more generally, for any operator satisfying some standard growth conditions. Furthermore, some natural unbounded solutions are excluded if one assumes $u \in L^\infty$. This is easily seen by noting that $u(x) = x_1$ is a one-dimensional unbounded solution of equation (7.3), with bounded and non-vanishing gradient. Clearly, this kind of solutions are taken into account by the approach that we have considered in this section.

(ii) The assumption $\nabla u \in L^\infty$ is necessary in the above results. This is shown by the following example taken from [25]. The function $u(x, y) = x - y^2/2$ satisfies:

$$\begin{cases} \Delta u + 1 = 0 & \text{in } \mathbb{R}^2, \\ \frac{\partial u}{\partial x} = 1 > 0 & \text{in } \mathbb{R}^2, \\ |\nabla u| = \sqrt{1 + y^2} \notin L^\infty(\mathbb{R}^2), \end{cases} \tag{7.30}$$

and it is not one-dimensional.

(iii) For $N = 2$, D. Danielli and N. Garofalo [18] proved the De Giorgi conjecture (but neither Theorem 7.1 nor Theorem 7.4) for monotone bounded critical points of the functional $\int_{\mathbb{R}^2} \Phi(u, \nabla u)\, dx$, where Φ satisfies standard structural assumptions and growth conditions. Their proof is based on a Liouville-type result described in the next section (cf. Proposition 8.2).

Now we turn to some remarks concerning Theorem 7.3.

REMARKS 7.6.

(i) Theorem 7.3 does not extend to the higher dimensional case. To construct the counterexamples we first recall that

$$v(x) := \left(\frac{\sqrt{N(N-2)}}{1 + |x|^2} \right)^{(N-2)/2},$$

is a positive solution of the Lane–Emden equation $-\Delta v = v^{(N+2)/(N-2)}$ on $\mathbb{R}^N$, for $N \geqslant 3$. Therefore the function $w(x) = \mathrm{e}^{-v(x)}$ satisfies:

$$\Delta w = V w, \quad 0 < w < 1 \text{ in } \mathbb{R}^N, \tag{7.31}$$

where $V \in C^\infty(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is the positive potential,

$$V(x) = \left[|\nabla v|^2 + v^{(N+2)/(N-2)}\right] > 0 \quad \text{in } \mathbb{R}^N. \tag{7.32}$$

Next we consider the function $u : \mathbb{R}^{N+1} \to \mathbb{R}$ defined by

$$u(x, x_{N+1}) = 1 + w(x)\cos(x_{N+1}).$$

A direct computation, using (7.31) and (7.32), shows that u is a positive, bounded and smooth solution of the linear *elliptic* equation in divergence form:

$$\operatorname{div}\big(B(x)\nabla u\big) = 0 \quad \text{in } \mathbb{R}^{N+1},\ N \geqslant 3, \tag{7.33}$$

where $B := \operatorname{Diag}(1, \ldots, 1, V)$ is a positive-definite diagonal matrix with bounded and smooth entries.

Thus the function u provides a counterexample to Theorem 7.3 in any dimension *greater than or equal to 4* (note that $N \geqslant 3$ in the above construction).

(ii) Observe that one cannot use the above construction to build a counterexample to Theorem 7.3 in the three-dimensional case. Indeed, such a construction would require the existence of a non-constant bounded function v, which also solves a *two-dimensional* elliptic problem of the form $-\operatorname{div}(A\nabla v) = f \geqslant 0$, for a symmetric, positive-definite matrix A with bounded coefficients. But the existence of such a kind of function is excluded by the following generalization of Theorem 3.1.

PROPOSITION 7.7. *Let $b \geqslant 0$ and $A = (a_{hk})$ be a symmetric real matrix, whose entries are bounded measurable functions defined on $\mathbb{R}^2$ and satisfying*:

$$\forall x \in \mathbb{R}^2,\ \forall \xi \in \mathbb{R}^2 \setminus \{0\}, \quad \sum_{h,k=1}^{2} a_{hk}(x)\xi_h\xi_k > 0. \tag{7.34}$$

Let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$ be a distribution solution of:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + b(A(x)\nabla u \cdot \nabla u) \geqslant 0 & \text{in } \mathbb{R}^2, \\ u(x) \geqslant -C & \text{a.e. in } \mathbb{R}^2, \end{cases} \tag{7.35}$$

for some positive constant C. Then u is a constant function.

PROOF. First we prove the proposition for $b = 0$. Clearly, it is sufficient to consider the case in which u is bounded from below by a positive constant. The proof is a variation on

the theme of the classical Caccioppoli inequality. For $R > 1$ and $x \in \mathbb{R}^2$ we set:

$$\phi_R(x) = \begin{cases} 1 & \text{if } |x| \leqslant R, \\ 1 - \frac{1}{\ln(R)} \ln(\frac{|x|}{R}) & \text{if } R \leqslant |x| \leqslant R^2, \\ 0 & \text{if } |x| \geqslant R^2. \end{cases} \tag{7.36}$$

Using $\phi_R^2 u^{-1}$ as a test function in (7.35) we find:

$$\int (A\nabla u \cdot \nabla u)\left(\frac{\phi_R}{u}\right)^2 \leqslant 2\int \frac{\phi_R}{u}(A\nabla u \cdot \nabla \phi_R)$$
$$\leqslant 2\left[\int (A\nabla u \cdot \nabla u)\left(\frac{\phi_R}{u}\right)^2\right]^{1/2}\left[\int (A\nabla \phi_R \cdot \nabla \phi_R)\right]^{1/2}.$$

Therefore:

$$\int (A\nabla u \cdot \nabla u)\left(\frac{\phi_R}{u}\right)^2 \leqslant 4\int (A\nabla \phi_R \cdot \nabla \phi_R) \leqslant C\int |\nabla \phi_R|^2 \leqslant \frac{C_1}{\ln(R)}$$

for some positive constant C_1 independent of R. The desired conclusion follows by letting $R \to +\infty$.

Next we consider the case $b > 0$. In this case the function $v = -\mathrm{e}^{-bu}$ satisfies:

$$v \in H^1_{\mathrm{loc}}\big(\mathbb{R}^2\big) \cap L^\infty\big(\mathbb{R}^2\big), \quad -\operatorname{div}\big(A(x)\nabla v\big) \geqslant 0 \quad \text{in } \mathcal{D}'\big(\mathbb{R}^2\big),$$

and the claim follows by applying the result for $b = 0$. □

(iii) In view of the above discussions it is natural to ask whether or not Theorem 7.3 holds in dimension $N = 3$. We are not aware of such a kind of result.

(iv) The above Proposition 7.7 provides a simple criterion to prove parabolicity for Riemannian manifolds of the form $(\mathbb{R}^2, g)$. For instance, a Riemannian manifold $(\mathbb{R}^2, g)$ is *parabolic* if the given Riemannian metric $g = (g_{hk}(x))$ satisfies the following property:

$$\exists C > 0: \quad g_{hk}^2 \leqslant C \det(g) \quad \text{in } \mathbb{R}^2 \ \forall h, k = 1, 2. \tag{7.37}$$

Indeed, setting $|g| = \det(g)$, the Laplace–Beltrami operator on $\mathbb{R}^2$ in the g-metric is given by:

$$\Delta_g = \frac{1}{\sqrt{|g|}} \operatorname{div}\big(\sqrt{|g|}G^{-1}\nabla \cdot\big)$$

and it is easy to check that (7.37) implies the boundedness of the coefficients of the positive-definite matrix $\sqrt{|g|}G^{-1}$. Hence the claim follows using Proposition 7.7 with $A = \sqrt{|g|}G^{-1}$.

We observe that condition (7.37) allows Riemannian metrics g which are not necessarily bounded and/or uniformly elliptic.

(v) For $N = 2$, Proposition 7.7 also shows a substantial difference between linear *elliptic* operators in divergence form and linear *elliptic* operators in non-divergence form. Indeed, the above result shows that Theorem 3.1 extends to linear *elliptic* operators in divergence form while, this result is not true for linear *elliptic* operators in non-divergence form, by virtue of the counterexample of E. Hopf (cf. Remarks 5.4).

(vi) For $N \geqslant 4$, the example given by equation (7.33) also shows that neither Theorem 2.2 nor Theorem 3.3 hold true for second order operators in divergence form, if one only assumes *ellipticity*.

8. De Giorgi and Liouville: Part II

In 2000 L. Ambrosio and X. Cabré [4] proved the De Giorgi conjecture in the three-dimensional case. This section is devoted to the proof of this result.

THEOREM 8.1 (L. Ambrosio and X. Cabré [4]). *Let us consider a solution $u \in C^2(\mathbb{R}^3, \mathbb{R})$ of*

$$\Delta u = u^3 - u \quad \text{in } \mathbb{R}^3 \tag{8.1}$$

such that

$$|u| \leqslant 1, \quad \frac{\partial u}{\partial z} > 0 \quad \text{in } \mathbb{R}^3.$$

Then there exist $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $g \in C^2(\mathbb{R})$ such that

$$u(x, y, z) = g(a_1 x + a_2 y + a_3 z), \quad \forall (x, y, z) \in \mathbb{R}^3.$$

It is interesting to observe that two crucial steps in the proof of Theorem 8.1 are based on the following Liouville-type theorem. This result is essentially due to H. Berestycki, L. Caffarelli and L. Nirenberg [6], who devised it in order to study symmetry properties of solutions to semilinear elliptic equations in unbounded Euclidean domains. The formulation of this result, that we are going to give below, is a slight generalization of the original one. This extension is due to L. Ambrosio and X. Cabré [4].

PROPOSITION 8.2. *Assume $N \geqslant 1$. Let $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ be a positive function. Suppose that $\sigma \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ satisfies*:

$$\sigma \operatorname{div}\left(\varphi^2 \nabla \sigma\right) \geqslant 0 \quad \text{in } \mathbb{R}^N \tag{8.2}$$

in the distributional sense. Let suppose that

$$\int_{\{|x| \leqslant R\}} (\varphi \sigma)^2 \leqslant C R^2, \quad \forall R > 1, \tag{8.3}$$

for some positive constant C independent of R.

Then σ is a constant function.

PROOF. Let $\phi \in C_c^\infty(\mathbb{R}^N)$ be a function such that $0 \leqslant \phi \leqslant 1$, and

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2. \end{cases} \tag{8.4}$$

For $R > 1$ and $x \in \mathbb{R}^N$, let $\phi_R(x) = \phi(x/R)$. Using ϕ_R^2 as a test function in (8.2), we find:

$$\begin{aligned} \int \phi_R^2 \varphi^2 |\nabla\sigma|^2 &\leqslant -2 \int \phi_R \varphi^2 \sigma (\nabla\phi_R \cdot \nabla\sigma) \\ &\leqslant C_1 \left[\int_{\{R \leqslant |x| \leqslant 2R\}} \phi_R^2 \varphi^2 |\nabla\sigma|^2 \right]^{1/2} \left[\frac{1}{R^2} \int_{\{|x| \leqslant R\}} (\varphi\sigma)^2 \right]^{1/2}, \end{aligned}$$

for some positive constant C_1 independent of R. Now, the assumption (8.3) yields:

$$\int \phi_R^2 \varphi^2 |\nabla\sigma|^2 \leqslant C_1 C^{1/2} \left[\int_{\{R \leqslant |x| \leqslant 2R\}} \phi_R^2 \varphi^2 |\nabla\sigma|^2 \right]^{1/2} \quad \forall R > 1, \tag{8.5}$$

which implies $\varphi^2 |\nabla\sigma|^2 \in L^1(\mathbb{R}^N)$. Using the latter information in (8.5) and letting $R \to +\infty$, we obtain $\sigma = const$. □

We turn now to the:

PROOF OF THEOREM 8.1. We split the proof into four steps.

Step 1. The function

$$\bar{u}(x, y) = \lim_{z \to +\infty} u(x, y, z) \tag{8.6}$$

is a bounded and stable solution of:

$$\Delta u = u^3 - u \quad \text{in } \mathbb{R}^2. \tag{8.7}$$

That is, $\bar{u}$ is a bounded solution of (8.7) *such that*

$$\int_{\mathbb{R}^2} |\nabla\phi|^2 + \left(3\bar{u}^2 - 1\right)\phi^2 \geqslant 0, \quad \forall \phi \in C_c^\infty\left(\mathbb{R}^2\right). \tag{8.8}$$

For every $t \in \mathbb{R}$ consider the function:

$$u^t(x, y, z) := u(x, y, z + t), \quad \forall (x, y, z) \in \mathbb{R}^3. \tag{8.9}$$

Clearly, for every $t \in \mathbb{R}$, the function u^t is a solution of (8.1) and, standard elliptic estimates [36] yield, $\|u^t\|_{C^2(\mathbb{R}^3)} \leqslant C$, for some constant C independent of t. Combining the latter together with the monotonicity assumption $\partial u/\partial z > 0$, we see that $u^t \to \bar{u}$

in $C^1_{\mathrm{loc}}(\mathbb{R}^3)$ (viewed as a function of the variables (x, y, z)). Therefore $\bar{u}$ is a bounded solution of (8.7) as claimed.

It remains to prove (8.8). To this end we first observe that u itself is a stable solution of the Allen–Cahn equation (8.1) in $\mathbb{R}^3$. Indeed, u satisfies:

$$\int_{\mathbb{R}^3} |\nabla \xi|^2 + \big(3u^2 - 1\big)\xi^2 \geqslant 0, \quad \forall \xi \in C_c^\infty\big(\mathbb{R}^3\big), \tag{8.10}$$

as one can easily establish by multiplying the derivative in z of equation (8.1) by the function $(\xi^2/(\partial u/\partial z))$, integrating by parts and using the Cauchy–Schwarz inequality.

Next, for every function $\phi \in C_c^\infty(\mathbb{R}^2)$ and every $r > 0$, consider the smooth function $\xi = \xi(x, y, z) = \phi(x, y)\psi_r(z)$, where $\psi_r \in C^\infty(\mathbb{R})$ satisfies $0 \leqslant \psi_r \leqslant 1$, $0 \leqslant \psi_r' \leqslant 2$, $\psi_r = 0$ in $\mathbb{R} \setminus [r, 2r + 2]$ and $\psi_r = 1$ in $[r + 1, 2r + 1]$. Inserting the function ξ in (8.10) and passing to the limit as $r \to +\infty$, we obtain the desired conclusion (8.8). Here we have used that fact that $u^t \to \bar{u}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^3)$.

Step 2. The are $\tau = (\tau_1, \tau_2) \in \mathbf{S^1}$ and $h \in C^2(\mathbb{R})$ such that

$$\bar{u}(x, y) = h(\tau_1 x + \tau_2 y), \quad \forall (x, y) \in \mathbb{R}^2. \tag{8.11}$$

Furthermore, either $h \equiv 1$ or there exists $\alpha \in \mathbb{R}$ such that $h(t) = \tanh(t/\sqrt{2} + \alpha)$ for every $t \in \mathbb{R}$.

The stability of $\bar{u}$ is equivalent to the existence of a positive solution $\varphi \in C^2$ of the linearized equation $\Delta\varphi = (3\bar{u}^2 - 1)\varphi$. This is a well-known result in the theory of linear elliptic partial differential equations (see for instance [2,60,32] and the references therein).

Now, for every $\nu \in \mathbf{S^1}$, we observe that the function $\bar{u}_\nu := \partial\bar{u}/\partial\nu$ also solves the linearized equation $\Delta v = (3\bar{u}^2 - 1)v$ in $\mathbb{R}^2$. Hence, the ratio $\sigma_\nu := ((\partial\bar{u}/\partial\nu)/\varphi)$ satisfies the partial differential equation $\operatorname{div}(\varphi^2\nabla\sigma_\nu) = 0$ in $\mathbb{R}^2$ and therefore, also the differential inequality (8.2).

Since $N = 2$ and $\varphi\sigma_\nu = \partial\bar{u}/\partial\nu \in L^\infty(\mathbb{R}^2)$ we immediately see that the integral condition (8.3) is fulfilled. Therefore, an application of Proposition 8.2 gives:

$$\forall \nu \in \mathbf{S^1}\ \exists c_\nu \in \mathbb{R}: \quad \frac{\partial\bar{u}}{\partial\nu} = c_\nu\varphi \quad \text{on } \mathbb{R}^2. \tag{8.12}$$

In particular, every partial derivative of $\bar{u}$ has a fixed sign on $\mathbb{R}^2$.

On the other hand, given the two-dimensional vector $\nabla\bar{u}(0)$, there exists $\nu_1 \in \mathbf{S^1}$ such that

$$0 = \nabla\bar{u}(0) \cdot \nu_1 = \frac{\partial\bar{u}}{\partial\nu_1}(0).$$

Therefore, applying the strong maximum principle to the linearized equation $\Delta\bar{u}_{\nu_1} = (3\bar{u}^2 - 1)\bar{u}_{\nu_1}$ in $\mathbb{R}^2$ we find:

$$\frac{\partial\bar{u}}{\partial\nu_1} \equiv 0 \quad \text{on } \mathbb{R}^2.$$

The latter implies the existence of $h \in C^2(\mathbb{R})$ and $\tau = (\tau_1, \tau_2) \in \mathbf{S^1}$ satisfying the identity (8.11). Moreover, using (8.11) and (8.12) we find that h must be a monotone solution of the ode $h'' = h^3 - h$ on $\mathbb{R}$.

Using once again the strong maximum principle we see that, either h is constant or h is a strictly monotone solution of $h'' = h^3 - h$. If h is constant, then either $h \equiv 0$ or $h \equiv \pm 1$. The case $h \equiv 0$ is immediately excluded by the stability property (8.8) while the case $h \equiv -1$ is ruled out by the monotonicity assumption $\partial u/\partial z > 0$. Hence $h \equiv 1$.

When h is not constant, it is easily seen that h can always be chosen strictly increasing in (8.11). Indeed, either h is strictly increasing, and we are done, or h is strictly decreasing and the representation (8.11) holds with τ replaced by $-\tau$ and $h = h(t)$ replaced by the function $t \to h(-t)$. To conclude the proof it is enough to observe that $h(t) = \tanh(t/\sqrt{2})$ is a strictly increasing solution of the ode $h'' = h^3 - h$ on $\mathbb{R}$.

Step 3. The Dirichlet energy of u satisfies the growth condition:

$$\int_{B(0,R)} |\nabla u|^2 \leqslant CR^2, \quad \forall R > 1, \tag{8.13}$$

for some positive constant C independent of R. Here $B(0, R)$ denotes the open ball centered at the origin and of radius R.

For every $R > 1$, we consider:

$$E_R\big(u^t\big) = \int_{B(0,R)} \left[\frac{1}{2}\big|\nabla u^t\big|^2 + \frac{((u^t)^2 - 1)^2}{4}\right] \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where u^t is the function defined in Step 1. Since $u^t \to \bar{u}$ in $C^1_{\mathrm{loc}}(\mathbb{R}^3)$ we get:

$$\forall R > 1 \quad \lim_{t \to +\infty} E_R\big(u^t\big) = E_R(\bar{u}). \tag{8.14}$$

On the other hand, we have:

$$\partial_t E_R\big(u^t\big) = \int_{B(0,R)} \nabla u^t \cdot \nabla\big(\partial_t\big(u^t\big)\big) + \int_{B(0,R)} \big(\big(u^t\big)^3 - u^t\big)\partial_t\big(u^t\big),$$

where ∂_t denotes the partial derivative with respect to t. We observe that $\partial_t(u^t) > 0$ everywhere by virtue of the monotonicity assumption $\partial u/\partial z > 0$. This fact, together with the uniform bound $\|u^t\|_{C^2(\mathbb{R}^3)} \leqslant C$, with C independent of t, leads to:

$$\partial_t E_R\big(u^t\big) = \int_{\partial B(0,R)} \frac{\partial u^t}{\partial \nu}\partial_t\big(u^t\big) \geqslant -C \int_{\partial B(0,R)} \partial_t\big(u^t\big).$$

Hence, for every $T > 0$ and every $R > 1$, we have:

$$\begin{aligned} E_R(u) &= E_R\big(u^T\big) - \int_0^T \partial_t E_R\big(u^t\big)\,\mathrm{d}t \\ &\leqslant E_R\big(u^T\big) + C\int_0^T \mathrm{d}t \int_{\partial B(0,R)} \partial_t u^t(s)\,\mathrm{d}\sigma(s) \end{aligned}$$

$$= E_R\big(u^T\big) + C\int_{\partial B(0,R)} \mathrm{d}\sigma(s)\int_0^T \partial_t\big[u^t(s)\big]\,\mathrm{d}t$$

$$= E_R\big(u^T\big) + C\int_{\partial B(0,R)} \big(u^T - u\big)(s)\,\mathrm{d}\sigma(s) \leqslant E_R\big(u^T\big) + C_1R^2, \qquad (8.15)$$

for some constant C_1 independent of T and R.

Letting $T \to +\infty$ and using (8.14), we have

$$E_R(u) \leqslant E_R(\bar{u}) + C_1R^2$$

$$\leqslant C_2R^2\int_{-R}^{R}\left[\frac{1}{2}\big(h'(t)\big)^2 + \frac{(h^2(t)-1)^2}{4}\right]\mathrm{d}t + C_1R^2, \quad \forall R > 1,$$

where C_2 is a constant independent of R. The desired conclusion follows observing that

$$\int_{\mathbb{R}}\left[\frac{1}{2}\big(h'(t)\big)^2 + \frac{(h^2(t)-1)^2}{4}\right]\mathrm{d}t$$

is finite, in view of the special form of h.

Step 4. End of the proof. Here we use again Proposition 8.2. For every $\eta \in \mathbf{S^2}$, we define the function $u_\eta := \partial u/\partial\eta$ and, as in Step 2, we see that the ratio $\sigma_\eta := ((\partial u/\partial\eta)/(\partial u/\partial z))$ satisfies the differential inequality (8.2) with $\sigma = \sigma_\eta$. Next, we notice that $\varphi\sigma_\eta = \partial u/\partial\eta$, so that the integral condition (8.3) is satisfied, by virtue of step 3. Hence, by applying Proposition 8.2 we obtain:

$$\forall\eta \in \mathbf{S^2}\ \exists c_\eta \in \mathbb{R}: \quad \frac{\partial u}{\partial\eta} = c_\eta\frac{\partial u}{\partial z} \quad \text{on } \mathbb{R}^3. \qquad (8.16)$$

From the latter it is immediate to see that there are η_1 and $\eta_2 \in \mathbf{S^2}$ such that

$$\eta_1 \perp \eta_2, \quad \frac{\partial u}{\partial\eta_1} = \frac{\partial u}{\partial\eta_2} \equiv 0 \quad \text{on } \mathbb{R}^3.$$

The latter concludes the proof of Theorem 8.1. □

An inspection of the proof of the above Theorem 8.1 shows that, for $N \geqslant 4$, De Giorgi's conjecture would be established if one could prove Proposition 8.2 with the exponent 2 in (8.3) replaced by an exponent greater than or equal to $N-1$. Thus, it is natural to look for the largest exponent $\delta = \delta(N) \geqslant 2$ such that the condition:

$$\int_{\{|x|\leqslant R\}} (\varphi\sigma)^2 \leqslant CR^\delta, \quad \forall R > 1, \qquad (8.17)$$

implies $\sigma = const.$ on $\mathbb{R}^N$.

F. Gazzola [33] answered the above question by exhibiting examples which prove that, for any $N \geqslant 2$, the largest exponent δ is 2. Here is the example.

EXAMPLE 8.3 ([33]). For any $N \geqslant 2$ and any $\delta > 2$, the functions,

$$\varphi(x) = \left(1 + |x|^2\right)^{-(N+\delta-4)/4}, \quad \sigma(x) = |x|^{\delta-2}, \quad \forall x \in \mathbb{R}^N$$

satisfy:

$$\varphi \in L^\infty\left(\mathbb{R}^N\right) \cap C^\infty\left(\mathbb{R}^N\right), \quad \varphi > 0 \quad \text{on } \mathbb{R}^N,$$
$$\sigma \in H^1_{\text{loc}}\left(\mathbb{R}^N\right) \cap C^0\left(\mathbb{R}^N\right), \quad \nabla\sigma \not\equiv 0,$$
$$\sigma \operatorname{div}\left(\varphi^2 \nabla\sigma\right) \geqslant 0 \quad \text{in } \mathcal{D}'\left(\mathbb{R}^N\right)$$

and

$$\int_{\{|x| \leqslant R\}} (\varphi\sigma)^2 \leqslant C R^\delta, \quad \forall R > 1,$$

for some positive constant C independent of R.

We conclude this section by the following:

REMARK 8.4. For $N = 3$, De Giorgi's conjecture was extended to the semilinear equation $\Delta u = F'(u)$, $F \in C^2$, by G. Alberti, L. Ambrosio and X. Cabré [1] in 2001. Their proof is still based on Proposition 8.2 combined with some results from the calculus of variations. This result has been extended by D. Danielli and N. Garofalo [18], to cover a class of quasilinear equations including the m-Laplace operator, with $1 < m < 2$, as well as the minimal surfaces operator.

For the m-Laplace operator, with $1 < m < +\infty$, and under the additional assumptions that

$$\forall x' \in \mathbb{R}^{N-1} \qquad \lim_{x_N \to +\infty} u(x', x_N) = \sup_{\mathbb{R}^N} u,$$
$$\forall x' \in \mathbb{R}^{N-1} \qquad \lim_{x_N \to -\infty} u(x', x_N) = \inf_{\mathbb{R}^N} u,$$

De Giorgi's conjecture was proved by D. Danielli and N. Garofalo [18] for $N \leqslant 3$ and, by E. Valdinoci, B. Sciunzi and O. Savin [85] for $N \leqslant 8$ in 2005.

9. Harnack and Liouville

Let $\Omega \subset \mathbb{R}^N$ be a domain containing the open ball centered at a point x_0 and of radius $4r > 0$. If one considers a non-negative harmonic function u defined on Ω and any two points y and z belonging to $B(x_0, r)$, an application of the mean value theorem gives,

$$\begin{aligned} u(y) &= \frac{1}{|B(y,r)|} \int_{B(y,r)} u(x)\,\mathrm{d}x \leqslant \frac{1}{|B(y,r)|} \int_{B(z,3r)} u(x)\,\mathrm{d}x \\ &= \frac{|B(z,3r)|}{|B(y,r)|} \frac{1}{|B(z,3r)|} \int_{B(z,3r)} u(x)\,\mathrm{d}x = 3^N u(z), \end{aligned}$$

which immediately implies that

$$\sup_{B(x_0,r)} u \leqslant 3^N \inf_{B(x_0,r)} u. \tag{9.1}$$

The latter inequality can be used to give another proof of the Liouville property stated in Theorem 3.3. Indeed, if u is a harmonic function bounded from below, with $m := \inf_{\mathbb{R}^N} u$, we can apply inequality (9.1) to the harmonic function $v := u - m \geqslant 0$ to find that

$$\sup_{B(x_0,r)} v \leqslant 3^N \inf_{B(x_0,r)} v.$$

By letting $r \to +\infty$, the right-hand side of the latter inequality tends to zero, hence $u \equiv m$ on $\mathbb{R}^N$.

Inequality (9.1) is the well-known *Harnack inequality* for non-negative harmonic functions. An interesting feature of this inequality is that it holds true for every second order uniformly elliptic operators both in divergence form and non-divergence form. More precisely we have:

THEOREM 9.1 (J. Moser [59]). *Let $N \geqslant 2$, $x_0 \in \mathbb{R}^N$, $r > 0$ and $A = (a_{hk})$ be a real symmetric matrix whose coefficients are bounded and measurable functions defined on $B(x_0, 4r) \subset \mathbb{R}^N$. Assume that there are $0 < \lambda \leqslant \Lambda < +\infty$ such that*

$$\forall x \in B(x_0, 4r),\ \forall \xi \in \mathbb{R}^N \setminus \{0\}, \quad \lambda|\xi|^2 \leqslant \sum_{h,k=1}^{N} a_{hk}(x)\xi_h\xi_k \leqslant \Lambda|\xi|^2. \tag{9.2}$$

Let $u \in H^1(B(x_0, 4r))$ be a distribution solution of

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = 0 & \text{in } B(x_0, 4r), \\ u(x) \geqslant 0 & \text{a.e. in } B(x_0, 4r), \end{cases} \tag{9.3}$$

then we have:

$$\operatorname*{ess\,sup}_{B(x_0,r)} u \leqslant C \operatorname*{ess\,inf}_{B(x_0,r)} u, \tag{9.4}$$

where $C = C(N, \Lambda/\lambda)$.

We also have the analogous result for uniformly elliptic operators in non-divergence form.

THEOREM 9.2 (N.V. Krylov and M.V. Safonov [48,49]). *Let $N \geqslant 2$, $x_0 \in \mathbb{R}^N$, $r > 0$ and $A = (a_{hk})$ be a real symmetric matrix whose coefficients are bounded and measurable functions defined on $B(x_0, 4r) \subset \mathbb{R}^N$. Assume that there are $0 < \lambda \leqslant \Lambda < +\infty$ such that*

$$\forall x \in B(x_0, 4r),\ \forall \xi \in \mathbb{R}^N \setminus \{0\}, \quad \lambda|\xi|^2 \leqslant \sum_{h,k=1}^{N} a_{hk}(x)\xi_h\xi_k \leqslant \Lambda|\xi|^2. \tag{9.5}$$

Let $u \in W^{2,N}(B(x_0, 4r))$ be a solution of

$$\begin{cases} -\sum_{h,k=1}^{N} a_{hk}(x) \frac{\partial^2 u}{\partial x_h \partial x_k} = 0 & a.e.\ in\ B(x_0, 4r), \\ u(x) \geqslant 0 & in\ B(x_0, 4r), \end{cases} \tag{9.6}$$

then we have:

$$\sup_{B(x_0,r)} u \leqslant C \inf_{B(x_0,r)} u, \tag{9.7}$$

where $C = C(N, \Lambda/\lambda)$.

These classical results are masterpieces and cornerstones of the modern theory of linear PDE. Theorem 9.1 was proved by J. Moser in 1961 while Theorem 9.2 is due to the work of N.V. Krylov and M.V. Safonov. For the problem (9.6), with $N = 2$ and $u \in C^2$, the Harnack inequality (9.7) was first proved by J. Serrin [76] in 1955.

We do not give the proofs of the above classical theorems. This would go beyond the scope of this work. For a proof of the above Harnack inequalities we refer to the original works of J. Moser [59], N.V. Krylov and M.V. Safonov [48,49] (see also [36]). The proof given by J. Serrin [76], for the two-dimensional case, is simple and it is based only on the maximum principle.

Let us mention that Moser's result also implies the Hölder continuity estimates for solutions of the linear uniformly elliptic problem (9.3). This celebrated result was first proved by E. De Giorgi [19] in 1957 and its discovery was crucial for the development of the higher-dimensional theory of quasilinear problems and also led to the solution of Hilbert's nineteenth problem [40].

Theorems 9.1 and 9.2 immediately imply the analogue of Theorem 3.3. Actually we have more:

THEOREM 9.3 (J. Moser [59], J. Serrin and H. Weinberger [79]). *Let $N \geqslant 2$ and $A = (a_{hk})$ be a real symmetric matrix whose coefficients are bounded and measurable functions defined on $\mathbb{R}^N$. Assume that there are $0 < \lambda \leqslant \Lambda < +\infty$ such that*

$$\forall x \in \mathbb{R}^N,\ \forall \xi \in \mathbb{R}^N \setminus \{0\}, \quad \lambda|\xi|^2 \leqslant \sum_{h,k=1}^{N} a_{hk}(x)\xi_h\xi_k \leqslant \Lambda|\xi|^2,$$

and let $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$ be a distribution solution of:

$$-Lu := -\operatorname{div}\big(A(x)\nabla u\big) = 0 \quad in\ \mathbb{R}^N. \tag{9.8}$$

(i) *Either u is constant or u must become both positively and negatively unbounded as $|x|$ tends to $+\infty$.*

(ii) *If u is not constant then there exist $\alpha = \alpha(N, \frac{\Lambda}{\lambda}, u) > 0$ and $\beta = \beta(N, \Lambda/\lambda) > 0$ such that*

$$M(R) \geqslant \alpha R^{\beta}, \quad m(R) \leqslant -\alpha R^{\beta}, \quad \forall R \gg 1, \tag{9.9}$$

where, for every $t \geqslant 0$, we have set $M(t) = \sup_{|x|<t} u(x)$ and $m(t) = \inf_{|x|<t} u(x)$. In particular, a non-constant solution of (9.8) *must go to infinity at least as fast as the power $|x|^{\beta}$, as $|x|$ tends to $+\infty$.*

The analogue for uniformly elliptic operators in non-divergence form is

THEOREM 9.4. *Let $N \geqslant 2$ and $A = (a_{hk})$ satisfy the assumptions of Theorem* 9.3. *Let $u \in W^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ be a solution of*:

$$-Mu := -\sum_{h,k=1}^{N} a_{hk}(x) \frac{\partial^2 u}{\partial x_h \partial x_k} = 0 \quad \text{a.e. in } \mathbb{R}^N. \tag{9.10}$$

Then, the conclusions of Theorem 9.3 *hold true.*

PROOF OF THEOREM 9.3. *Proof of* (i). To obtain the desired conclusion it is sufficient to prove that a non-negative solution of (9.8) is constant. To this end, it is enough to use the argument employed for the Laplace operator, at the beginning of this section (one has to apply Harnack's inequality (9.4) instead of (9.1)).

Proof of (ii). Assume that u is not constant. The function $M(R)$ is increasing and the function $m(R)$ is decreasing for $R > 0$. Furthermore, by the result of the previous step, there exists $R_0 > 0$ such that $M(R) > 0$ and $m(R) < 0$ for $R \geqslant R_0$.

For every $t > 0$ the functions $M(4t) - u$ and $u - m(4t)$ are non-negative on the ball $B(0, 4t)$. Therefore, an application of Harnack's inequality (9.4) yields:

$$M(4t) - m(t) \leqslant C\big[M(4t) - M(t)\big], \quad \forall t > 0, \tag{9.11}$$

$$M(t) - m(4t) \leqslant C\big[m(t) - m(4t)\big], \quad \forall t > 0, \tag{9.12}$$

where $C = C(N, \Lambda/\lambda) > 1$ (since $u \not\equiv const.$) is the constant appearing in Theorem 9.1.

Adding the above inequalities we have,

$$M(4t) - m(4t) \geqslant \left(\frac{C+1}{C-1}\right)\big[M(t) - m(t)\big] \quad \forall t > 0, \tag{9.13}$$

and by iterating we obtain:

$$M\big(4^n t\big) - m\big(4^n t\big) \geqslant \left(\frac{C+1}{C-1}\right)^n \big[M(t) - m(t)\big], \quad \forall t > 0, \ \forall n \in \mathbf{N}^{\star}. \tag{9.14}$$

The latter inequality, together with the monotonicity of the function $t \to M(t) - m(t)$ (the oscillation of u over the ball $B(0,t)$), immediately imply:

$$M(R) - m(R) \geqslant \gamma R^{\beta}, \quad \forall R \geqslant 4R_0, \tag{9.15}$$

with

$$\beta = \ln_4\left(\frac{C+1}{C-1}\right) > 0, \qquad \gamma = \frac{M(R_0) - m(R_0)}{(4R_0)^{\beta}} > 0. \tag{9.16}$$

On the other hand, from (9.11)–(9.12) and (9.15) it follows that

$$(C-1)M(4t) \geqslant CM(t) - m(t) \geqslant \gamma t^{\beta}, \quad \forall t > 4R_0, \tag{9.17}$$

$$(C-1)m(4t) \leqslant -\big[M(t) - Cm(t)\big] \leqslant -\gamma t^{\beta}, \quad \forall t > 4R_0. \tag{9.18}$$

Hence, the desired conclusion (9.9) follows by taking $\alpha = \gamma/(4^{\beta}(C-1)) > 0$ and β as in (9.16). □

The same proof also applies to the case of a uniformly elliptic operator of non-divergence form (one has only to use Harnack's inequality (9.7), instead of inequality (9.4), and the maximum principle of Alexandrov, Bakelman and Pucci [36] for solutions of class $W^{2,N}$) and for this reason we omit it.

Let L and M be the uniformly elliptic operators considered in Theorems 9.3 and 9.4 respectively. For every $\delta \geqslant 0$ we define the vector spaces:

$$\mathcal{H}_{\delta}(L) := \big\{u \in H^1_{\mathrm{loc}}\big(\mathbb{R}^N\big)\colon\ -Lu = 0,\ u(x) = \mathrm{O}\big(|x|^{\delta}\big) \text{ as } |x| \to +\infty\big\}, \tag{9.19}$$

and

$$\mathcal{H}_{\delta}(M) := \big\{u \in W^{2,N}_{\mathrm{loc}}\big(\mathbb{R}^N\big)\colon\ -Mu = 0,\ u(x) = \mathrm{O}\big(|x|^{\delta}\big) \text{ as } |x| \to +\infty\big\}. \tag{9.20}$$

From Theorems 9.3 and 9.4 we know that there exists $\beta = \beta(N, \frac{\Lambda}{\lambda}) > 0$ such that

$$\forall \delta < \beta \quad \dim \mathcal{H}_{\delta}(L) = 1 \quad (\text{resp. } \dim \mathcal{H}_{\delta}(M) = 1).$$

If one compares the above results with those proved in Section 2, for the special case of the Laplace's operator, the following questions arise in a natural way:

QUESTION 9.5. Find the greatest exponent $\bar{\delta} \in (0, +\infty]$ such that

$$\forall \delta < \bar{\delta} \quad \dim \mathcal{H}_{\delta}(L) = 1 \quad (\text{resp. } \dim \mathcal{H}_{\delta}(M) = 1). \tag{9.21}$$

QUESTION 9.6. Let $\delta > 0$ be a fixed constant. What is the dimension of $\mathcal{H}_{\delta}(L)$? Is it finite? In the affirmative, can one estimate $\dim \mathcal{H}_{\delta}(L)$ in terms of N and Λ/λ? Can one describe the structure of the vector space $\mathcal{H}_{\delta}(L)$? Same questions for an operator M of non-divergence form.

The value $\bar{\delta}$ gives the least growth at infinity of every non-constant solution of (9.8) (resp. of (9.10)). The quantity $\bar{\delta}$ is always greater than or equal to β in (9.16) and, for an operator in non-divergence form is always less than or equal to 1 (equation (9.10) always admits non-constant linear solutions). Moreover, it is clear that it is equal to 1 for the special case of the Laplace's operator (cf. Theorem 2.3). The following examples, due to N. Meyers [57] and J. Serrin and H. Weinberger [79], show that, in general, the value $\bar{\delta}$ depends on both dimension N and on ratio Λ/λ.

EXAMPLE 9.7 ([57,79]). Let $N \geqslant 2$. For $\eta \in \mathbb{R}$ consider the real symmetric matrix $A = (a_{hk})$, with bounded and measurable coefficients given by:

$$a_{kh} = a_{kh}(x) := \left(1+\eta^2\right)^{-1/2}\left(\delta_{hk} + \eta^2 \frac{x_h x_k}{|x|^2}\right), \quad h,k = 1,\dots,N.$$

The corresponding linear partial differential equation (9.8) is uniformly elliptic with $\lambda = (1+\eta^2)^{-1/2}$ and $\Lambda = (1+\eta^2)^{1/2}$ and it has the solution:

$$u(x) = x_1|x|^{-\theta}, \quad \theta = \frac{1}{2}\left\{N - \left[(N-2)^2 + 4(N-1)(1+\eta^2)^{-1}\right]^{1/2}\right\} \in (0,1),$$

which goes to infinity at the rate $|x|^{1-\theta}$. Hence

$$\bar{\delta} \leqslant 1-\theta = \frac{1}{2}\left\{\left[(N-2)^2 + 4(N-1)\left(\frac{\Lambda}{\lambda}\right)^{-1}\right]^{1/2} - (N-2)\right\} \in (0,1),$$

which goes to zero when the ratio Λ/λ goes to $+\infty$.

The most general result concerning the above questions is the following one. It was proved in 1997 by T. Colding and W.P. Minicozzi [17] and by P. Li [52].

THEOREM 9.8 (T. Colding and W.P. Minicozzi [17], P. Li [52]). *Let $N \geqslant 2$ and $A = (a_{hk})$ be a real symmetric matrix whose coefficients are bounded and measurable functions defined on $\mathbb{R}^N$. Assume that there are $0 < \lambda \leqslant \Lambda < +\infty$ such that*

$$\forall x \in \mathbb{R}^N,\ \forall \xi \in \mathbb{R}^N \setminus \{0\}, \quad \lambda|\xi|^2 \leqslant \sum_{h,k=1}^{N} a_{hk}(x)\xi_h\xi_k \leqslant \Lambda|\xi|^2.$$

Then, $\dim \mathcal{H}_\delta(L)$ is finite for all $\delta > 0$. Furthermore, there exists a positive constant C, depending only on N and Λ/λ, such that

$$\forall \delta \geqslant 1 \quad \dim \mathcal{H}_\delta(L) \leqslant C\delta^{N-1}. \tag{9.22}$$

Theorem 9.8 has the following analogue for the operator M:

THEOREM 9.9 (P. Li [52]). *Let $N \geqslant 2$ and $A = (a_{hk})$ be a real symmetric matrix whose coefficients are bounded and measurable functions defined on $\mathbb{R}^N$. Assume that there are $0 < \lambda \leqslant \Lambda < +\infty$ such that*

$$\forall x \in \mathbb{R}^N,\ \forall \xi \in \mathbb{R}^N \setminus \{0\}, \quad \lambda|\xi|^2 \leqslant \sum_{h,k=1}^{N} a_{hk}(x)\xi_h\xi_k \leqslant \Lambda|\xi|^2.$$

Then, there exists a positive constant C, depending only on N and Λ/λ, such that

$$\forall \delta \geqslant 1 \quad \dim \mathcal{H}_\delta(M) \leqslant C\delta^{N-1}. \tag{9.23}$$

The proof of the above theorems is based on the well-known *doubling property* of the N-dimensional Lebesgue measure $\mathcal{L}_N$:

$$\exists C_1 = C_1(N) > 0\colon\ \forall x_0 \in \mathbb{R}^N,\ \forall r > 0 \quad \mathcal{L}_N\big(B(x_0, 2r)\big) \leqslant C_1\mathcal{L}_N\big(B(x_0, r)\big),$$

and on the generalized *mean value inequality* for non-negative subsolutions:

$$\exists C_2 = C_2\left(N, \frac{\Lambda}{\lambda}\right) > 0\colon\ u(x_0) \leqslant \frac{C_2}{r^N}\int_{B(x_0,r)} u(x)\,\mathrm{d}x,$$

$$\forall u,\ \forall x_0 \in \mathbb{R}^N,\ \forall r > 0\colon\ u \geqslant 0, \quad -Lu \leqslant 0 \quad (\text{resp. } -Mu \leqslant 0) \quad \text{on } B(x_0, r).$$

The latter inequality is a by-product of Moser's iteration method (resp. of Krylov–Safonov analysis). See for instance [36] and [73]. Actually, the arguments used in [17] and [52] also work in a more general geometrical setting. For instance, they can be used to study Liouville-type properties for the Laplace–Beltrami operator of some complete Riemannian manifolds. For these topics we refer the interested reader to [17,52] and [53], as well as to the references therein.

Harnack's inequality has been extended to quasilinear elliptic equations (possibly degenerate) by J. Serrin [77] in 1964 and by N.S. Trudinger [84] in 1967. Their proofs are based on a modification (and an extension) of Moser's iteration scheme [59] originally introduced to prove Theorem 9.1.

Consider the quasilinear second order partial differential equation of the form:

$$-\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u). \tag{9.24}$$

We assume that the Carathéodory functions $A = (A_1, \ldots, A_N)$ and B are defined on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ and satisfy the following standard structural assumptions:

$$\big|A(x, u, p)\big| \leqslant a_0|p|^{m-1}, \quad \forall (x, u, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \tag{9.25}$$

$$p \cdot A(x, u, p) \geqslant |p|^m, \quad \forall (x, u, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \tag{9.26}$$

$$|B(x, u, p) \leqslant b_0|p|^m, \quad \forall (x, u, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N, \tag{9.27}$$

where $m > 1$, a_0 and b_0 are non-negative constants.

Under these assumptions we have:

THEOREM 9.10 (J. Serrin [77], N. Trudinger [84]). *Let $N \geqslant 1$, $x_0 \in \mathbb{R}^N$, $r > 0$ and $M > 0$. Let $K(x_0, \rho)$ denote the cube in $\mathbb{R}^N$ of side ρ and center x_0 and whose sides are parallel to the coordinate axes.*

Let $u \in W^{1,m}(K(x_0, 4r))$ be a distribution solution of (9.24) *in $K(x_0, 4r)$ such that $0 \leqslant u < M$ a.e. in $K(x_0, 4r)$. Then we have*:

$$\operatorname*{ess\,sup}_{K(x_0,r)} u \leqslant C \operatorname*{ess\,inf}_{K(x_0,r)} u, \tag{9.28}$$

where $C = C(m, N, a_0, b_0 M)$.

Furthermore, when $b_0 = 0$ in (9.27), *there is no need to assume boundedness of u in $K(x_0, 4r)$. In this case the above constant C only depends on m, N and a_0.*

We are now in position to prove the following:

THEOREM 9.11. *Let $N \geqslant 1$ and assume that conditions* (9.25)–(9.27) *are satisfied.*

(i) *Let $u \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N)$ be a distribution solution of*:

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) = 0 & \text{in } \mathbb{R}^N, \\ u(x) \geqslant 0 & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{9.29}$$

Then u is a constant function.

(ii) *Let $u \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a distribution solution of*:

$$-\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \mathbb{R}^N. \tag{9.30}$$

Then u is a constant function.

PROOF. *Proof of* (i). For all $(x, u, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ we set:

$$l = \inf_{\mathbb{R}^N} u, \quad v(x) = u(x) - l, \quad \mathcal{A}(x, p) = A\big(x, u(x), p\big), \tag{9.31}$$

and observe that $v \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N)$ is a non-negative distribution solution of the equation,

$$-\operatorname{div} \mathcal{A}(x, \nabla v) = 0 \quad \text{in } \mathbb{R}^N.$$

Since $\mathcal{A}$ satisfies the structural assumptions (9.25) and (9.26), an application of Theorem 9.10 (with $b_0 = 0$) yields:

$$\forall r > 0, \quad \operatorname*{ess\,sup}_{K(x_0,r)} v \leqslant C(m, N, a_0) \operatorname*{ess\,inf}_{K(x_0,r)} v.$$

As in the linear case, the desired conclusion follows by letting $r \to +\infty$.

Proof of (ii). We proceed as before. We define l, v and $\mathcal{A}$ as in (9.31) and set, for all $(x, u, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$, $\mathcal{B}(x, p) = B(x, u(x), p)$. With these definitions, the function $v \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ solves the quasilinear equation $-\operatorname{div} \mathcal{A}(x, \nabla v) = \mathcal{B}(x, \nabla v)$ on $\mathbb{R}^N$. Since $\mathcal{A}$ and $\mathcal{B}$ satisfy (9.25)–(9.27) we can apply Harnack's inequality (9.28), with M any real number satisfying $M > \|u\|_{L^\infty} + |l|$, and conclude as before. □

The next result is a straightforward generalization of Theorem 3.1 and Proposition 7.7 to quasilinear equations of the form (9.24).

THEOREM 9.12. *Let* $1 \leqslant N \leqslant m$ *and assume that conditions* (9.25)–(9.26) *are satisfied. Let* $u \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N)$ *be a distribution solution of*:

$$\begin{cases} -\operatorname{div} A(x, u, \nabla u) \geqslant 0 & \text{in } \mathbb{R}^N, \\ u(x) \geqslant -C & \text{a.e. in } \mathbb{R}^N, \end{cases} \tag{9.32}$$

for some positive constant C. *Then* u *is a constant function.*

PROOF. For all $(x, u, p) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ we set:

$$v(x) = u(x) + C + 1, \qquad \mathcal{A}(x, p) = A\big(x, u(x), p\big),$$

and observe that $v \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N)$ is a distribution solution of,

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla v) \geqslant 0 & \text{in } \mathbb{R}^N, \\ v(x) \geqslant 1 & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{9.33}$$

The function v^{1-m} belongs to $W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, since $v \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N)$ and v is bounded from below by 1. Hence, a standard density argument implies that the function $\phi^m v^{1-m}$ can be used as a test function in the distributional inequality (9.33), whenever $\phi \in C^\infty_c(\mathbb{R}^N)$ and $\phi \geqslant 0$ everywhere on $\mathbb{R}^N$. With this choice of test function we have:

$$(m-1) \int \mathcal{A}\big(x, \nabla v(x)\big) \cdot \nabla v \phi^m v^{-m} \leqslant m \int \mathcal{A}\big(x, \nabla v(x)\big) \cdot \nabla \phi \phi^{m-1} v^{1-m}.$$

The assumptions (9.25) and (9.26) then yields:

$$\begin{aligned} (m-1) \int |\nabla v|^m \phi^m v^{-m} &\leqslant m a_0 \int |\nabla v|^{m-1} |\nabla \phi| \phi^{m-1} v^{1-m} \\ &\leqslant m a_0 \left[\int |\nabla v|^m \phi^m v^{-m} \right]^{1-1/m} \left[\int |\nabla \phi|^m \right]^{1/m}. \end{aligned}$$

Therefore, for any non-negative $\phi \in C^\infty_c(\mathbb{R}^N)$ we have:

$$\int |\nabla v|^m \phi^m v^{-m} \leqslant \left[\frac{m a_0}{m-1} \right]^m \int |\nabla \phi|^m.$$

Hence, for every $r > 0$,

$$\int_{B(0,r)} |\nabla v|^m \phi^m v^{-m} \leqslant \left[\frac{ma_0}{m-1}\right]^m \inf_{\phi \in \mathcal{D}(r,\mathbb{R}^N)} \int_{\mathbb{R}^N} |\nabla \phi|^m, \tag{9.34}$$

where we have set:

$$\mathcal{D}\big(r,\mathbb{R}^N\big) := \big\{\phi \in C_c^\infty\big(\mathbb{R}^N\big)\colon\ 0 \leqslant \phi \leqslant 1 \text{ and } \phi = 1 \text{ in a neighborhood of } \overline{B(0,r)}\big\}.$$

We observe that the right-hand side of (9.34) is equal to $[ma_0/(m-1)]^m \operatorname{cap}_m(B(0,r))$, where $\operatorname{cap}_m(B(0,r))$ denotes the m-capacity of $\overline{B(0,r)}$ (see for instance [39]). Also we notice that the assumption $N \leqslant m$ implies $\operatorname{cap}_m(B(0,r)) = 0$, for every $r > 0$ (cf. [39]). The latter result concludes the proof. □

It is worthwhile to observe that the structural assumptions (9.25)–(9.27) are satisfied by the well-known m-Laplace operator $-\operatorname{div}(|\nabla \cdot|^{m-2}\nabla\cdot)$, $m > 1$. Hence, the following extension of the Liouville properties, established in Sections 2 and 3 for the classical Laplace operator ($m = 2$), are an immediate consequence of Theorems 9.11 and 9.12.

COROLLARY 9.13. *Let $N \geqslant 1$ and $m > 1$.*

(i) *Let $u \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N)$ be a distribution solution of $-\operatorname{div}(|\nabla u|^{m-2}\nabla u) = 0$ in $\mathbb{R}^N$. If u is bounded from below (or above) then u is constant.*

(ii) *Assume further that $N \leqslant m$ and let $u \in W^{1,m}_{\mathrm{loc}}(\mathbb{R}^N)$ be a distribution solution of $-\operatorname{div}(|\nabla u|^{m-2}\nabla u) \geqslant 0$ in $\mathbb{R}^N$. If u is bounded from below then u is constant.*

The special case $m = 2$, in part (ii) of Theorem 9.11 and in Theorem 9.12, was proved by M. Meier [56] in 1979. All the others assertions of Theorems 9.11, 9.12 and Corollary 9.13 are well-know results in the communities of PDE's and of non-linear potential theory [39] (even though it is not always easy to find a bibliographical reference). The above proof of Theorem 9.11 is essentially the one devised by M. Meier in [56], while the one of Theorem 9.12 is inspired by the proof of Proposition 7.7.

The second part of Corollary 9.13 implies that $\mathbb{R}^N$, endowed with the standard flat metric, is a *m-parabolic Riemannian manifold* if $N \leqslant m$. The *m-parabolicity* generalizes the notion of *parabolic* Riemannian manifold considered in Section 3. A Riemannian manifold (M, g) is *m-parabolic* if the only non-negative super-m-harmonic functions are the constants, that is, the conditions: $-\Delta_{m,g} u \geqslant 0$ and $u \geqslant 0$ on M imply $u = const$. (here $\Delta_{m,g}$ denotes the m-Laplace–Beltrami operator, i.e., the m-Laplace operator associated with the metric g). In particular, a Riemannian manifold (M, g) is *parabolic* if and only if it is *2-parabolic*.

For $N > m$, the function

$$u(x) := \left(\frac{C}{1 + |x|^{m/(m-1)}}\right)^{(N-m)/m}$$

is a positive, non-constant solution of $-\operatorname{div}(|\nabla u|^{m-2}\nabla u) \geqslant 0$ on $\mathbb{R}^N$, provided that C is a positive constant. This proves that Corollary 9.13 is sharp. In particular we have that, $\mathbb{R}^N$ endowed with the standard flat metric, is a *m-parabolic Riemannian manifold* if and only if $N \leqslant m$.

To conclude the section we observe that Theorem 9.3 can be extended to quasilinear equations of the form $-\operatorname{div} A(x, u, \nabla u) = 0$ where A satisfies the structural assumptions (9.25) and (9.26).

10. Moser and Liouville

This section is devoted to the classification of solution to quasilinear elliptic equations involving the minimal surfaces operator. We aim:

(i) to prove some further Liouville-type (Bernstein-type) results concerning the minimal surfaces equation (5.9),

(ii) to classify all the entire solutions of the equation $-\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2}) + f(u) = 0$ in $\mathbb{R}^N$, for any continuous non-decreasing function f, with $f \not\equiv 0$.

To this end the next result, proved by J. Moser [59] in 1961, plays a crucial rôle. Its proof is based on Liouville-type theorems considered in the previous section.

THEOREM 10.1 (J. Moser [59]). *Let $N \geqslant 2$ and $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a solution of*:

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } \mathbb{R}^N, \\ \nabla u \in L^\infty(\mathbb{R}^N). \end{cases} \tag{10.1}$$

Then u is an affine function, i.e., the graph of u is a hyperplane in $\mathbb{R}^{N+1}$.

PROOF. We already know (cf. the proof of Theorem 5.1) that any solution of the minimal surface equation is smooth (real-analytic) hence, by differentiating equation (10.1) with respect to x_j, we find that

$$-\operatorname{div}\big(A(x)\nabla u_j\big) = 0 \quad \text{in } \mathbb{R}^N, \tag{10.2}$$

where u_j denotes the function $\partial u/\partial x_j$, for any $j = 1, \ldots, N$ and $A = (a_{hk})$ is the real symmetric matrix whose entries are given by:

$$a_{hk} = a_{hk}(x) := \frac{\delta_{hk}}{(1+|\nabla u|^2)^{1/2}} - \frac{u_h u_k}{(1+|\nabla u|^2)^{3/2}}. \tag{10.3}$$

The smallest eigenvalue of the above matrix is $1/(1+|\nabla u|^2)^{3/2}$ while the largest one is $1/(1+|\nabla u|^2)^{1/2}$. Since the assumption $\nabla u \in L^\infty(\mathbb{R}^N)$ is in force, we see that the above linear equation (10.2) is uniformly elliptic on the entire Euclidean space $\mathbb{R}^N$ with $\lambda = 1/(1+\|\nabla u\|^2_{L^\infty})^{3/2}$ and $\Lambda = 1$.

Using once again the assumption $\nabla u \in L^\infty(\mathbb{R}^N)$ we infer, from Theorem 9.3, that every u_j is a constant function. This concludes the proof. □

The above result of J. Moser says that a non-affine minimal graph must have unbounded gradient. This suggests that some informations about the growth of the gradient of the solutions to the minimal surfaces equation could be useful to obtain further results of Bernstein type. The following deep result provides a universal interior gradient estimate for solutions to the minimal surfaces equation. It was proved by E. Bombieri, E. De Giorgi and M. Miranda [10] in 1969 (for the two-dimensional case this kind of bound was first proved by R. Finn [28] in 1954).

THEOREM 10.2 (E. Bombieri, E. De Giorgi and M. Miranda [10]). *Let $N \geqslant 2$, $x_0 \in \mathbb{R}^N$ and $r > 0$. Let $u \in C^2(B(x_0, r), \mathbb{R})$ be a solution of*:

$$\begin{cases} -\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } B(x_0, r), \\ u \geqslant 0 & \text{in } B(x_0, r). \end{cases} \tag{10.4}$$

Then, we have the estimate:

$$\left|\nabla u(x_0)\right| \leqslant C_1 \mathrm{e}^{[C_2 u(x_0)/r]}, \tag{10.5}$$

where C_1 and C_2 are constants depending only on N.

For the proof of the above theorem we refer the reader to the original work of E. Bombieri, E. De Giorgi and M. Miranda [10] (see also [47] and Chapter 16 of [36]).

A combination of the above two results leads to the following extensions of the classical Bernstein's theorem.

THEOREM 10.3 (E. Bombieri, E. De Giorgi and M. Miranda [10]). *Assume $N \geqslant 2$ and let $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a solution of*:

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } \mathbb{R}^N. \tag{10.6}$$

(i) *If u is bounded from below (above) then u is constant.*
(ii) *If u satisfies*,

$$u(x) \geqslant -K\big(|x|+1\big) \quad \forall x \in \mathbb{R}^N, \tag{10.7}$$

for some positive constant K, then u is an affine function.

PROOF. *Proof of* (i). It suffices to prove the result for a positive solution of (10.6). Pick $x_0 \in \mathbb{R}^N$. For every $r > 0$, we apply the gradient estimate (10.5) to u. This gives:

$$\forall r > 0 \quad \left|\nabla u(x_0)\right| \leqslant C_1 \mathrm{e}^{C_2[u(x_0)/r]}.$$

By letting $r \to +\infty$ in the latter, we obtain the uniform gradient bound:

$$\forall x_0 \in \mathbb{R}^N \quad \left|\nabla u(x_0)\right| \leqslant C_1 = C_1(N) < +\infty.$$

The desired conclusion then follows by applying Moser's Theorem 10.1.

Proof of (ii). We observe that, under the assumption (10.7), for every $x_0 \in \mathbb{R}^N$ and every $r > 0$ we have:

$$u(x_0) - \inf_{B(x_0,r)} u \leqslant u(x_0) + K\big[|x_0| + r + 1\big],$$

therefore an application of Theorem 10.2 to the function $u(x_0) - \inf_{B(x_0,r)} u$, on the ball $B(x_0, r)$ yields,

$$\forall x_0 \in \mathbb{R}^N,\ \forall r > 0, \quad \big|\nabla u(x_0)\big| \leqslant C_1 \mathrm{e}^{[C_2 \frac{u(x_0)+K|x_0|+K}{r} + C_2 K]}.$$

The latter implies $\|\nabla u\|_{L^\infty} \leqslant C_1 \mathrm{e}^{C_2 K} < +\infty$, and the claim follows, once again, from Theorem 10.1. □

Next we turn to the study of the equation:

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + f(u) = 0 \quad \text{in } \mathbb{R}^N \tag{10.8}$$

where f is a continuous non-decreasing function. The above equation includes some models which are important for applications. For instance, when $f \equiv H = const.$, the above equation (10.8) describes entire graphs (non-parametric hypersurfaces in $\mathbb{R}^{N+1}$) with constant mean curvature equal to H [16] while, for $f(t) = at + b$, $a > 0$, equation (10.8) gives the well-know capillarity equation [29–31]. More generally, equation (10.8) says that the mean curvature of the graph of u (view as a non-parametric hypersurfaces in $\mathbb{R}^{N+1}$) is equal to $f(u(x))/N$ at every point $(x, u(x))$.

For this class of equations we have the following result:

THEOREM 10.4. *Assume $N \geqslant 1$ and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, non-decreasing function, $f \not\equiv 0$. Let $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a solution of* (10.8).

Then u is a constant function, say $u \equiv c \in \mathbb{R}$, and $f(c) = 0$.

PROOF. Let $\{\rho_n\}_{n\in\mathbf{N}^\star}$ be a sequence of standard *mollifiers* and set $f_n = f * \rho_n$. Clearly, for every $n \in \mathbf{N}^\star$, f_n is a smooth and non-decreasing function on $\mathbb{R}$.

For every $\phi \in C_c^\infty(\mathbb{R}^N)$, $\phi \geqslant 0$ on $\mathbb{R}^N$, and every $n \in \mathbf{N}^\star$ we multiply equation (10.8) by $|f_n(u)|^{N-1} f_n(u)\phi^{N+1}$ and integrate by parts. This leads to:

$$\begin{aligned}
&\int f(u)\big|f_n(u)\big|^{N-1} f_n(u)\phi^{N+1} \\
&\quad = -(N+1)\int \phi^N \big|f_n(u)\big|^{N-1} f_n(u) \frac{(\nabla u \cdot \nabla \phi)}{\sqrt{1+|\nabla u|^2}}
\end{aligned}$$

$$
\begin{aligned}
&- N \int \left| f_n(u) \right|^{N-1} f_n'(u) \phi^{N+1} \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}} \\
&\leqslant (N+1) \int \phi^N \left| f_n(u) \right|^N |\nabla \phi|,
\end{aligned}
$$

where we have used $f_n' \geqslant 0$. By letting $n \to +\infty$ we have:

$$
\int \left| f(u) \right|^{N+1} \phi^{N+1} \leqslant (N+1) \int \left| f(u) \right|^N \phi^N |\nabla \phi|.
$$

By the Schwarz–Hölder inequality we see that

$$
\begin{aligned}
&\int \left| f(u) \right|^{N+1} \phi^{N+1} \\
&\quad \leqslant (N+1) \left[\int \left| f(u) \right|^{N+1} \phi^{N+1} \right]^{N/(N+1)} \left[\int |\nabla \phi|^{N+1} \right]^{1/(N+1)}
\end{aligned}
$$

and consequently

$$
\int \left| f(u) \right|^{N+1} \phi^{N+1} \leqslant (N+1)^{N+1} \int |\nabla \phi|^{N+1}. \tag{10.9}
$$

Now, for every $R > 0$, we consider the function $\phi = \phi_R(x) = \varphi(|x|/R)$, where $\varphi \in C_c^\infty(\mathbb{R})$, $0 \leqslant \varphi \leqslant 1$ everywhere on $\mathbb{R}$, and

$$
\varphi(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1, \\ 0 & \text{if } |t| \geqslant 2. \end{cases}
$$

Inserting $\phi = \phi_R$ in (10.9) we obtain:

$$
\int_{B(0,R)} \left| f(u) \right|^{N+1} \leqslant C(N) \int_{B(0,2R)} \frac{1}{R^{N+1}} \to 0 \quad \text{as } R \to +\infty.
$$

Therefore

$$
\forall x \in \mathbb{R}^N \quad f\big(u(x)\big) = 0.
$$

The latter implies that u must be bounded on one side (otherwise we would have $f \equiv 0$, a contradiction). Thus we can apply Theorem 10.3 to infer that u is constant, which concludes the proof. □

Some remarks are in order.

REMARKS 10.5.

(i) The above Theorem 10.4 is essentially due to V.G. Tkachev [82]. When $N \leqslant 7$ one can drop the assumption $f \not\equiv 0$. In this case the conclusion is: *u is an affine function* (as in Bernstein's theorem). Indeed, the proof of the theorem yields $f(u) \equiv 0$ on $\mathbb{R}^N$. Consequently u is a minimal graph and hence an affine function, since $N \leqslant 7$ (cf. Section 5).

A similar result is not available for $N \geqslant 8$ since, in this case, non-affine entire minimal graphs do exist [9].

Theorem 10.4 improves an earlier result of S.Y. Cheng and S.T. Yau [15], established for $N = 2$ and for a non-negative and non-decreasing function f. Their method was entirely different from the one given here. It was based on the fact that, under the above assumptions on f, the graph of any solution of the considered equation is a parabolic Riemannian manifold in $\mathbb{R}^3$.

(ii) The above result also says that an entire graph with constant mean curvature must be an entire minimal graph. This was first proved by S.S. Chern [16] in 1965.

(iii) Note that the monotonicity assumption on f is necessary to obtain the desired conclusion in the above theorem. Indeed, the non-affine function $u = \sqrt{1 + x_1^2}$ satisfies equation (10.8) with $N \geqslant 1$ and

$$f(t) := \begin{cases} 1, & t \leqslant 1, \\ \frac{1}{(2u^2-1)^{3/2}}, & t > 1. \end{cases}$$

It is easily checked that f is not monotone increasing.

We conclude this section by considering the differential inequality:

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \geqslant 0 \quad \text{in } \mathbb{R}^N. \tag{10.10}$$

For the above inequality we have the following result:

THEOREM 10.6.

(i) *Assume $N \leqslant 2$ and let $u \in C^2(\mathbb{R}^N, \mathbb{R})$ be a solution of*:

$$\begin{cases} -\operatorname{div}(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) \geqslant 0 & \text{in } \mathbb{R}^N, \\ u(x) \geqslant -C & \forall x \in \mathbb{R}^N, \end{cases}$$

for some positive constant C. Then u is a constant function.

(ii) *For $N \geqslant 3$ the function*:

$$u(x) = \gamma\left(1 + |x|^2\right)^{-1/3}, \quad \forall x \in \mathbb{R}^N$$

is a positive solution of class C^2 to (10.10), *provided that γ is a small enough, positive constant.*

PROOF. *Proof of* (i). Apply Proposition 7.7 with $b = 0$ and the diagonal matrix $A = (1/\sqrt{1+|\nabla u|^2})\mathrm{Id}$, where Id denotes the $N \times N$ identity matrix.

Proof of (ii). This follows by a direct computation. □

Acknowledgements

We are indebted to Gérard Tronel for his aid in locating the works of A. Cauchy, J. Hadamard and J. Liouville. We are also grateful to Gérard Tronel for a careful reading of a first version of this work and for many interesting comments. This work was partially supported by the ESF-Programme "Global and geometrical aspects of nonlinear partial differential equations (GLOBAL)".

References

[1] G. Alberti, L. Ambrosio, and X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math. **65** (1–3) (2001), 9–33. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday.

[2] W. Allegretto, *On the equivalence of two types of oscillation for elliptic operators*, Pacific J. Math. **55** (1974), 319–328.

[3] F.J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2) **84** (1966), 277–292.

[4] L. Ambrosio and X. Cabré, *Entire solutions of semilinear elliptic equations in* $\mathbb{R}^3$ *and a conjecture of De Giorgi*, J. Amer. Math. Soc. **13** (4) (2000), 725–739.

[5] M.T. Barlow, R.F. Bass and C. Gui, *The Liouville property and a conjecture of De Giorgi*, Comm. Pure Appl. Math. **53** (8) (2000) 1007–1038.

[6] H. Berestycki, L. Caffarelli and L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1–2) (1997), 69–94 (1998). Dedicated to Ennio De Giorgi.

[7] H. Berestycki, F. Hamel and R. Monneau, *One-dimensional symmetry of bounded entire solutions of some elliptic equations*, Duke Math. J. **103** (3) (2000), 375–396.

[8] S. Bernstein, *Sur un théorème de géométrie et son application aux équations aux dérivées partielles du type elliptique*, Comm. Soc. Math. de Kharkov **2** (15) (1915–1917), 38-45; German translation: *Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus*, Math. Z. **26** (1) (1927), 551–558.

[9] E. Bombieri, E. De Giorgi and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.

[10] E. Bombieri, E. De Giorgi and M. Miranda, *Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche*, Arch. Ration. Mech. Anal. **32** (1969), 255–267.

[11] H. Brezis, *Semilinear equations in* $\mathbb{R}^N$ *without condition at infinity*, Appl. Math. Optim. **12** (3) (1984), 271–282.

[12] E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Michigan Math. J. **5** (1958), 105–126.

[13] A. Cauchy, *Mémoires sur les fonctions complémentaires*, C. R. Acad. Sci. Paris **19** (1844), 1377–1384 and Oeuvres complètes, Ier série, tome **VIII**, 378–383.

[14] A. Cauchy, C. R. Acad. Sci. Paris **32** (1851), 452–454 and Oeuvres complètes, Ier série, tome **XI**, 373–376.

[15] S.Y. Cheng and S.T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (3) (1975), 333–354.

[16] S.-S. Chern, *On the curvatures of a piece of hypersurface in Euclidean space*, Abh. Math. Sem. Univ. Hamburg **29** (1965), 77–91.

[17] T.H. Colding and W.P. Minicozzi II, *Harmonic functions on manifolds*, Ann. of Math. (2) **146** (3) (1997) 725–747.
[18] D. Danielli and N. Garofalo, *Properties of entire solutions of non-uniformly elliptic equations arising in geometry and in phase transitions*, Calc. Var. Partial Differential Equations **15** (4) (2002), 451–491.
[19] E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43.
[20] E. De Giorgi, *Una estensione del teorema di Bernstein*, Ann. Scuola Norm. Sup. Pisa (3) **19** (1965), 79–85.
[21] E. De Giorgi, *Convergence problems for functionals and operators*, Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978), Pitagora, Bologna (1979), 131–188.
[22] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI (1998).
[23] A. Farina, *Symmetry for solutions of semilinear elliptic equations in $\mathbb{R}^N$ and related conjectures*, Ricerche Mat. **48** (Suppl.) (1999), 129–154. Papers in memory of Ennio De Giorgi.
[24] A. Farina, *Symmetry for solutions of semilinear elliptic equations in $\mathbb{R}^N$ and related conjectures*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **10** (4) (1999), 255–265.
[25] A. Farina, *One-dimensional symmetry for solutions of quasilinear equations in $\mathbb{R}^2$*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **6** (3) (2003), 685–692.
[26] A. Farina, *Rigidity and one-dimensional symmetry for semilinear elliptic equations in the whole of $\mathbb{R}^N$ and in half spaces*, Adv. Math. Sci. Appl. **13** (1) (2003), 65–82.
[27] R. Finn, *Sur quelques généralisations du théorème de Picard*, C. R. Acad. Sci. Paris **235** (1952), 596–598.
[28] R. Finn, *On equations of minimal surface type*, Ann. of Math. (2) **60** (1954), 397–416.
[29] R. Finn, *New estimates for equations of minimal surface type*, Arch. Ration. Mech. Anal. **14** (1963), 337–375.
[30] R. Finn, *Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature*, J. Analyse Math. **14** (1965), 139–160.
[31] R. Finn, *On the Laplace formula and the meniscus height for a capillary surface*, Z. Angew. Math. Mech. **61** (3) (1981), 165–173, 175–177. With addenda by the author.
[32] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in* 3*-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. **33** (2) (1980), 199–211.
[33] F. Gazzola, *The sharp exponent for a Liouville-type theorem for an elliptic inequality*, Rend. Istit. Mat. Univ. Trieste **34** (1–2) (2002) (2003), 99–102.
[34] N. Ghoussoub and C. Gui, *On a conjecture of De Giorgi and some related problems*, Math. Ann. **311** (3) (1998), 481–491.
[35] D. Gilbarg and J. Serrin, *On isolated singularities of solutions of second order elliptic differential equations*, J. Analyse Math. **4** (1955/56), 309–340.
[36] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin (2001).
[37] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, vol. 80, Birkhäuser, Basel (1984).
[38] J. Hadamard, *Oeuvres de Jacques Hadamard. Tomes I, II, III, IV*, Comité de publication des oeuvres de Jacques Hadamard: M. Fréchet, P. Levy, S. Mandelbrojt and L. Schwartz, Éditions du Centre National de la Recherche Scientifique, Paris (1968).
[39] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York (1993).
[40] D. Hilbert, *Mathematical problems*, Bull. Amer. Math. Soc. **37** (2000), 407–436; Reprinted from Bull. Amer. Math. Soc. **8** (July 1902), 437–479; Originally published as *Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematike-Congress zu Paris 1900*, Gött. Nachr. 1900, 253–297, Vandenhoeck & Ruprecht, Göttingen; Translated for the Bulletin, with the author's permission, by Dr. Mary Winston Newson, 1902.
[41] E. Hopf, *Bemerkungen zu einem Satze von S. Bernstein aus der Theorie der elliptischen Differentialgleichungen*, Math. Z. **29** (1) (1929), 744–745.
[42] E. Hopf, *On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature*, Proc. Amer. Math. Soc. **1** (1950), 80–85.

[43] D. Jerison and R. Monneau, *Towards a counter-example to a conjecture of De Giorgi in high dimensions*, Ann. Mat. Pura Appl. (4) **183** (4) (2004), 439–467.
[44] K. Jörgens, *Über die Lösungen der Differentialgleichung* $rt - s^2 = 1$, Math. Ann. **127** (1954), 130–134.
[45] T. Kato, *Schrödinger operators with singular potentials*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972), Israel J. Math. **13** (1972) (1973), 135–148.
[46] J.B. Keller, *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), 503–510.
[47] N. Korevaar, *An easy proof of the interior gradient bound for solutions to the prescribed mean curvature equation*, Nonlinear Functional Analysis and its Applications, Part 2 (Berkeley, CA, 1983), Proc. Sympos. Pure Math., vol. 45, Part 2, Amer. Math. Soc., Providence, RI (1986), 81–89.
[48] N.V. Krylov, *Nonlinear Elliptic and Parabolic Equations of the Second Order*, Mathematics and its Applications (Soviet Series), vol. 7, D. Reidel Publishing Co., Dordrecht (1987). Translated from the Russian by P.L. Buzytsky [P.L. Buzytskiĭ].
[49] N.V. Krylov and M.V. Safonov, *A property of the solutions of parabolic equations with measurable coefficients*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1) (1980), 161–175, 239.
[50] O.A. Ladyzhenskaya and N.N. Uraltseva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York–London (1968); Translated from the Russian by Scripta Technica, Inc., translation ed. L. Ehrenpreis.
[51] H.B. Lawson, Jr. and R. Osserman, *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*, Acta Math. **139** (1–2) (1977), 1–17.
[52] P. Li, *Harmonic sections of polynomial growth*, Math. Res. Lett. **4** (1) (1997), 35–44.
[53] P. Li, *Curvature and function theory on Riemannian manifolds*, Surveys in Differential Geometry, Surv. Differ. Geom., vol. VII, Int. Press, Somerville, MA (2000), 375–432.
[54] J. Liouville, C. R. Acad. Sci. Paris **19** (1844), 1262.
[55] J. Liouville, J. Math. Pures et Appl. **20** (1855), 201–208.
[56] M. Meier, *Liouville theorems for nonlinear elliptic equations and systems*, Manuscripta Math. **29** (2–4) (1979), 207–228.
[57] N.G. Meyers, *An* L^p*-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 189–206.
[58] E.J. Mickle, *A remark on a theorem of Serge Bernstein*, Proc. Amer. Math. Soc. **1** (1950), 86–89.
[59] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.
[60] W.F. Moss and J. Piepenbrink, *Positive solutions of elliptic equations*, Pacific J. Math. **75** (1) (1978), 219–226.
[61] Ch.H. Müntz, *Die Lösung des Plateauschen Problems über konvexen Bereichen*, Math. Ann. **94** (1) (1925), 53–96.
[62] E. Nelson, *A proof of Liouville's theorem*, Proc. Amer. Math. Soc. **12** (1961), 995.
[63] W.M. Ni, *On the elliptic equation* $\Delta u + K(x)\mathrm{e}^{2u} = 0$ *and conformal metrics with prescribed Gaussian curvatures*, Invent. Math. **66** (2) (1982), 343–352.
[64] J.C.C. Nitsche, *Elementary proof of Bernstein's theorem on minimal surfaces*, Ann. of Math. (2) **66** (1957), 543–544.
[65] R. Osserman, *On the inequality* $\Delta u \geqslant f(u)$, Pacific J. Math. **7** (1957), 1641–1647.
[66] R. Osserman, *Le théorème de Bernstein pour des systèmes*, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A571–A574.
[67] R. Osserman, *Minimal varieties*, Bull. Amer. Math. Soc. **75** (1969), 1092–1120.
[68] R. Osserman, *A Survey of Minimal Surfaces*, second ed., Dover Publications, Inc., New York (1986).
[69] A.V. Pogorelov, *On the improper convex affine hyperspheres*, Geom. Dedicata **1** (1) (1972), 33–46.
[70] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York (1984); Corrected reprint of the 1967 original.
[71] T. Rado, *Über den analytischen Charakter der Minimalflächen*, Math. Z. **24** (1) (1926), 321–327 (in German).
[72] R. Redheffer, *On the inequality* $\Delta u \geqslant f(u, |\mathrm{grad} u|)$, J. Math. Anal. Appl. **1** (1960), 277–299.
[73] L. Saloff-Coste, *Aspects of Sobolev-Type Inequalities*, London Mathematical Society Lecture Note Series, vol. 289, Cambridge University Press, Cambridge (2002).

[74] D.H. Sattinger, *Conformal metrics in* $\mathbb{R}^2$ *with prescribed curvature*, Indiana Univ. Math. J. **22** (1972/73), 1–4.

[75] O. Savin, *Phase transitions: Regularity of flat level sets*, Ph.D. thesis at the University of Texas at Austin (2003).

[76] J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Analyse Math. **4** (1955/56), 292–308.

[77] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302.

[78] J. Serrin, *Entire solutions of nonlinear Poisson equations*, Proc. London. Math. Soc. (3) **24** (1972), 348–366.

[79] J. Serrin and H.F. Weinberger, *Isolated singularities of solutions of linear elliptic equations*, Amer. J. Math. **88** (1966), 258–272.

[80] L. Simon, *The minimal surface equation*, Geometry, vol. V, Encyclopaedia Math. Sci., vol. 90, Springer, Berlin (1997), 239–272.

[81] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105.

[82] V.G. Tkachev, *Some estimates for the mean curvature of nonparametric surfaces defined over domains in* $\mathbb{R}^N$, Ukrain. Geom. Sb. **35** (1992), 135–150, 166; Translation in J. Math. Sci. **72** (4) (1994), 3250–3260.

[83] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1) (1984), 126–150.

[84] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[85] E. Valdinoci, B. Sciunzi and V.O. Savin, *Flat level set regularity of p-Laplace phase transitions*, Mem. Amer. Math. Soc. **182** (2006), 858.

CHAPTER 3

Similarity and Pseudosimilarity Solutions of Degenerate Boundary Layer Equations

Mohammed Guedda

Lamfa, CNRS UMR 6140, Université de Picardie Jules Verne, Faculté de Mathématiques et d'Informatique, 33, rue Saint-Leu 80039 Amiens, France
E-mail: Mohamed.guedda@mathinfo.u-picardie.fr

Contents

Chapter 3.1. Similarity solutions of degenerate boundary layer equations 120
1. Introduction 120
2. Similarity reduction 122
3. A shooting method and preliminary results 128
4. The effects of deceleration of the surface velocity 132
5. Global behavior of solutions 137
6. Conclusion 143

Chapter 3.2. Heat transfer in non-Newtonian fluid laminar boundary layer flow along a moving surface 144
1. Introduction and laminar boundary layer equations 144
2. Main result and "finite propagation" 146
3. Existence of a solution to GB problem and large η-behavior 148
4. Uniqueness results of the GB equation 153

Chapter 3.3. Exact solution for the heat transfer past a vertical plate with an applied magnetic field . 157
1. Introduction 157
2. The exact analytic form for negative ω 160
3. Blow-up profiles and pseudosimilarity solutions 162
   3.1. The unbounded solutions 162
   3.2. The missing solutions 164
   3.3. Summarizing remarks 173

Chapter 3.4. Mixed convection on a wedge embedded in a porous medium 176
1. Introduction 176
2. Pseudosimilarity or similarity reductions? 179
3. Eigensolutions 187
   3.1. Eigensolutions for $-1 \leqslant \beta_0 < 0$ 189

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 4
Edited by M. Chipot
© 2007 Elsevier B.V. All rights reserved

4. Asymptotic solution ($\varepsilon \gg 1$) 193
5. The limit case $\beta_0 = -\infty$ 194
Acknowledgements 198
References 198

Abstract

The aim of this work is to introduce, review and discuss similarity and pseudosimilarity solutions to a class of problems in the boundary layer theory. The boundary layer technique is encountered in many aspects of fluid dynamics and aerodynamics. This technique has been created for finding the flow of certain fluids. In 1904 Ludwig Prandtl[1] introduced the concept that, at high Reynolds numbers, the flow about a solid can be divided into two regions. A very thin region adjacent to the body in which the viscosity of the fluid exerts an influence on the motion of the fluid. In this region the velocity gradient $\partial u/\partial y$ is very large. This region is called the "boundary layer". In the remaining region the viscosity is negligible. The external flow is determined by the displacement of streamlines about the body and the pressure field is developed. Consequently, the full Navier–Stokes equations are simplified in the boundary layer. In fluid dynamical problems the main questions treated are the study of exact or particular solutions of the PDE approximations – called Prandtl's equations – to the Navier–Stokes equations. A class of these solutions has proved to play an important role for describing the behavior of the fluid in the boundary layer and in the development of mathematical theories as well as in numerical computational schemes. Most remarkable is that certain particular solutions, called similarity solutions, are exhibited by solving problems which are expressed in terms of a lower order PDEs or an ODEs. In fact, the idea of finding similarity or exact solutions is connected with the transformation of the PDEs to a set of equations which are easier to analyze, in general.

The model for non-Newtonian fluid are highly non-linear. Indeed, unlike the situation of a Newtonian fluid, which satisfies a linear relationship between its stress and its rate of strain, the elastic effects are relatively unimportant compared to viscous effect, and then a model that deals with these non-linear effects is required. In addition, the constant viscosity of Newtonian fluids is defined as the ratio between a given shear stress and the resultant rate of strain. For a simple unidirectional flow, such as the flow between two parallel plates, this relationship is expressed as:

$$\tau = \nu \frac{\mathrm{d}u}{\mathrm{d}y}.$$

This relation, known as Newtonian law of viscosity, is a mathematical statement, and there is no reason to believe that all real fluids, as polymer melts, paints and foams for example, should obey it. However one may define in a similar way an apparent viscosity for non-Newtonian fluid by

$$\tau = \mu_{\mathrm{app}} \frac{\mathrm{d}u}{\mathrm{d}y}.$$

[1] L. Prandtl, *Über Flüssigkeitsbewegungen bei sehr kleiner Reibung*, Verhandlg. III Intern. Math. Kongr. Heidelberg (1905), 484–491.

Here, however, the apparent viscosity is not constant but depends on the rate of strain. One of the ways to describe the flow behavior of a non-Newtonian fluid is the Ostwald–de Waele power-law model.[2,3]

$$\mu_{\text{app}} = \nu \left| \frac{\mathrm{d}u}{\mathrm{d}y} \right|^{n-1}, \quad \nu > 0,$$

where $n > 0$ is called the power-law index. The case $n < 1$ is referred to as the pseudo-plastic fluid, the case $n > 1$ is known as dilatant or shear-thickening fluids. The Newtonian fluid is, of course, a special case where the power-law index n is one.

The physical origin of a non-Newtonian behavior relates to the microstructure of the material. Polymer materials (solutions and melts) contains molecules having molecular weights of many hundreds of millions. In a fluid at rest, these long chain molecules will tend to exist in a coiled state and will usually entangle each other. During bulk deformation the molecules interact in a complex and a non-linear way. The resulting flow phenomena depend on this interaction, and in general will produce viscoelastic behavior.

The range of non-Newtonian fluid behavior exhibited by industrial liquids is very large. A broad description of the behavior in both steady and unsteady flow situations, together with mathematical model, can be found for example in Barnes,[4] Bird[5] and Tanner.[6]

In this work we consider some layer models of laminar non-Newtonian fluids, with a power-law viscosity, past a semi-infinite flat plate. With analogy of Blasius, Falkner–Skan, Goldstein and Mangler Methods, partial differential equations are transformed into an autonomous third-order non-linear degenerate equation. We establish the existence of a family of unbounded global solutions. The asymptotic behavior is also discussed. Some properties of solutions depend on the power-law index.

This work reflects the scientific interests of the author, during the last three years, on the boundary layer equations of non-Newtonian fluids. Many results presented here have not published before and are obtained with the helpful discussions and remarks of M. Benlahsen, B. Brighi, R. Kersner and A. Gmira. The final version of this book was written during the author's visit to Department of Mathematics of University of A. Essaadi, Tétouan (Maroc). This work has been partially supported in part by PAI No MA/05/116 for France–Maroc scientific cooperation and by Direction des Affaires Internationales, UPJV, Amiens France. The Newtonian case of Chapter 3.4 was presented at the Conference "Self-Similar Solutions in Nonlinear PDE's", Bedlow, Poland (2005). The author wishes to thank the organizers for the invitation and their kind hospitality.

[2] G. Astarita and G. Marrucci, *Principals of Non-Newtonian Fluid Mechanics*, McGraw-Hill (1974).

[3] A. Jabbarzadeh, J.D. Atkinson and R.I. Tanner, *Nanorheology of molecular thin films of n-hexadecane in couette shear flow by molecular dynamics simulation*, J. Non-Newtonian Fluid. Mech. **77** (1998), 53–78.

[4] H.A. Barnes, *A brief history of the yield stress*, Appl. Rheol. **9** (6) (1999), 262–266.

[5] R.B. Bird, *Non-Newtonian behavior of polymeric liquids*, Physica A: Statistical and Theoretical Physics **118** (1–3) (1988), 3–16.

[6] R.I. Tanner, *Engineering Rheology*, Calderon Press, Oxford (1988).

CHAPTER 3.1

Similarity Solutions of Degenerate Boundary Layer Equations

1. Introduction

We investigate a one layer model of laminar non-Newtonian fluids, with a "power-law" viscosity, past a semi-infinite vertical flat plate. Referred to a Cartesian system of coordinates Oxy, the boundary-layer approximation leads to the system of equations [20,93]

$$u\partial_x u + v\partial_y u = \partial_y\big(\mu\big(|\partial_y u|^2\big)\partial_y u\big) + u_e\partial_x u_e, \qquad \partial_x u + \partial_y v = 0, \tag{1.1}$$

where $y = 0$ is the plate. The x-axis is directed upwards to the plate and the y-axis is normal to it. The functions u and v are the velocity components along the x and y axes and $u_e \equiv u_e(x)$ is a given free stream velocity such that $u(x, y) \to u_e(x)$ as $y \to \infty$. Here it is assumed that the temperature-independent viscosity obeys the power law model;

$$\mu(\zeta) = \nu\zeta^{(n-1)/2}, \quad \nu > 0, \tag{1.2}$$

for all $\zeta > 0$, where the power-law index n is positive: the Ostwald–de Waele Model.

The boundary conditions to be applied to (1.1) are given by

$$\begin{aligned} &u(x, 0) = u_w x^m, \quad u_w > 0,\ m < 0, \\ &v(x, 0) = 0, \qquad u(x, y) \to u_e \equiv 0 \quad \text{as } y \to \infty. \end{aligned} \tag{1.3}$$

The constant u_w and the exponent m can be regarded as materially dependent constants [7]. The second equation in (1.1) (the continuity equation or the mass-conservation equation) ensures that $u\,\mathrm{d}y - v\,\mathrm{d}x$ is an exact differential, equal to $\mathrm{d}\psi$ say. Hence

$$u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x}.$$

A line in the fluid whose tangent is everywhere parallel to (u, v) is given by $\psi = const$. These lines are often called lines of flow or streamlines (paths of motion of the fluid particles). The function ψ is termed the stream function and is associated with the name of Lagrange. The function ψ satisfies

$$\partial_y\psi\,\partial_{yx}\psi - \partial_x\psi\,\partial_{yy}\psi = \nu\partial_y\big(|\partial_{yy}\psi|^{n-1}\partial_{yy}\psi\big), \tag{1.4}$$

and

$$\begin{aligned} &\partial_y \psi(x,0) = u_w x^m, \qquad \partial_x \psi(x,0) = 0, \\ &\partial_y \psi(x,y) \to 0 \quad \text{as } y \to \infty. \end{aligned} \tag{1.5}$$

Equation (1.4) can be written as

$$\partial_y \psi \partial_{yx} \psi - \partial_x \psi \partial_{yy} \psi = \nu \Delta_{n+1}(\partial_y \psi),$$

where Δ_{n+1} is the $\{n+1\}$-Laplace operator (see for example [50]).

If we rewrite (1.4) as

$$\partial_y \psi \partial_{yx} \psi - \partial_x \psi \partial_{yy} \psi = n\nu |\partial_{yy} \psi|^{n-1} \partial_{yyy} \psi,$$

then we see that this PDE is degenerate or singular in the sense that the coefficient $|\partial_{yy}\psi|^{n-1}$ of the highest-order derivative $\partial_{yyy}\psi$ vanishes or blows up at $\partial_{yy}\psi = 0$. It is well known that degenerate or singular equations do not have classical solutions, in general. So, one of the important questions is to obtain a family of exact or explicit solutions (traveling wave, similarity solutions, ...) which play a fundamental role in finding certain properties (asymptotic behavior, blow-up, finite speed of propagation, optimal regularity, ...) of the general solutions of the PDE under consideration. These solutions serve also as a laboratory for testing physical properties and mathematical theories.

In the present work we consider the problem of finding particular "exact" solutions to (1.4), (1.5). Research on this subject has a long history, which dates to the pioneering works for the Newtonian case ($n = 1$) by Blasius [17] and Falkner and Skan [37] in which the external velocity is given by

$$u_e(x) = u_\infty x^m, \quad m \in \mathbb{R} \ (u_\infty > 0). \tag{1.6}$$

Blasius with $m = 0$ and Falkner and Skan with $m \neq 0$ obtained the family of particular solutions of (1.1), where $n = 1$, such that the Prandtl velocity profile, $u(x,y)/u_e(x)$, depends only on a single variable $\eta = \eta(x,y)$. In particular, they showed that the stream function can be written as

$$\psi(x,y) = x^{(m+1)/2} f(\eta), \qquad \eta(x,y) = y x^{-(1-m)/2}, \tag{1.7}$$

or equivalently

$$\psi(x,y) = x^s f(\eta), \qquad \eta(x,y) = y x^{-r}, \tag{1.8}$$

with $s - r = m$ and for the Newtonian fluid $s + r = 1$.

Generally speaking, this expression has the form

$$\psi(x,y) = h(x) f\big(y\chi(x)\big), \tag{1.9}$$

where h and χ are smooth functions. From this, we see that

$$\psi(x, y) = \frac{h(x)}{h(x_0)}\psi\left(x_0, y\frac{\chi(x)}{\chi(x_0)}\right). \tag{1.10}$$

This means that $\psi(x, y)$, for x fixed, is similar to $\psi(x_0, y)$ at a certain x_0. A class of solutions (1.9) are called invariant or similarity solutions and $\eta = y\chi$ is called similarity variable. The PDE for ψ is reduced to the single third-order ODE – called the Falkner–Skan (FS) equation – satisfied by the function f;

$$f''' + \frac{m+1}{2}ff'' + m\left(1 - f'^2\right) = 0, \tag{1.11}$$

where the prime indicates the differentiation with respect to the variable η. The solution f, if exists, is called the shape function or the dimensionless stream function and its first derivative, after suitable normalization, represents the velocity parallel to the plate; $u(x, y) = u_e(x)f'(\eta)$. The broad goal of this chapter is to obtain similarity solutions to (1.4), (1.5) and their properties.

2. Similarity reduction

As it is said in the introduction, this work deals with similarity solutions to (1.1), (1.2). First, we shall characterize a family of external velocity such that the PDE

$$\partial_y\psi\partial^2_{xy}\psi - \partial_x\psi\partial^2_{yy}\psi = \nu\partial_y\left(\left|\partial^2_{yy}\psi\right|^{n-1}\partial^2_{yy}\psi\right) + u_e\partial_x u_e \tag{2.1}$$

accompanied by the boundary condition

$$u(x, y) \to u_e(x) \quad \text{as } y \to \infty, \tag{2.2}$$

may have similarity solutions or a similarity reduction. The main problems, arising in the study of similarity solutions, are related to the existence of the exponents s and r such that (1.8) scales (2.1) and to the rigorous study of the differential equation satisfied by the profile f, which is, in general, non-linear. For boundary-layer equations the classical approach for identifying s and r is the scaling transformation or the Lie-group. The essential idea is to seek a and b such that the function $\psi_\lambda : (x, y) \to \lambda^a\psi(\lambda^b x, \lambda y)$ is a solution to (2.1) at the time as ψ. The parameters a and b may depend on s, r and n.

Note that if ψ satisfies the invariance property $\psi_\lambda \equiv \psi$, where $b \neq 0$, we deduce

$$\psi\left(\lambda^b, \lambda y\right) = \lambda^{-a}\psi(1, y),$$

hence, by taking $x = \lambda^b$ and $Y = \lambda y$, we get

$$\psi(x, Y) = x^{-a/b}\psi\left(1, \frac{Y}{x^{1/b}}\right).$$

Therefore

$$\psi(x, y) = x^{-a/b}\psi\left(1, \frac{y}{x^{1/b}}\right) = x^s f\left(yx^{-r}\right),$$

where

$$s = -\frac{a}{b}, \qquad r = \frac{1}{b}.$$

This leads to (1.8).

It is expected that similarity solutions describe large-x behavior of general solutions to boundary layer equations [8]. Indeed, assume that the limit

$$\lim_{\lambda\to\infty} \lambda^a \psi\left(\lambda x, \lambda^b y\right) = \Psi(x, y)$$

exists in an appropriate sense. Then $x^a\psi(x, x^b y)$ tends to $\Psi(1, y)$ as x goes to infinity and Ψ satisfies the invariance property $\Psi_\lambda \equiv \Psi$.

The following theorem shows that the condition

$$u_e(x) = cx^m, \tag{2.3}$$

is necessary and sufficient for (2.1), (2.2) to admit similarity solutions under the form (1.8). First we prove the following.

THEOREM 2.1. *Assume that equation* (2.1) *has a similarity solution in the form* (1.8) *where*

$$s(2-n) + r(2n-1) = 1.$$

Then, there exists a constant c such that

$$u_e(x) = cx^m, \tag{2.4}$$

for all $x > 0$, *where* $m = s - r$.

PROOF. Let ψ be a stream function to (2.1) defined by (1.8) where $s(2-n) + r(2n-1) = 1$. Assume that $r \neq 0$. We choose $a = -s/r$, $b = 1/r$, and define $\psi_\lambda(x, y) = \lambda^a\psi(\lambda^b x, \lambda y)$. Hence $a(2-n) + b = 2n - 1$, $\psi \equiv \psi_\lambda$ and

$$L(\psi_\lambda)(x, y) = \lambda^{an+2n+1} L(\psi)\left(\lambda^b x, \lambda y\right)$$

for any $\lambda > 0$, where L is the operator defined by

$$L(\psi) = \partial_y\psi\,\partial^2_{xy}\psi - \partial_x\psi\,\partial^2_{yy}\psi - \nu\partial_y\left(\left|\partial^2_{yy}\psi\right|^{n-1}\partial^2_{yy}\psi\right).$$

According to the PDE (2.1) we deduce

$$h(x) = \lambda^{an+2n+1} h\big(\lambda^b x\big),$$

where

$$h = u_e \partial_x u_e.$$

In particular, for fixed $x_0 > 0$, $h(\lambda^b x_0) = \lambda^{-(an+2n+1)} h(x_0)$. Setting $x = \lambda^b x_0$ we see that

$$h(x) = x^{-(an+2n+1)/b} x_0^{(an+2n+1)/b} h(x_0).$$

Solving the equation $u_e \partial_x u_e = h$ yields

$$u_e^2(x) = c_1 x^{2m} + c_2, \tag{2.5}$$

for all $x > 0$, where $m = s - r$, c_1 and c_2 are constants if $r \neq 0$, since $-(an + 2n + 1)/b + 1 = 2(s - r)$.

For $r = 0$, hence $s = m = 1/(2 - n)$, $n \neq 2$, the new function

$$\psi_\lambda(x, y) = \lambda^s \psi\big(\lambda^{-1} x, y\big)$$

which is equivalent to ψ satisfies

$$L(\psi_\lambda)(x, y) = \lambda^{sn} L(\psi)\big(\lambda^{-1} x, y\big),$$

for any $\lambda > 0$. Arguing as in the case $r \neq 0$ one arrives at (2.5) with $m = 1/(2 - n)$.

Next, because $\lim_{y\to\infty} x^{2m} f'(yx^{-r})^2 = c_1 x^{2m} + c_2$, the function f'^2 has a finite limit at infinity, which is unique and given by $c_1 x^{2m} + c_2$. This is acceptable only for $c_2 = 0$. The proof is finished. □

REMARK 2.1. The above theorem indicates, in particular, that for a prescribed external velocity $u_e(x) = u_\infty x^m$, the real numbers s and r such that (1.8) scales (2.1) are given by

$$s = \frac{1 + m(2n - 1)}{n + 1}, \qquad r = \frac{1 + m(n - 2)}{n + 1}.$$

REMARK 2.2. We note also that if $u_e(x) = cx^m$, it is possible to obtain, in addition to similarity solutions, other particular solutions. For instance for $n = 1, u_e(x) = u_\infty x^{-1} (s = 0,\ r = 1)$ the function $\psi(x, y) = g(yx^{-1}) + c_2 \log x$ is also a solution which is not similarity (see [67] and Chapter 3.1). The profile g satisfies the following ordinary differential equation (ODE)

$$g''' + g'^2 + c_2 g'' - c_1^2 = 0.$$

REMARK 2.3. In passing, we stress that the boundary conditions on the plate are not required in the above analysis. This means, in particular, that our argument works even if we have a continuous stretching surface. In this case, if the stretching velocity is given by $v(x,0) = v_w x^j$, $v_w \neq 0$ and $u(x,0) = u_w x^z$, we deduce $j = s-1$, $z = s-r$ which is the same exponent as that of the external velocity and the profile f satisfies $f'(0) = u_w$, $sf(0) = -v_w$.

In the next analysis we shall identify s and r such that (1.4) and (1.5) have a similarity reduction (a necessary condition).

Here, it is assumed that $u_e \equiv 0$ and $u(x,0) = u_w x^m$. Thus we are looking for the stream function in the "standard" form

$$\psi(x,y) = bx^s f(\eta), \quad \eta = \lambda \frac{y}{x^r},$$

where $s, r > 0, b > 0$ and $\lambda > 0$ such that $b\lambda = u_w$ and $\nu\lambda^{2(n-1)} u_w^{n-2} = 1$.

Using (1.4) and (1.5) we find that the profile function f satisfies

$$x^{(s-2r)n-r}\left(|f''|^{n-1}f''\right)' + sx^{2(s-r)+1} ff'' = (s-r)x^{2(s-r)+1} f'^2. \tag{2.6}$$

Equation (2.7) is an ODE if and only if

$$m = s - r, \qquad s(2-n) + r(2n-1) = 1,$$

the scaling relation, i.e.

$$s = \frac{1+m(2n-1)}{n+1}, \qquad r = \frac{1+m(n-2)}{n+1}.$$

So, the function f satisfies the following boundary value problem

$$\left(|f''|^{n-1}f''\right)' + sff'' = (s-r)f'^2, \tag{2.7}$$

and

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = \lim_{\eta\to\infty} f'(\eta) = 0, \tag{2.8}$$

Equation (2.7) with $m = 0$ ($s = r = 1/(n+1)$) is referred to as Generalized Blasius equation; the GB equation:

$$\left(|f''|^{n-1}f''\right)' + \frac{1}{n+1} ff'' = 0. \tag{2.9}$$

The existence, uniqueness and η-behavior of solutions to (2.9) will be investigated in Chapter 3.2. So, we suppose $m \neq 0$ and we shall be concerned with a class of the autonomous third order non-linear differential equations

$$\left(|f''|^{n-1}f''\right)' + \frac{1+m(2n-1)}{n+1} ff'' - mf'^2 = 0, \quad \text{on } (0,\infty), \tag{2.10}$$

where $-1/(3n) < m < 0$ and $n > 0$.

For the Newtonian case equation (2.10) reads

$$f''' + \frac{1+m}{2} f f'' - m f'^2 = 0. \tag{2.11}$$

The above equation with suitable boundary conditions, also arises in industrial manufacturing processes [7], in the excitation of liquid metals when placed in a high-frequency magnetic field [77] and in the context of boundary-layer flow on stretching permeable surfaces with mass transfer parameter $a \neq 0$ [28,68]. In the last situation the boundary conditions are

$$f(0) = a, \quad f'(0) = 1, \quad f'(\infty) = 0. \tag{2.12}$$

The real a is referred also to as the suction/injection parameter. The case $a > 0$ corresponds to the suction and $a < 0$ to the injection of the fluid. In connection with the free convection, along an impermeable vertical flat plate embedded in a fluid-saturated porous medium, the real a is 0 and the range for which the problem has some physical meaning is $-1/3 \leqslant m \leqslant 1$ (see for example [30,12]).

Results concerning problem (2.11), (2.12), where $a = 0$, can be found in [30] in which the numerical solution has been performed in the case where $-1/3 < m < 0$. For $m > -1/2$ numerical investigations are in [58] and [6]. The mathematical analysis is also considered in [58]. The authors showed the non-existence of solutions to (2.11), (2.12) if $m < -1/2$ satisfying

$$\lim_{\eta \to \infty} f' f^2(\eta) = 0. \tag{2.13}$$

A similar argument is used in [28] to prove the non-existence of non-trivial solutions to (2.11) satisfying

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 0, \tag{2.14}$$

where $m \neq -1/2$.

Recently some new results have been obtained in [13,12,22,25]. The authors showed non-existence of solutions to (2.11), (2.12) for $m \leqslant -1/2$ without condition (2.13). They also proved that this problem has an infinite number of solutions when $m = -1/3$, one bounded solution for $m \geqslant -1/3$ and uniqueness holds for $0 \leqslant m \leqslant 1/3$.

Very recently, the existence of multiple solutions to (2.11), (2.12) is considered by Guedda [44,46]. By using a shooting argument, it is shown that there are multiple unbounded solutions to (2.11), (2.12), with $f'(0) = b \geqslant 0$ instead of $f'(0) = 1$, for any $a \in \mathbb{R}$ and any fixed $m \in (-1/3, 0)$. Moreover, it is proved that, as η tends to plus infinity, f behaves like $\eta^{(1+m)/(1-m)}$.

The problem to which we now address is to solve equation (2.10) subject to the boundary conditions (permeable surface)

$$f(0) = a, \quad f'(0) = b \geqslant 0, \quad f'(\infty) = 0, \tag{2.15}$$

where $a \in \mathbb{R}$ and $b \geqslant 0$. It should be noticed that if f is a solution of (2.10) then it is for

$$\eta \to \kappa f(\aleph \eta),$$

for any $\kappa, \aleph > 0$, such that $\kappa^{n-2}\aleph^{2n-1} = 1$. This invariance property allows to reduce (2.15) to $b = 0$ and $b = 1$. This problem corresponds to some imposed injection or suction (permeable flat plate) which is initiated by Sakiadis [91] and Ericson et al. [36]. The y direction velocity component is given by $v(x, y) = v_w x^{(m(2n-1)-n)/(n+1)}$.

The analysis is devoted to study the existence, non-uniqueness and the asymptotic behavior at infinity of solutions. It may be noted that the global behavior and the question of uniqueness of similarity solutions have not been investigated before for the non-Newtonian fluids. On the other hand, a large number of papers were devoted to numerical similarity solutions. The numerical results were produced for the equation

$$n|f''|^{n-1}f''' + \frac{1+m(2n-1)}{1+n} ff'' - mf'^2 = 0$$

instead of (2.10). Nevertheless, any solution to (2.10) is not necessarily of class C^3. We note, in passing, that for $n \neq 1$, equation (2.10) can be either degenerate or singular at the point η_0 where $f''(\eta_0) = 0$. The existence of the real η_0 is done for $f''(0) > 0$ and $n > 1$. We shall see also that f''' is not bounded at this point if $n > 1$; that is f is not classical. So, by a solution to (2.10), (2.15) we mean a function $f \in C^2(0, \infty)$ such that $|f''|^{n-1}f'' \in C^1(0, \infty)$, $f'(\infty) = 0$ and the following

$$\begin{aligned} &|f''|^{n-1}f''(\eta) + \kappa f'(\eta) f(\eta) \\ &\quad = |f''|^{n-1}f''(0) + \kappa ab + \frac{1+3mn}{1+n} \int_0^\eta f'(s)^2 \, \mathrm{d}s, \end{aligned} \tag{2.16}$$

where

$$\kappa = \frac{1+m(2n-1)}{1+n}, \tag{2.17}$$

holds for all $\eta > 0$. Note also that any solution is classical if $0 < n \leqslant 1$ and for $n > 1$ any solution is classical on any interval where the second derivative is of constant sign.

The purpose of this chapter is to show that problem (2.10), (2.15) has a multiple unbounded solutions and these solutions behave like $\eta^{(1+m(2n-1))/(1+m(n-2))}$, as $\eta \to \infty$. Returning to the original (physical) problem we find that, for $-1 < m(2n-1)$ and $-1/(3n) \leqslant m < 0$ the stream function satisfies

$$\psi(x, y) \sim y^{(1+m(2n-1))/(1+m(n-2))} \quad \text{as } yx^{(m(2-n)-1)/(n+1)} \to \infty.$$

The plan of the present chapter is as follows. In Section 2, we consider problem (2.10), (2.15) with $f''(0) = \gamma$ instead of $f'(\infty) = 0$ and establish some preliminaries which will be needed later in the proofs. Thereafter, in Section 3, we prove that, for each selected γ,

the initial value problem has a unique global unbounded solution and the first derivative of this solution tends to 0 as η goes to infinity. Section 4 deals with the asymptotic behavior results for any possible global unbounded solution.

3. A shooting method and preliminary results

In order to prove the existence of solutions we use the so-called shooting method: the condition at infinity is replaced by one at $\eta = 0$; $f''(0) = \gamma$. Therefore, we consider the initial value problem

$$\begin{cases} (|f''|^{n-1} f'')' + \frac{1+m(2n-1)}{n+1} ff'' - mf'^2 = 0, \quad \eta > 0, \\ f(0) = a, \quad f'(0) = b, \quad f''(0) = \gamma, \end{cases} \tag{3.1}$$

in which $n > 0$, $a \in \mathbb{R}$, $b \geqslant 0$, $-1/(3n) \leqslant m < 0$ and $\gamma \in \mathbb{R}$ is the shooting parameter. The real number γ has also a physical meaning, since it is proportional to the drag experienced by the fluid [93, p. 159]. This initial value problem has the following equivalent system

$$\begin{cases} f' = g, \\ g' = |h|^{(1-n)/n} h, \\ h' = -\frac{1+m(2n-1)}{n+1} f|h|^{(1-n)/n} h + mg^2, \end{cases}$$

with the initial condition

$$\big(f(0), g(0), h(0)\big) = \big(a, b, |\gamma|^{n-1}\gamma\big).$$

By the usual theorems the above problem has local solutions and they are uniquely determined by γ for $\gamma \neq 0$. The solution is defined for all $\eta \geqslant 0$ or else over the maximal interval of existence $(0, \eta_\gamma)$, $\eta_\gamma < \infty$. The real number η_γ (the existence time) is characterized by

$$\lim_{\eta \uparrow \eta_\gamma} \big|f(\eta)\big| + \big|g(\eta)\big| + \big|h(\eta)\big| = \infty,$$

if η_γ is finite.

Let us denote the solution to (3.1) by f_γ. The function f_γ satisfies identity (2.16) for all $\eta < \eta_\gamma$; that is

$$|f_\gamma''|^{n-1} f_\gamma''(\eta) + \kappa f_\gamma'(\eta) f_\gamma(\eta) = |\gamma|^{n-1}\gamma + \kappa ab + \frac{1+3mn}{n+1} \int_0^\eta f_\gamma'(s)^2 \, \mathrm{d}s, \quad \forall \eta < \eta_\gamma, \tag{3.2}$$

where the positive real number κ is given by (2.17). We shall vary γ such that f_γ is global ($\eta_\gamma = \infty$) and satisfies the desired condition at infinity. It is clear from the above system that $f_\gamma \in C^3[0, \eta_\gamma)$ when $0 < n \leqslant 1$. Let us note that if we require classical solution of

(3.1) for $n > 1$, that is $f \in C^3$, it is possible that f ceases to exist at some $\eta_1 < \infty$ and such that f, f' and f'' remain bounded on $[0, \eta_1)$. More precisely we have the following result.

PROPOSITION 3.1. *Let f_γ be the local solution to* (3.1), *where $n > 1$ and $\gamma \neq 0$. Assume that there exists $\eta_0 \in (0, \eta_\gamma)$ such that $f_\gamma''(\eta_0) = 0$. Then $\gamma > 0$ and f_γ''' is unbounded on* $(0, \eta_0)$.

PROOF. First assume that $\gamma < 0$. Therefore f_γ satisfies, for some small ε,

$$n|f_\gamma''|^{n-1} f_\gamma''' + \kappa f_\gamma f_\gamma'' - m f_\gamma'^2 = 0, \quad \forall 0 < \eta < \varepsilon, \tag{3.3}$$

which leads to

$$\left(f_\gamma'' \mathrm{e}^F\right)' = \frac{1}{n} m \mathrm{e}^F |f_\gamma''|^{1-n} f_\gamma'^2, \quad t \in (0, \varepsilon), \tag{3.4}$$

where

$$F(\eta) = \frac{\kappa}{n} \int_0^\eta f_\gamma |f_\gamma''|^{1-n}(s)\, \mathrm{d}s.$$

From this, one sees that the function $f_\gamma'' \mathrm{e}^F$ decreases, and so $f_\gamma''(\eta) < 0$ for all $\eta \in [0, \eta_\gamma)$, a contradiction. Then $\gamma > 0$. In fact we have $f_\gamma'' > 0$ and equation (3.3) is satisfied on $(0, \eta_0)$. Suppose that f_γ''' is bounded on $(0, \eta_0)$. From the equation of f_γ we deduce that $f_\gamma'(\eta_0) = 0$, and this is not possible, since $f_\gamma'(0) > b$. □

The above proof indicates, in particular, that $f_\gamma'' < 0$ on $(0, \eta_\gamma)$ for $\gamma < 0$. In this case $f_\gamma \in C^\infty([0, \eta_\gamma))$. For the case $\gamma > 0$ the function f_γ is not classical (see the next section).

PROPOSITION 3.2. *Assume $\gamma \neq 0$. Let f_γ be the local solution to* (3.1), *where $n \geqslant 1$, such that $\eta_\gamma < \infty$. Then* $\lim_{\eta \uparrow \eta_\gamma} f_\gamma(\eta) = -\infty$.

PROOF. First we show that $\sup_{[0,\eta_\gamma)} |f_\gamma(\eta)| = \infty$. Suppose not and $f_\gamma''(\eta_0) = 0$ holds, for some $\eta_0 \in (0, \eta_\gamma)$. From (3.2) we infer

$$-(-f_\gamma'')^n(\eta) + \kappa f_\gamma'(\eta) f_\gamma(\eta) = \kappa f_\gamma'(\eta_0) f_\gamma(\eta_0) + \frac{1 + 3mn}{n+1} \int_{\eta_0}^\eta f_\gamma'(s)^2\, \mathrm{d}s,$$
$$\forall \eta_0 < \eta < \eta_\gamma.$$

Hence

$$\frac{\kappa}{2} f_\gamma^2(\eta) - \kappa f_\gamma'(\eta_0) f_\gamma(\eta_0)(\eta - \eta_0) - \frac{\kappa}{2} f_\gamma^2(\eta_0)$$
$$= \frac{1 + 3mn}{n+1} \int_{\eta_0}^\eta \int_{\eta_0}^\tau f_\gamma'^2(s)\, \mathrm{d}s\, \mathrm{d}\tau + \int_{\eta_0}^\eta \left(-f_\gamma''(s)\right)^n \mathrm{d}s.$$

Because the right-hand side of the above is positive and monotonic increasing with respect to η, the function f_γ has a finite limit as $\eta \to \eta_\gamma$. Therefore $(-f''_\gamma)^n$ and then f''_γ are integrable on (η_0, η_γ), since $n \geqslant 1$. Consequently the function f'_γ is bounded. Next we use (3.2) to deduce that f''_γ is bounded too. A contradiction. It remains to prove that the hypothesis $f''_\gamma > 0$ on $(0, \eta_\gamma)$ leads also to a contradiction. In fact, in this case, we know that f_γ is classical and satisfies, using (3.3),

$$(f''_\gamma)^{n-2} f'''_\gamma \leqslant -\frac{\kappa}{n} f_\gamma,$$

and then

$$(f''_\gamma)^{n-2} f'''_\gamma \leqslant \frac{\kappa}{n} \sup_{\eta\in[0,\eta_\gamma)} \big|f_\gamma(\eta)\big|.$$

Therefore f''_γ and (then) f'_γ are bounded, a contradiction.

Next, since f_γ is a monotonic function on (τ, η_γ), for some $0 < \tau < \eta_\gamma$, we deduce that $|f_\gamma(\eta)|$ goes to ∞ as $\eta \to \eta_\gamma$. Finally, to show that $f_\gamma(\eta) \to -\infty$ as $\eta \to \eta_\gamma$ we suppose not; that is $\lim_{\eta\to\eta_\gamma} f_\gamma(\eta) = \infty$. Hence the functions f_γ and f'_γ are positive on (τ, η_γ). On the other hand, using the equation of f_γ we can see that the function

$$E = \frac{n}{n+1} |f''_\gamma|^{n+1} - \frac{m}{3} f'^3_\gamma, \tag{3.5}$$

satisfies

$$E' = -\kappa f_\gamma f''^2_\gamma \leqslant 0.$$

Consequently, f''_γ and f'_γ are bounded. It follows from this that f_γ is bounded. A contradiction. In conclusion, if η_γ is finite the function f_γ goes to minus infinity as η approaches η_γ. □

The discussion for $n < 1$ is more difficult. The following result eliminates the possibility that f_γ is global, for any $b < 0$ and any $\gamma < 0$, in the case $n < 2$.

PROPOSITION 3.3. *Assume* $0 < n < 2$. *Let* f_γ *be the local solution to* (3.1), *where* $b < 0$. *For any* $\gamma < 0$ η_γ *is finite.*

PROOF. We assume for the sake of contradiction that f_γ is global; that is $\eta_\gamma = \infty$. Because $f'_\gamma(\eta) < b$, for all $\eta > 0$, $f_\gamma(\eta) < a + b\eta$ and tends to $-\infty$ for $\eta \to \infty$. Using this we deduce that the function E, is increasing on $(-a/b, \infty)$, therefore, the following holds

$$\big|f''_\gamma(\eta)\big|^{n+1} \geqslant \frac{m(n+1)}{3n}\big(f'_\gamma(\eta)^3 - b^3\big) + \big|f''_\gamma(-a/b)\big|^{n+1},$$

for all $\eta \geqslant -a/b$. Setting $g = -f_\gamma$, one readily verifies that g is positive on $(-a/b, \infty)$, increasing and $g(\eta), g'(\eta)$ tend to ∞ with η. It is concluded that, for some positive constant, C,

$$g''(\eta) \geqslant C g'(\eta)^{3/(n+1)},$$

for η large. A simple analysis of this inequality implies that g' is not global, since the exponent $3/(n+1)$ is greater than one. A contradiction. □

REMARK 3.1. The condition $n < 2$ can hardly be avoided since for any $T > 0, 0 \leqslant n \neq 2$, the function

$$f_\star(\eta) = A(T - c\eta)^{-\tau}, \tag{3.6}$$

where

$$\tau = \frac{1-2n}{n-2}, \qquad c^{2n-1} = \frac{A|A|^{1-n}}{n(n-2)|\tau(1+\tau)|^{n-1}(1+\tau)(\tau+2)}, \tag{3.7}$$

satisfies $(3.1)_1$. Therefore for $n > 2$ and $A < 0$ we have $c < 0$ and then $f_\star$ is negative and global. Moreover $f_\star$, $f'_\star$ and $f''_\star$ tend to $-\infty$ as η goes to infinity.

If $1/2 < n < 2$ and $A < 0$ the function $f_\star$ blows up at $\eta = T/c$. However if $A > 0$ this function is positive and tends to 0 at infinity. The situation is different and important if $A > 0$ and $n > 2$ or $n < 1/2$. In this case the function

$$f_\star(\eta) = A(T - c\eta)_+^{-\tau},$$

where $(\cdot)_+ = \max\{\cdot, 0\}$, is global and supported by $[0, T/c]$.

For the critical case $n = 2$ equation $(3.1)_1$ has a global solution of the form

$$f(\eta) = A\mathrm{e}^{\tau\eta},$$

where $A \neq 0$ and $\tau^3 = -\frac{A}{|A|}\frac{1}{6}$. Observe that if $A < 0$ this solution is negative and goes to $-\infty$ as $t \to \infty$.

The last result of this section provides a non-existence result for $m < 0, m(2n-1) \leqslant -1$ and $a \geqslant 0$. Let us consider the last possibility more closely.

THEOREM 3.1. *Let $a \geqslant 0$. Assume $m < 0, m(2n-1) \leqslant -1$. Let f be a solution to problem* (2.10), (2.15). *Then $f \equiv a$.*

PROOF. Suppose that (2.10), (2.15) has a solution $f \not\equiv a$. Because $f'(0) \geqslant 0$ and $f'(\infty) = 0$ it is easily seen that $f''(\eta) < 0$ on (η_0, ∞), for some $\eta_0 > 0$, and then $f'(\eta) > 0$ for any η large. Next, since $m < 0$ f satisfies

$$\left(|f''|^{n-1} f''\right)' + \kappa f f'' \leqslant 0. \tag{3.8}$$

Assume first that $f(\eta) > 0$ for η large. In this case we deduce that the function $|f''|^{n-1}f''$ is monotonic decreasing and tends to 0 at infinity. Therefore f'' is positive for η large, a contradiction. Next we assume that $f(\eta) < 0$ for all $\eta \geqslant \eta_1$, η_1 large. Since $f(0) \geqslant 0$ and $f'(0) > 0$ the function f has a positive local maximum at some $\eta_2 < \eta_1$. Hence $f''(\eta) < 0$ on $[\eta_2, \infty)$ and $f'(\eta_2) = 0$. We again obtain a contradiction since $f'(\infty) = 0$. □

4. The effects of deceleration of the surface velocity

Because equation (2.11) is of the third order, we may suspect that (2.11) has a unique solution satisfying (2.12). This holds in the case $0 \leqslant m \leqslant 1$, see [12]. For the case $m < 0$ it is shown that the problem has multiple solutions [44,46]. It is reported in [93, p. 99] that the solutions to Navier–Stokes equations do not have to be unique for given initial and boundary conditions. The non-linearity of the differential equations and the variation of fluid mechanical parameters can lead to bifurcation phenomena and thus to multiple solutions.

In a different context Guedda and Veron [50] studied the structure of solutions to $-(|u_x|^{n-1}u_x)_x + f(u) = \lambda|u|^{n-1}u$ on $(0, 1)$ such that $u(0) = u(1) = 0$ where $n > 0$. Under some assumptions it is proved that the solutions with k zeros are unique when $n \leqslant 1$ and are not unique for $n > 1$ (secondary bifurcation).

Returning to our occupation, can we get new phenomena if the boundary-layer equations contains the $\{n + 1\}$-Laplace operator?

So, the object of this section is to investigate in detail how the multiple solutions is related to the parameters n and m. From mathematical viewpoint problem (2.10), (2.15) will be considered for $a, b \in \mathbb{R}$. We shall exhibit sufficient conditions for the existence of multiple solutions. In particular, we prove that the local solution f_γ is global and satisfies the condition $f'_\gamma(\infty) = 0$, for appropriate $\gamma = f''(0)$. In the hydrodynamical problem the initial value of f'' determines the skin friction on the plate. It can be inferred from Section 2 that if $\gamma < 0$, $b < 0$ and $n < 2$, then f_γ is not global. So, in the first part of the present section, we restrict our analysis to the case $b \geqslant 0$. We show that, for any γ such that $|\gamma|^{n-1}\gamma > -((1 + m(2n - 1))/(n + 1))ab$, f_γ exists on the entire positive axis $\mathbb{R}^+$ and satisfies $f'_\gamma(\infty) = 0$. For our analysis, we need to distinguish two cases for the initial data $a = f_\gamma(0)$; namely $a \geqslant 0$ and $a < 0$.

We begin by a simple observation that, if $1 + 3mn > 0$ and

$$|\gamma|^{n-1}\gamma > -\frac{1 + m(2n - 1)}{n + 1}ab,$$

identity (2.16) yields the important fact that f_γ cannot have a local maximum. Thus, since a and b are positive f_γ is positive and monotonic increasing. To show that f_γ is global ($\eta_\gamma = \infty$) it suffices to show that f_γ remains bounded on any bounded interval $[0, T]$. Let us consider the Lyapunov function E for f_γ defined by (3.5). Since

$$E' = -\kappa f_\gamma f''^2_\gamma \leqslant 0,$$

we have

$$\frac{n}{n+1}\left|f_\gamma''(\eta)\right|^{n+1} - \frac{m}{3}f_\gamma'(\eta)^3 \leqslant \frac{n}{n+1}|\gamma|^{n+1} - \frac{m}{3}b^3, \quad \forall \eta < \eta_\gamma.$$

This in turn implies that $f_\gamma''(\eta)$, $f_\gamma'(\eta)$ and then $f_\gamma(\eta)$ are bounded in $[0, T]$. Hence f_γ is global. In order to obtain that $f_\gamma'(\eta)$ tends to zero at infinity we recall first that f_γ' and f_γ'' are bounded, by using the energy E and f_γ' is monotonic on (η_1, ∞), η_1 large enough. Hence there exists $l \in \mathbb{R}_+$ such that

$$\lim_{\eta\to\infty} f_\gamma'(\eta) = l.$$

Moreover there exists a sequence (η_j) tending to ∞ with j such that $\lim_{j\to\infty} f_\gamma''(\eta_j) = 0$. It follows that E has a finite limit at infinity, E_∞, say. And this limit is given by $E_\infty = \lim_{\eta\to\infty} E(\eta) = \lim_{j\to\infty} E(\eta_j)$; that is $E_\infty = -(m/3)l^3$ and then $\lim_{\eta\to\infty} f_\gamma''(\eta) = 0$, by using again E.

Suppose that $l > 0$. Together with identity (2.16) we get, as η approaches infinity

$$|f_\gamma''|^{n-1} f_\gamma''(\eta) = -\kappa l^2\eta + (\kappa + m)l^2\eta + o(\eta),$$
$$|f_\gamma''|^{n-1} f_\gamma''(\eta) = ml^2\eta + o(\eta),$$

as $\eta \to \infty$. This is only possible if $ml^2 = 0$. Then $l = 0$. This show that f_γ satisfies the problem (2.10), (2.15). Using again identity (2.16) it may be shown that f_γ is not bounded.

Our next task will be to determine solutions for $a < 0$. Let f_γ be the local solution to (3.1) where $a < 0$, $b \geqslant 0$ and $-1/(3n) < m < 0$. Since f_γ cannot have a local maximum, we have two possibilities: either there exists $\eta_1 > 0$, such that $f_\gamma(\eta) < 0$ on $(0, \eta_1)$, f_γ vanishes at η_1 and remains positive after this point, or $f_\gamma(\eta) < 0$ for all $\eta \geqslant 0$. We will assume that $f_\gamma(\eta)$ is negative for all $\eta \geqslant 0$ and deduce a contradiction. Because f_γ is monotonic increasing and negative f_γ is global, bounded and tends to a finite limit in $(a, 0]$. Hence, there exists a sequence (η_j) converging to infinity with j such that $f_\gamma'(\eta_j) \to 0$, as $j \to \infty$. On the other hand, using (2.16) one sees $f_\gamma'' > 0$. Therefore $f_\gamma'(\eta) \to 0$ as $\eta \to \infty$. But this implies that $f_\gamma'(\eta) < 0$ for all $\eta \geqslant 0$. A contradiction. Hence f_γ has exactly one zero, say η_1. Next, since the ordinary differential equation in (3.1) is autonomous the function

$$h(\eta) = f_\gamma(\eta + \eta_1)$$

is a solution and satisfies

$$h(0) \geqslant 0, \qquad h''(0) > -\kappa h(0)h'(0).$$

Finally we argue as before to deduce that h is global, unbounded and $h'(\eta)$ tends to 0 as $\eta \to \infty$. Therefore f_γ is a solution to problem (2.10), (2.15). Altogether we have proved

THEOREM 4.1. *Let $a \in \mathbb{R}$, $b \in \mathbb{R}^+$ and $-1/(3n) < m < 0$. For any γ such that $|\gamma|^{n-1}\gamma > -\kappa ab$, where $\kappa = (1 + m(2n-1))/(n+1)$, there exists a unique global solution, f_γ,*

to (3.1). *Moreover* $f_\gamma(\eta)$ *goes to infinity with* η, *and the first and the second derivative of* f_γ *tend to* 0 *as* η *approaches infinity. Moreover, if* (*in addition*) $\gamma \leqslant 0$ *and* $b > 0$ *we have*

$$0 \leqslant f'_\gamma \leqslant b; \tag{4.1}$$

that is f_γ *is a physical solution* [12].

Here by a physical solution we mean a solution f satisfying (4.1) if $b > 0$.

REMARK 4.1. Since γ is arbitrary we deduce that problem (2.10), (2.15) has an infinite number of global unbounded solutions.

We consider next the existence of multiple solutions for $b < 0$, $a > 0$ and $n \geqslant 1$. In this case the real number γ takes place in $(0, \infty)$ and satisfies

$$a\gamma^n - \frac{1}{2}b^2\gamma^{n-1} + \kappa a^2 b > 0. \tag{4.2}$$

If f_γ is the local solution to (3.1), there exists a real number $\eta_0 > 0$ such that f_γ is positive, decreasing and convex in $(0, \eta_0)$. Define $\Lambda \in [\eta_0, \infty]$ by

$$\Lambda = \sup\{\eta:\ f_\gamma(s) > 0,\ f'_\gamma(s) < 0,\ f''_\gamma(s) > 0,\ \text{for all } s \in (0, \eta)\}.$$

Assume that $\Lambda = \infty$. Hence f_γ has a finite limit at infinity, $f'_\gamma(\eta)$ and $f''_\gamma(\eta)$ tend to 0 as $\eta \to \infty$. On the other hand, it follows from (3.1) that the function

$$H = f_\gamma |f''_\gamma|^{n-1} f''_\gamma - \frac{1}{2} {f'_\gamma}^2 |f''_\gamma|^{n-1} + \kappa f_\gamma^2 f'_\gamma$$

satisfies

$$H' = f_\gamma {f'_\gamma}^2 \left[m + 2\kappa + \frac{\kappa(n-1)}{2n} \right] - \frac{m(n-1)}{2n} {f'_\gamma}^4 {f''_\gamma}^{-1}.$$

Therefore, H is monotonic increasing on $(0, \infty)$. In particular $H(0) < H(1) < H(\eta)$, for all $\eta > 1$. Letting $\eta \to \infty$ we see that

$$H(0) < \lim_{\eta \to \infty} H(\eta) = 0,$$

$$a\gamma^n - \frac{1}{2}b^2\gamma^{n-1} + \kappa a^2 b < 0,$$

a contradiction. It follows that Λ is finite. Note that the assumption

$$f_\gamma(\Lambda) = 0 \quad \text{or} \quad f''_\gamma(\Lambda) = 0$$

also leads to a contradiction. In conclusion, if condition (4.2) holds the function f_γ has a local positive minimum at some $\eta_1 > 0$. In the same way as before the function $h(\eta) = f_\gamma(\eta + \eta_1)$ is global, unbounded and satisfies $h'(\infty) = h''(\infty) = 0$. We capture this conclusion in the following.

THEOREM 4.2. *Let* $b < 0$, $a > 0$, $-1/(3n) < m < 0$ *and* $n \geqslant 1$. *For any* $\gamma > 0$ *such that* (4.2) *holds the unique local solution*, f_γ, *to* (3.1), *is global*, *unbounded and* $\lim_{\eta\to\infty} f'_\gamma(\eta) = 0$.

The argument used to prove Theorem 4.1 can also be applied to prove the existence of physical solution in the case $n \leqslant 1$ and $m < -1/(3n)$. More precisely we have

THEOREM 4.3. *Let* $a > 0$, $b > 0$, $1/3 < n \leqslant 1$ *and*

$$-\frac{5n-1}{12n^2-5n+1} < m < -\frac{1}{3n}. \tag{4.3}$$

For any $\gamma < 0$ *such that*

$$\kappa a^2 b \geqslant a|\gamma|^n + \frac{1}{2}b^2|\gamma|^{n-1}, \tag{4.4}$$

the unique local solution, f_γ, *to* (3.1), *is global and unbounded. Moreover*,

$$\lim_{\eta\to\infty} f'_\gamma(\eta) = 0,$$

and

$$0 < f'_\gamma(\eta) < b,$$

for all $\eta > 0$.

Note that the real number κ is positive and the first inequality in (4.3) can be read as $m + 2\kappa + \kappa(n-1)/(2n) > 0$.

PROOF. The proof of the theorem will be conducted with the help of the function H as before. Let f_γ be the local solution to (3.1). Because $b > 0$, f_γ is increasing in $(0, \eta_0)$, η_0 small. Assume that f_γ has a local maximum at the point η_0, and then $f'_\gamma(\eta_0) = 0$, $f''_\gamma(\eta_0) < 0$. Since the function H is monotonic increasing on $(0, \eta_0)$ we get

$$H(0) < f_\gamma(\eta_0)\left|f''_\gamma(\eta_0)\right|^{n-1} f''_\gamma(\eta_0) < 0,$$

where,

$$H(0) = \kappa a^2 b - a|\gamma|^{n-1} - \frac{1}{2}b^2|\gamma|^n,$$

and the latter implies that

$$\kappa a^2 b < a|\gamma|^{n-1} + \frac{1}{2} b^2 |\gamma|^n,$$

which contradicts (4.4). Therefore $f'_\gamma > 0$. Using again the function E we deduce that f_γ is global, $f'_\gamma(\eta) \to l$ and $f''_\gamma(\eta)$ tends to 0 as $\eta \to \infty$. To show that $f_\gamma(\eta)$ tends to infinity with η we assume that f_γ is bounded and use the function H to get,

$$-\frac{1}{2} \lim_{\eta\to\infty} f'_\gamma(\eta)^2 \left|f''_\gamma(\eta)\right|^{n-1}$$
$$= H(0) + \int_0^\infty \left[f f'^2_\gamma \left[m + 2\kappa + \frac{\kappa(n-1)}{2n} \right] - \frac{m(n-1)}{2n} f'^4_\gamma f''^{-1}_\gamma \right] \mathrm{d}s.$$

The left-hand side takes place in $[-\infty, 0]$ while the right-hand side is positive (recall that $n < 1$) or infinite. A contradiction. The remainder of the proof follows easily. □

To summarize, the main results of the preceding analysis are:

1. For every $a \in \mathbb{R}$, $b \geqslant 0$ and $-1/(3n) < m < 0$ there exist multiple unbounded solutions to problem (2.10), (2.15) including physical solutions for $b > 0$. The case $b < 0$ is also considered provided $a > 0$ and $n \geqslant 1$.
2. If $-(5n-1)/(12n^2 - 5n + 1) < m < -1/(3n)$, where $1/3 < n \leqslant 1$ there are multiple physical solutions. In this case solutions are obtained by taking $\gamma < 0$ small enough.
3. For $n = 1$ and $-1/2 < m < -1/3$ solutions are obtained for any $a > \sqrt{b/(m+1)}$ (and any $\gamma \geqslant 0$ see [44]).
4. For $-(5n-1)/(12n^2 - 5n + 1) < m < -1/(3n)$ and $n > 1$ we have been unable to prove the existence of a solution. A partial answer can be obtained from Section 5.
5. The existence of global unbounded solutions to (2.10), (2.15), via the shooting method, indicates that for any $\gamma < 0$, we have $f_\gamma \in C^3(\mathbb{R}^+)$. If $\gamma < 0$ there exists a finite unique point, η_0 say such that the second derivative of f_γ vanishes at this point. Therefore $f_\gamma \notin C^3(\mathbb{R}^+)$ if $n > 1$. But $f_\gamma \in C^3(\eta_0, \infty)$.

In the next section we shall derive the large η-behavior of solutions to (2.10) converging to infinity. Note that if $m = -1/(3n)$ identity (2.16) reads

$$|f''|^{n-1} f'' + \frac{1}{3n} f f' = A, \tag{4.5}$$

where

$$A = \left|f''(0)\right|^{n-1} f''(0) + \frac{1}{3n} ab.$$

Therefore if $A > 0$ and $b \geqslant 0$ f is global, unbounded and satisfies

$$f_\gamma(\eta) \sim \eta^{1/2}$$

as $\eta \to \infty$. Observe that if $n = 1$ equation (4.5) leads to Riccati equation

$$f'' + \frac{1}{6} f^2 = A\eta + B.$$

For $A = 0$ any solution is given by [44]

$$f(\eta) = \begin{cases} \varepsilon\sqrt{6B}(c\mathrm{e}^{\varepsilon\sqrt{2B}\,\eta/3} + 1)/(c\mathrm{e}^{\varepsilon\sqrt{2B}\,\eta/3} - 1) & \text{if } B > 0, \\ \frac{6}{\eta + c} & \text{for } B = 0, \end{cases}$$

where $\varepsilon = \pm 1$, $c \in \mathbb{R}$.

5. Global behavior of solutions

In the present section we shall be concerned with the large η-behavior of global solutions to (2.10) such that

$$\lim_{\eta \to \infty} f(\eta) = \infty. \tag{5.1}$$

For the GB equation, $m = 0$;

$$\left(|f''|^{n-1} f''\right)' + \frac{1}{n+1} f f'' = 0,$$

where $n > 1$, it is shown in [15] (see Chapter 3.2) that any global solution is "linear" for a large η, i.e. there exists a finite $\eta_0 > 0$, depending on initial data, such that the following holds

$$f(\eta) = \lambda\eta + \Gamma, \tag{5.2}$$

for any $\eta \geqslant \eta_0$ and for some real numbers λ and Γ. The Newtonian case (Blasius equation) was studied by Belhachmi, Brighi and Taous [14]. The authors proved that any solution such that $f''(0) \neq 0$ its second derivative never vanishes and the function approaches, at infinity, the "linear" solution (given by (5.2)). Recently, the asymptotic behavior of global unbounded solutions to (2.11) is discussed in [46]. The author showed that for $-1/3 < m < 0$, any solution to (2.11), (5.1) has the following estimate

$$f(\eta) \sim \eta^{(1+m)/(1-m)},$$

for large η.

It is proposed here to extend the above results to the non-Newtonian case. First we state the following two results.

PROPOSITION 5.1. *Let f be a (positive) solution to* (2.10), (5.1), *where* $0 < 1+m(2n-1)$ *and* $m < 0$. *Then* $f''(\eta)$ *is negative for large* η *and*

$$\lim_{\eta\to\infty} f'(\eta) = 0.$$

PROOF. Let f be a solution to (2.10), (5.1). As f' is monotonic function on some (η_0, ∞), η_0 large, we deduce that f' and f are positive on (η_1, ∞), $\eta_1 \geqslant \eta_0$. On the other hand, by using the function E (see Section 3) we infer that f' and f'' are bounded. Hence, f' has a finite limit at infinity. Finally, arguing as in Section 4, we get $f'(\eta) \to 0$ as $\eta \to \infty$ and (then) $f''(\eta) < 0$ for large η. □

PROPOSITION 5.2. *Let f be a (positive) solution to* (2.10), (5.1), *where* $0 < 1+m(2n-1)$ *and* $m < 0$. *Then*

$$\lim_{\eta\to\infty} f(\eta) f''(\eta) = \lim_{\eta\to\infty} \left(|f''|^{n-1} f''\right)'(\eta) = 0.$$

PROOF. According to Proposition 5.1 $f'(\eta) > 0$ and $f''(\eta) < 0$ for all $\eta > \eta_0$, η_0 large enough, $f'(\eta)$ tends to 0 as $\eta \to \infty$ and $\lim_{j\to\infty} f''(\eta_j) = 0$, for some sequence (η_j) converging to infinity with j.

We re-write equation (2.10) into the following form

$$f''' + \frac{\kappa}{n} f f''|f''|^{1-n} = \frac{m}{n}|f''|^{1-n} f'^2, \quad \forall \eta > \eta_0.$$

Recall that $\kappa = (1 + m(2n+1))/(n+1)$. We get by differentiation

$$f^{(iv)} + f'''\left[\frac{\kappa(2-n)}{n}|f''|^{1-n} f - \frac{m(1-n)}{n}|f''|^{-n-1} f'' f'^2\right] = \frac{2m-\kappa}{n} f' f''|f''|^{1-n}. \tag{5.3}$$

Then the function $f'''\mathrm{e}^G$ is monotonic increasing on (η_0, ∞), where

$$G' = \frac{\kappa(2-n)}{n}|f''|^{1-n} f - \frac{m(1-n)}{n}|f''|^{-n-1} f'' f'^2.$$

This indicates, in particular, that the function f''' has at most one zero and (then) f'' goes to 0 at infinity. Because f'' is negative, we deduce that $f'''(\eta) > 0$ on (η_1, ∞), η_1 large. On the other hand, from the differential equation

$$\left(|f''|^{n-1} f''\right)'' + (\kappa - 2m) f' f'' = -\kappa f f''',$$

which is obtained from equation (2.10) by differentiation, the function

$$\left(|f''|^{n-1} f''\right)' + \frac{\kappa - 2m}{2} f'^2$$

is positive and monotonic decreasing on $(\inf\{\eta_0, \eta_1\}, \infty)$. Together with the fact that f' tends to 0 as $t \to \infty$ we deduce that

$$\lim_{\eta\to\infty} \left(|f''|^{n-1} f''\right)'(\eta) = 0$$

and then $f(\eta) f''(\eta) \to 0$ as $\eta \to \infty$, thanks to (2.10). □

PROPOSITION 5.3. *Let f be a solution to* (2.10), (5.1), *where* $1 + m(2n-1) > 0$ *and* $m < 0$, *such that the following holds*

$$|f''|^{n-1} f''(\eta_0) + \kappa f f'(\eta_0) > 0, \tag{5.4}$$

for some $\eta_0 \geqslant 0$. *Then*

$$\lim_{\eta\to\infty} f f'(\eta) = \begin{cases} \infty, & \text{if } -\frac{1}{3n} < m, \\ L \in (0, \infty), & \text{if } -\frac{1}{3n} = m, \\ 0, & \text{if } -\frac{1}{3n} > m. \end{cases}$$

PROOF. To prove this proposition we use the identity

$$|f''|^{n-1} f''(\eta) + \kappa f f'(\eta) = |f''|^{n-1} f''(\eta_0) + \kappa f f'(\eta_0) + \frac{1+3mn}{n+1} \int_{\eta_0}^{\eta} (f')^2(s)\, \mathrm{d}s,$$

to deduce that $f f'$ has a limit $L \in [0, \infty]$ at infinity. This limit is finite for $m = -1/(3n)$. Assume that $m \neq -1/(3n)$ and $0 < L < \infty$. Because $f''(\eta)$ goes to 0 at infinity, we have $\lim_{\eta\to\infty} f(\eta) f'(\eta) = L$. Therefore $f'(\eta)^2 \sim L\eta^{-1}$ as $\eta \to \infty$. It follows from this and from the above identity that $f f'$ is unbounded, since the integral $\int_{\eta_0}^{\eta} f'(s)^2\, \mathrm{d}s$ tends to infinity with η. A contradiction, hence $L \in \{0, \infty\}$. Using again the above identity to get $L = \infty$ if $-1/(3n) < m$ and L is zero if $m < -1/(3n)$. □

REMARK 5.1. Remember that in Theorem 4.1 we have already argued that condition (5.4) leads to a global unbounded solution for $-1/(3n) < m < 0$ and $n > 0$. In passing, we stress that if $n \geqslant 1$ the existence of global solution converging to infinity required condition (5.4). To see this we assume

$$|f''|^{n-1} f''(\eta) + \kappa f f'(\eta) \leqslant 0,$$

for all $\eta \geqslant 0$. Since $f''(\eta) \to 0$ as $\eta \to \infty$ and $n \geqslant 1$ the following

$$f'' + \kappa f f' \leqslant 0$$

holds on some (η_1, ∞), η_1 large. Consequently the function $f' + \frac{\kappa}{2} f^2$ is decreasing and goes to infinity with η, which is absurd.

Now, we are ready to give the large η asymptotic of solutions. We proceed in two further cases; namely (i) $m \geqslant -1/(3n)$, $n > 0$. (ii) $-2/(5n-1) < m < -1/(3n)$, $n \geqslant 1$.

THEOREM 5.1. *Let f be a solution to* (2.10), (5.1), *where $-1/(3n) \leqslant m < 0$ and $n > 0$. Assume that condition* (5.4) *holds. Then there exists a constant, $A > 0$, such that*

$$f(\eta) = \eta^{(1+m(2n-1))/(1+m(n-2))}\big(A + \mathrm{o}(1)\big), \tag{5.5}$$

as $\eta \to \infty$.

PROOF. For the case $m = -1/(3n)$ estimate (5.5) is given in Section 4. So, we assume that $m > -1/(3n)$.

Let η_1 be a real such that $f'' < 0$ and $f''' > 0$ on (η_1, ∞). We divide equation (2.10) by ff' and get

$$\frac{(|f''|^{n-1}f'')'}{ff'} = m\frac{f'}{f} - \kappa\frac{f''}{f'}.$$

Integrating over (η_2, η), $\eta_2 > \eta_1$ gives

$$\int_{\eta_2}^{\eta} \frac{(|f''|^{n-1}f'')'}{ff'}\,\mathrm{d}s = \log\big(f^m(\eta)f'^{-\kappa}(\eta)\big) - \log\big(f^m(\eta_2)f'^{-\kappa}(\eta_2)\big).$$

Since ff' tends to infinity with η the LHS is integrable up to infinity. Hence, letting $\eta \to \infty$ we see that $\log(f^m(\eta)f^{-\kappa}(\eta))$ tends to a finite limit. Thus, $f^m(\eta)f'^{-\kappa}(\eta)$ has a finite limit and this limit must be positive. The asymptotic behavior of f is then obtained by integration. □

THEOREM 5.2. *Let f be a solution to* (2.10), (5.1), *where $-2/(5n-1) \leqslant m < -1/(3n)$ and $n \geqslant 1$. Assume that* (5.4) *is satisfied. Then relation* (5.5) *holds.*

PROOF. Define an auxiliary function

$$\Psi = \varphi(f)|f''|^{n-1}f'' - \frac{1}{2}\varphi'(f)f'^2|f''|^{n-1} + \kappa\varphi(f)ff',$$

where φ is a smooth function. Then if follows from the differential equation of f

$$\Psi' = f'^2\big[\kappa f\varphi'(f) + (\kappa+m)\varphi\big] - \frac{1}{2}\varphi''(f)f'^3|f''|^{n-1} - \frac{n-1}{2}\varphi'(f)f'^2|f''|^{n-3}f''f'''.$$

Let the function φ be defined by

$$\varphi(s) = s^{-(1+3mn)/(1+m(2n-1))}.$$

The function φ satisfies the differential equation

$$\kappa s\varphi' + (\kappa + m)\varphi = 0.$$

This in turn implies

$$\Psi' = -\frac{1}{2}\varphi''(f)f'^3|f''|^{n-1} - \frac{n-1}{2}\varphi'(f)f'^2|f''|^{n-3}f''f''',$$

which is non-negative,

$$\Psi = \varphi(f)\left[|f''|^{n-1}f'' - \frac{\kappa + m}{2\kappa f}f'^2|f''|^{n-1} + \kappa ff'\right],$$

and

$$\Psi' \leqslant \varepsilon\left[f^{-(3\kappa+m)/\kappa}f' + (n-1)(-f'')^{n-2}f'''\right],$$

on (η_1, ∞), η_1 large. Therefore Ψ' is integrable on $[0, \infty)$ and then Ψ has a finite limit at infinity, say L. The next step is to show that $L > 0$. It will be sufficient to show that $\Psi(\eta_2) > 0$ for some η_2 large. Suppose not; that is for any $\eta > \eta_3$, η_3, may be large,

$$|f''|^{n-1}f'' - \frac{\kappa + m}{2\kappa f}f'^2|f''|^{n-1} + \kappa ff' \leqslant 0.$$

Hence,

$$f'' + \kappa ff' \leqslant 0,$$

since $n > 1$ and $\kappa + m < 0$, and this leads to a contradiction as above. Finally, as $\lim_{\eta\to\infty}\Psi(\eta) = \lim_{\eta\to\infty}\kappa f^{-m/\kappa}f'$, we deduce that $f^{-m/\kappa}f'$ converges to L/κ as $\eta \to \infty$ and then estimate (5.5) follows by a simple integration. □

The last results of the chapter deal with the asymptotic behavior of solutions which tend to $-\infty$. According to the proof of Theorem 3.1 this family of solutions may exist only for $n \geqslant 2$. The behavior of these solutions is quite different from the above results. This is shown by the following

THEOREM 5.3. *Assume $n \geqslant 2$. Let $-1/(2n-1) < m < 0$. Let f be a global concave solution converging to minus infinity as η goes to plus infinity. Then the following estimate*

$$f(\eta) \leqslant \begin{cases} C\eta^{(2n-1)/(n-2)}, & \text{if } n > 2, \\ Ce^{(|m|/2)\eta}, & \text{if } n = 2, \end{cases}$$

holds for some negative constant C.

PROOF. Let f be a global concave solution converging to $-\infty$. Since f'' is monotonic in some (η_1, ∞), η_1 large (see the proof of Proposition 5.2), there exists η_0 such that f, f' and f'' are negative on $[\eta_0, \infty)$. Since the function

$$E = \frac{n}{n+1}(-f'')^{n+1} - \frac{m}{3}f'^3$$

is monotonic increasing on $[\eta_0, \infty)$ f' and f'' tend to minus infinity at infinity and

$$(-f'')^{n+1}(\eta) \geqslant \frac{(n+1)m}{3n}f'^3(\eta) + (-f'')^{n+1}(\eta_0) - \frac{(n+1)m}{3n}f'^3(\eta_0),$$

for any $\eta \geqslant \eta_0$. Assume first that there exists $\eta_1 \geqslant \eta_0$ such that

$$(-f'')^{n+1}(\eta_1) - \frac{(n+1)m}{3n}f'^3(\eta_1) \geqslant 0.$$

Thus

$$(-f'')^{n+1}(\eta) \geqslant \frac{(n+1)m}{3n}f'^3(\eta)$$

and the required estimate follows by a simple integration. Next assume that $E(\eta) \leqslant 0$ for all $\eta > 0$. Since E is monotonic increasing there exists a real number $l \leqslant 0$ such that

$$\lim_{\eta\to\infty} \frac{n}{n+1}(-f'')^{n+1} - \frac{m}{3}f'^3 = l.$$

Thereafter

$$\lim_{\eta\to\infty} (-f'')^{n+1}f'^{-3} = \frac{m(n+1)}{3n},$$

and the asymptotic form of f follows by integration. □

A similar argument can be used to obtain an upper bound for any global positive solution converging to 0 at infinity. It is easily verified that $f'' + \frac{\kappa}{2}f^2 \leqslant 0$ if $n \leqslant 1$. Thus

THEOREM 5.4. *Assume* $m(2n-1) > -1$ *and* $0 < n \leqslant 1$. *Let* f *be a positive solution converging to* 0 *at infinity. Then, for some constant* c *the following*

$$f(\eta) \geqslant \frac{2}{\kappa\eta + c},$$

holds on (η_0, ∞), η_0 *large.*

To end our analysis we consider non-global solutions in the case where $1/2 < n < 2$. From Theorem 3.1 we know that if f is not global and $1 \leqslant n < 2$, there exists $T > 0$

finite such that f tends to minus infinity as η approaches T. Arguing as in the proof of Theorem 5.3 we get the following result.

THEOREM 5.5. *Assume* $1/2 < n < 2$, $-1/(2n-1) < m < 0$. *Let* f *be a non-global solution defined on* $(0, T)$ *such that* $\lim_{\eta \to T} f(\eta) = -\infty$. *Then there exists a real number* B *such that*

$$f(\eta) \leqslant \frac{2n-1}{n-2}\left(\frac{2-n}{n+1}\right)^{(n+1)/(n-2)}\left(\frac{(n+1)|m|}{3n}\right)^{1/(n-2)} \times (T-\eta)^{(2n-1)/(n-2)} + B,$$

for all $\eta \in (0, T)$.

6. Conclusion

From physical viewpoint, the main results of this chapter deal with exact or similarity solutions to (1.4), (1.5). It has been shown that multiple solutions arise in the case $u(x, 0) = u_w x^m$, $m < 0$. These multiple solutions being obtained by solving the following ODE

$$\left(|f''|^{n-1} f''\right)' + \frac{1+m(2n-1)}{n+1} f f'' - m f'^2 = 0, \quad \eta > 0,\ n > 0,$$
$$f(0) = a, \quad f'(0) = b \geqslant 0, \quad f'(\infty) = 0.$$

It is shown that for $-1/(3n) \leqslant m < 0$ there exist multiple solutions of the above problem, including multiple physical solutions if $b > 0$. The case $m < -1/(3n)$ was also considered for $a > 0$, $b > 0$, $m > -(5n-1)/(12n^2 - 5n + 1)$ and $1/3 < n \leqslant 1$. All solutions obtained are unbounded and satisfy

$$f(\eta) \sim \eta^{(1+m(2n-1))/(1+m(n-2))},$$

as $\eta \to \infty$.

CHAPTER 3.2

Heat Transfer in Non-Newtonian Fluid Laminar Boundary Layer Flow Along a Moving Surface

In this chapter, we examine the momentum and energy occurring in laminar boundary layer flow, of a power-law fluid over a vertical continuously moving plate in an unbounded porous medium, utilizing a similarity reduction and a shooting argument. The problem contains the power-law exponent $n > 0$ and the velocity ratio parameter ζ. We prove a general global existence, non-existence and uniqueness theorem accordingly to n and ζ.

1. Introduction and laminar boundary layer equations

The physical problem being examined is as follows. We consider a vertical flat plate with a uniform power law flow at a constant speed $u_\infty > 0$ ($u_e(x) = u_\infty,\ m = 0$) which is embedded in a saturated porous media and moving in the direction of the stream at constant speed $u(x, 0) = u_w < u_\infty$. The temperature distribution of the plate and far from the plate (the outer temperature) is assumed to be constant, namely T_w, T_∞ such that $T_w \neq T_\infty$. The coordinates (x, y) are measured along the plate and normal to it, with the origin at the leading edge. The x-axis being parallel to the direction of gravity but directed upwards, the system, under some approximation at large Rayleigh number [35,104],

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial T}{\partial y}\right), \tag{1.1}$$

where u, v are the velocity components, describes the 2D stationary heat convection. Here, the fluid is assumed to have the following transport property

$$q = -k_m\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial T}{\partial y}, \tag{1.2}$$

where q is the heat flux normal to the plate, k_m is the modified thermal conductivity and n is the exponent expressing the heat transfer (the power law exponent). The longitudinal diffusivity is assumed to be constant and $\partial^2 T/\partial x^2$ is neglected. In porous media, u and v obey the Darcy's law:

$$u = -k\mu^{-1}\left(\frac{\partial p}{\partial x} + \rho g\right), \qquad v = -k\mu^{-1}\frac{\partial p}{\partial y},$$

where ρ is the T-dependent density, defined from the Boussinesq approximation [104]

$$\rho = \rho_0\big(1 - \beta(T - T_0)\big), \tag{1.3}$$

μ, β, k, α, g are constants (viscosity, thermal expansion coefficient, permeability, thermal diffusivity, gravitational acceleration), p is the pressure, ρ_0 is the density at a reference temperature T_0. For the present work the reference temperature is T_∞, and then $\rho_0 = \rho_\infty$ is the value of ρ far from the plate, the reference density [93].

The following dimensionless variables are introduced (the characteristic length is assumed to be one)

$$U = \frac{u}{u_\infty}, \qquad V = \frac{v}{u_\infty}.$$

The boundary layer equations then become

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \qquad U\frac{\partial T}{\partial x} + V\frac{\partial T}{\partial y} = \frac{1}{Pr}\frac{\partial}{\partial y}\left(\left|\frac{\partial U}{\partial y}\right|^{n-1}\frac{\partial T}{\partial y}\right), \tag{1.4}$$

and

$$U = -k\mu^{-1}u_\infty^{-1}\left(\frac{\partial p}{\partial x} + \rho g\right), \qquad V = -k\mu^{-1}u_\infty^{-1}\frac{\partial p}{\partial y}, \tag{1.5}$$

where $Pr = \alpha^{-1}u_\infty^{2-n}$ which is called the Prandtl number [93, p. 211], [35].

Eliminating the pressure from (1.5) we arrive at the following equation

$$\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} = -k\mu^{-1}u_\infty^{-1}g\frac{\partial \rho}{\partial y}.$$

Introducing the stream function $\psi(x, y)$ as usual ($U = \partial\psi/\partial y$, $V = -\partial\psi/\partial x$), using the boundary layer approximation ($\partial^2\psi/\partial x^2 = 0$), we obtain the system

$$\begin{cases} \frac{\partial^2\psi}{\partial y^2} = b\frac{\partial T}{\partial y}, \\ \frac{1}{Pr}\frac{\partial}{\partial y}\left(\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial T}{\partial y}\right) = \frac{\partial T}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial T}{\partial y}\frac{\partial \psi}{\partial x}, \end{cases} \tag{1.6}$$

where $b = \rho_\infty\beta g k\mu^{-1}u_\infty^{-1}$.

We are looking for similarity solutions of (1.6) in standard form (see Chapter 3.1)

$$\psi(x, y) = ax^s f(\eta), \quad T(x, y) = (T_w - T_\infty)\theta(\eta) + T_\infty,$$

where $\eta = \lambda y/x^r$, $a\lambda = 1$ and $\lambda^{n+1} = Pr$. According to Section 2 of Chapter 3.1 we have $s = r = 1/(n+1)$. It can be checked that the shape functions (f, θ) satisfy the ODEs system

$$\begin{cases} f'' - b(T_w - T_\infty)\theta' = 0, \\ (|f''|^{n-1} f'')' + \frac{1}{n+1} f f'' = 0. \end{cases} \tag{1.7}$$

So, this chapter is devoted to the study of properties of solutions to the following problem

$$\begin{cases} (|f''|^{n-1} f'')' + \frac{1}{n+1} f f'' = 0, \\ f(0) = 0,\ f'(0) = \zeta,\ f'(\infty) = 1, \end{cases} \tag{1.8}$$

where $\zeta = u_w / u_\infty$. If $\zeta = 0$ we obtain the Generalized Blasius Problem (GB) [60,80, 81,105].

We recall that it was shown in Chapter 3.1 that the above problem describes also the motion for steady non-Newtonian flow over a plane in a uniform stream. For $n = 1$ and $\zeta = 0$ we get the classical Blasius problem [17,33,102].

2. Main result and "finite propagation"

The title of this section contains "finite propagation" which is frequently used in the mathematical theory of the Porous Medium Equations (PME for short). However, there are, under favorable circumstance, some connections between boundary layer equations (BLE) and PME (see Chapter 3.4) and researchers in BLE and PME are closely related (Scaling, similarity or self-similarity solutions, mass balance, Darcy law, ...) see [8]. In the BLE – at least in the case of laminar flows – the analog of the free boundary is the boundary-layer thickness, denoted by δ, which divides the flow into two regions (the boundary-layer concept of Prandtl). For the laminar plate boundary layers the boundary-layer thickness is defined by $u(x, \delta(x)) = u_e(x)$, or $u(x, y) = u_e(x)$ for all $y \geqslant \delta(x)$, where u_e is the velocity of the outer flow. Since the transition from the boundary-layer flow to outer flow takes place continuously, the function δ cannot be given with precision, in general. So, frequently δ is given where u reaches $0.99u_e$. This new boundary-layer thickness is noted δ_{99}. For the Blasius problem ($n = 1$), where the plate is at zero incidence δ_{99} is given by [93, p. 31]

$$\delta_{99}(x) = 5\sqrt{\frac{\nu x}{u_\infty}},$$

where ν is the kinematic viscosity and $u_e(x) = u_\infty$. As in PME, BLE may have explicit "fundamental solutions". System (1.6) has the explicit formula, see Chapter 3.1,

$$\psi(x, y) = a x^{1/(n+1)} \left[A - C\lambda \frac{y}{x^{1/(n+1)}} \right]_+^{(2n-1)/(n-2)}, \tag{2.1}$$

where $n > 2$, $C = C(n, a)$ and A is an arbitrary constant.

This solution does not satisfy the boundary condition at infinity. However, if the boundary condition at infinity is given by $\lim_{y\to\infty} \partial_y \psi(x, y) = 0$, as in Chapter 3.1, the boundary-layer thickness δ_{99} is defined by $|u(x, \delta_{99}(x))| = 0.01$. Formula (2.1) leads to

$$\delta_{99}(x) = Bx^{1/(n+1)},$$

where $B = B(n, A, C, \lambda)$ is a constant. An analysis of the explicit solution (2.1) shows that GB equation may have solutions which are compactly supported and then the finite propagation occurs. In this case δ can be precisely obtained. For the explicit solution we have $\delta_{99}(x) = (A/(C\lambda))x^{1/(n+1)}$.

In practice, for problem (1.1) (or (1.8)), the finite propagation is to be understood in the sense that $u(x, y) = u_\infty$ for all $y \geqslant \delta(x)$; the velocity u outside the boundary layer is uniform and equal to u_∞, which implies that f'' is compactly supported from the right. Let us note that the ODE in (1.8) has no explicit solution under the form (2.1), i.e. $f(\eta) = a[A - c\eta]_+^{(2n-1)/(n-2)}$, satisfying $f(0) = 0$ and $f'(0) = \zeta > 0$.

The purpose of this chapter is to explore the effect of n on the global existence and the finite propagation property. For a mathematical consideration we shall be interested in solutions to the more general problem

$$\begin{cases} (|f''|^{n-1} f'')' + \kappa f f'' = 0, \\ f(0) = 0, \quad f'(0) = \zeta, \quad f'(\infty) = \Gamma, \end{cases} \tag{2.2}$$

where $\zeta < \Gamma$ are non-negative real numbers and $\kappa > 0$. In particular, for the Generalized Blasius equation (GB)

$$\begin{cases} (|f''|^{n-1} f'')' + \kappa f f'' = 0, \\ f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = \Gamma > 0, \end{cases} \tag{2.3}$$

we summarize the main result in the following.

THEOREM 2.1. *Let $n > 0$. Then there exists a unique solution of problem* (2.3). *Furthermore this solution has the following asymptotic behavior*:

(1) *If $n > 1$, there exist $\eta_0 > 0$ and $\Lambda \in (-\infty, 0)$ such that $f(\eta) = \Gamma\eta + \Lambda$, for all $\eta \geqslant \eta_0$, and $f'' > 0$ on $[0, \eta_0)$.*
(2) *If $0 < n < 1$, then $\eta^{1/(n-1)} f''(\eta)$ is positive and bounded for all $\eta > 0$. For $1/2 < n \leqslant 1$ there exists $-\infty < \Lambda < 0$ such that $\lim_{\eta\to\infty}(f(\eta) - \Gamma\eta) = \Lambda$.*

REMARK 2.1. As a consequence of Theorem 2.1 is that if $n > 1$ the boundary-layer thickness $\delta = \delta_{100}$ (i.e. $u(x, \delta_{100}(x)) = u_\infty$) is given by

$$\delta(x) = \eta_0 Pr^{1/(n+1)} x^{1/(n+1)}.$$

3. Existence of a solution to GB problem and large η-behavior

Our goal in this chapter is to analyze the large η behavior of solutions of problem (2.2) and to rigorously prove the existence and uniqueness of solutions to GB equation. The existence and the uniqueness will be established via the general Crocco transformation and a shooting method as in the work by Nachman and Callegari [80]. In the last paper the authors studied the existence, uniqueness and some analytical results for

$$\begin{cases} f''' + f f''^{2-n} = 0, \\ f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \end{cases} \tag{3.1}$$

where $0 < n < 1$, which can be obtained from (1.6) (or from (2.7) of Chapter 3.1) under the restriction $f'' > 0$. In fact, paper [80] investigated the positive solution. Hence, since $f(0) = 0$ and $f'(0) = 0$, $f''(0)$ must be positive and then (see (3.4) below) f'' is positive. Assuming $f'' > 0$ and $f''(\infty) = 0$ analytical and numerical solutions of (2.2), where $0 < n \leqslant 1$, $f'(0) = \zeta \in [0, 1)$ and $\Gamma = 1$, are reported by Zheng and Zhang [105]. In fact, since $f'(\infty) = 1$ it is easy to see that f'' is strictly positive if $n \leqslant 1$ and then the ODEs in (2.2) and (3.1) are equivalent. However, for $n > 1$, the ODE in (2.2) can be degenerate at the point η_0 where $f''(\eta_0) = 0$. So, as in the precedent chapter, a solution to (2.2) is a function such that $|f''|^{n-1} f'' \in C(0, \infty)$, $f(0) = 0$, $f'(0) = \zeta$, $f'(\infty) = \Gamma$ and the equality

$$|f''|^{n-1} f''(\eta) + \kappa f'(\eta) f(\eta) = \left|f''(0)\right|^{n-1} f''(0) + \kappa \int_0^\eta f'(\tau)^2 \, \mathrm{d}\tau, \tag{3.2}$$

holds for all $\eta > 0$. The existence result for the GB problem will be established by showing that the local solution f_γ to

$$\begin{cases} (|f''|^{n-1} f'')' + \kappa f f'' = 0, \\ f(0) = 0, \quad f'(0) = 0, \quad f''(0) = \gamma, \end{cases} \tag{3.3}$$

for appropriate γ, satisfies (2.2). First we recall that f_γ, $\gamma \neq 0$, exists on some (maximal) interval $(0, \eta_\gamma)$, $\eta_\gamma \leqslant \infty$, and satisfies

$$f''_\gamma(\eta) = \gamma \exp\left(-\frac{\kappa}{n} \int_0^\eta f_\gamma(s) |f''_\gamma|^{1-n}(\tau) \, \mathrm{d}\tau \right) \tag{3.4}$$

for all $0 < \eta < \eta^\star < \eta_\gamma$ as long as $|f''_\gamma| > 0$ in $(0, \eta^\star)$. Therefore, if $0 < n \leqslant 1$ $f_\gamma \neq 0$ and then $f_\gamma \in C^\infty([0, \eta_\gamma))$.

Observe that if γ is negative then $f''_\gamma < 0$ on $(0, \eta_\gamma)$. Therefore, the boundary condition at infinity cannot be satisfied. From now on we suppose that γ is positive. For the existence of a solution we need to distinguish two cases for n; namely $0 < n \leqslant 1$ and $n > 1$. The proof for the case $n \leqslant 1$ follows the argument given in Chapter 3.1. For the reader convenience, we give here a sketch of the proof. Since γ is positive we deduce that f_γ, f'_γ and f''_γ

are positive on $(0, \eta_\gamma)$. Next, to show that f_γ is global, we consider the function $E = (n/(n+1))\, f_\gamma''^{n+1}$ which satisfies

$$E' = -\kappa f_\gamma f_\gamma''^2 \leqslant 0.$$

Therefore

$$f_\gamma''(\eta)^{n+1} \leqslant \gamma^{n+1}.$$

This shows, in particular, that if η_γ is finite the functions f_γ, f_γ' and f_γ'' are bounded, in $(0, \eta_\gamma)$, respectively by $\eta_\gamma^2\gamma, \eta_\gamma\gamma$ and γ, a contradiction. Next, as f_γ' is positive $f_\gamma(\eta)$ tends to a limit as $\eta \to \infty$. This limit is not finite, otherwise, we get from (3.2), as in Chapter 3.1,

$$0 = \gamma^n + \kappa \int_0^\infty f_\gamma'(s)^2 \, \mathrm{d}s,$$

which is absurd. It remains to prove that f_γ' has a finite limit at infinity. To this end observe that f_γ'' has a finite limit as $\eta \to \infty$, since f_γ'' is monotonic decreasing, and that the function

$$H = f_\gamma''^n + \kappa f_\gamma f_\gamma',$$

which is monotonic increasing ($H' = \kappa f_\gamma'^2$), tends to infinity with η. Next, as the function $(|f_\gamma''|^{n-1} f_\gamma'')'/(f_\gamma f_\gamma')$ is integrable on $(1, \infty)$, there exists a real number l such that

$$\lim_{\eta\to\infty} \int_1^\eta \frac{(|f_\gamma''|^{n-1} f_\gamma'')'}{f_\gamma f_\gamma'}(\tau)\, \mathrm{d}\tau = l.$$

From the ODE satisfied by f_γ one sees

$$\lim_{\eta\to\infty} \int_1^\eta \frac{f_\gamma''}{f_\gamma'}(\tau)\, \mathrm{d}\tau = -\frac{l}{\kappa}.$$

So, there exists $\lambda_\gamma > 0$ such that

$$\lim_{\eta\to\infty} f_\gamma'(\eta) = \lambda_\gamma$$

and then $f_\gamma''(\eta) \to 0$ as $\eta \to \infty$.

Let us notice that in the above argument we only used the fact that $f_\gamma'(0) \geqslant 0$. In conclusion, we have proved the following.

THEOREM 3.1. *Let $0 < n \leqslant 1$ and $\zeta, \gamma > 0$. Then the local solution f_γ to* (3.3), *with $f'_\gamma(0) = \zeta$ instead of $f'_\gamma(0) = 0$, is global, goes to infinity with η, and the second derivative of f_γ tends to 0 as $\eta \to \infty$. Moreover there exists a positive constant $\lambda_\gamma > \zeta$ such that*

$$\lim_{\eta\to\infty} f'_\gamma(\eta) = \lambda_\gamma.$$

Let us now consider the GB equation (2.3). Let γ_0 be a fixed positive real number and let f_{γ_0} be the solution to (3.3). If $\lambda_{\gamma_0} = f'_{\gamma_0}(\infty)$ it is easy to see that the new function

$$g(\eta) = \left(\frac{\lambda_{\gamma_0}}{\Gamma}\right)^{-(2n-1)/(n+1)} f_{\gamma_0}\left(\eta\left(\frac{\lambda_{\gamma_0}}{\Gamma}\right)^{(n-2)/(n+1)}\right),$$

where $\Gamma > 0$, is a solution to (2.3). This gives, in particular, a value of γ such that the local solution f_γ, is a solution to (2.3). It will be

$$\gamma = \left(\frac{\Gamma}{\lambda_{\gamma_0}}\right)^{3/(n+1)} \gamma_0. \tag{3.5}$$

We stress that (3.5) will express the necessary and sufficient condition of the existence of the solution to GB problem (see Section 4).

The following result deals with the large η-behavior of solutions to (2.2) for the case $0 < n \leqslant 1$.

THEOREM 3.2. *Let f be a solution to* (2.2), *where $0 \leqslant \zeta \leqslant \Gamma$. Then $\eta^{1/(1-n)} f''$ is bounded for $0 < n < 1$. Moreover, if $1/2 < n \leqslant 1$, there exists $\Lambda \in (-\infty, 0]$ such that*

$$\lim_{\eta\to\infty} \big(f(\eta) - \Gamma\eta\big) = \Lambda \tag{3.6}$$

and, if $1/2 < n < 1$, there exist constants c_1 and c_2 such that

$$\Gamma\eta + \frac{1-n}{2n-1}\left(\frac{1-n}{n}\right)^{n/(n-1)} (\eta + c_2)^{(2n-1)/(n-1)} + c_1 \leqslant f(\eta), \tag{3.7}$$

for all η large.

PROOF. Estimate (3.6) was obtained in [14] for the Blasius equation ($n = 1$) and for the concave solutions ($\zeta > \Lambda$). See also [53] for $n = 1$, $\Gamma = 1$ and $0 \leqslant \zeta < 1$.

The proof of the present theorem for $\zeta = \Gamma$ is obvious. So, we consider the case $n < 1$ and $\zeta < \Gamma$. Since f satisfies (recall that $f'' > 0$)

$$\big(f''^{n-1}\big)' = -\frac{(n-1)\kappa}{n} f,$$

we get, by the l'Hôpital rule, that $f''(\eta)^{n-1}/\eta$ goes to infinity with η. From this we deduce that there exists η_0 large, such that

$$f''(\eta) \leqslant \frac{1}{2}\eta^{1/(n-1)}, \quad \forall \eta \geqslant \eta_0, \tag{3.8}$$

and, hence $\eta^{1/(1-n)} f''$ is bounded.

Now we are going to show (3.6) for $1/2 < n < 1$. Since $f'' > 0$ the function $f' - \Gamma$ is negative. Therefore there exists $\Lambda \in [-\infty, 0)$ such that

$$\lim_{\eta\to\infty} \big(f(\eta) - \Gamma\eta\big) = \Lambda.$$

Note that this limit is finite if (3.7) holds. To prove (3.7), we first deduce from the ODE satisfied by f and the l'Hôpital rule,

$$\lim_{\eta\to\infty} \frac{f''(\eta)^n}{f'(\eta) - \Gamma} = -\infty.$$

Consequently, the following

$$f''^n \geqslant \big(\Gamma - f'(\eta)\big),$$

holds for all $\eta \geqslant \eta_1$, η_1 large enough. Thus

$$f''(\Gamma - f')^{-1/n} \geqslant 1,$$

and, by integrations,

$$\Gamma\eta - f(\eta) \leqslant c_1 + \frac{n-1}{2n-1}\left(\frac{1-n}{n}\right)^{n/(n-1)} (\eta + c_2)^{(2n-1)/(n-1)}, \quad \forall \eta \geqslant \eta_1,$$

where $1/2 < n < 1$ and c_1, c_2 and constants. □

Now we consider the case $n > 1$. The result of Theorems 3.1 and estimate (3.6) still valid, but the proofs are slightly different. In fact, we shall see that the second derivative of any solution has a compact support; that is f'' vanishes at some $[\eta_0, \infty)$, $\eta_0 > 0$.

THEOREM 3.3. *Let $n > 1$. Let f be a solution to* (2.2), *where $\zeta < \Gamma$. Then, there exist $\eta_0 > 0$ and $\Lambda \in (-\infty, 0]$ such that $f''(\eta_0) = 0$ and*

$$f(\eta) = \Gamma\eta + \Lambda,$$

for all $\eta \geqslant \eta_0$.

PROOF. Clearly, since $\zeta < \Gamma$, we must have $f''(0) > 0$ and then there exists a real number $\eta_1 > 0$ such that f'' is positive on $[0, \eta_1)$. Suppose that f'' is positive on $[0, \infty)$. Arguing as above, we can show that f'' goes to 0 at infinity. On the other hand, from the ODE in (2.2) the following ODE

$$nf''^{n-2}f''' + \kappa f = 0,$$

is satisfied on $(0, \infty)$. Hence, we get by integration over $(0, \eta)$,

$$\frac{n}{n-1}f''(\eta)^{n-1} + \kappa \int_0^\eta f(s)\,\mathrm{d}s = \frac{n}{n-1}\gamma^{n-1},$$

where $\gamma = f''(0)$. Passing to the limit as $\eta \to \infty$ we find that $f \in L^1(\mathbb{R}_+)$, hence f vanishes at infinity, a contradiction. Therefore there exists a finite real number $\eta_0 > 0$ such that $f''(\eta_0) = 0$. Assume that there exists $\eta_1 > \eta_0$ such that $f'' < 0$ on (η_0, η_1). Using again the ODE one sees

$$|f''|^{n-1}f''(\eta) = -\kappa \int_{\eta_0}^\eta f(s)f''(s)\,\mathrm{d}s, \quad \eta_0 < \eta < \eta_1. \tag{3.9}$$

The right-hand side of the above is positive while the left-hand side is negative, a contradiction. Therefore, f'' is non-negative and then $f'' \equiv 0$ on (η_0, ∞), by using again (3.9). □

REMARK 3.1. The proofs of the preceding theorems can also be used to obtain the large η behavior of any possible solution to (2.2), where $\zeta > \Gamma > 0$. In particular, the second derivative of any solution such that $n > 1$ is compactly supported.

REMARK 3.2. In [105] the authors considered the following problem

$$\begin{cases} (|f''|^{n-1}f'')' + ff'' = 0, \quad \text{for } \eta > 0, \\ f(0) = 0, \quad f'(0) = \zeta, \quad f'(\infty) = 1, \\ \Phi'' + N_{Pr}f\Phi' = 0, \\ \Phi(0) = 0, \quad \Phi(\infty) = 1, \end{cases} \tag{3.10}$$

arising in modeling heat transfer in power-law fluid along a moving surface. Φ is the dimensionless temperature, $\zeta \in [0, 1)$ is the velocity ratio parameter and N_{Pr} is the Prandtl number. According to the above results we deduce that if $n > 1$ there exists $\eta_0 > 0$ such that $f(\eta) = \eta + \Lambda$, for all $\eta \geqslant \eta_0$. Thereafter,

$$\Phi(\eta) = \Phi(\eta_0) + \Phi'(\eta_0)\mathrm{e}^{(N_{Pr}/2)(\eta_0+\Lambda)^2} \int_{\eta_0}^\eta \mathrm{e}^{-(N_{Pr}/2)(s+\Lambda)^2}\,\mathrm{d}s.$$

We close this section with a simple remark concerning the existence of solutions to (2.2). The analysis of the present section cannot allow us to demonstrate that (2.2) possesses a solution for any $\zeta < \Gamma$ except for the case $\zeta = \chi\Gamma$, where $\chi \in \{0, 1, f'_\gamma(0)/f'_\gamma(\infty)\}$.

Note that if $\zeta = \Gamma$ we have an explicit solution given by $f_0(\eta) = \Gamma\eta$. The uniqueness of solutions for the GB equation will be established in the next section, by using the Crocco variables approach.

4. Uniqueness results of the GB equation

In this section we are concerned with the question of existence and uniqueness of solution to (2.2). The case $n = 1$ and $\kappa = 1/2$ (the well known Blasius equation) was treated in [53] and [102]. It is shown, by using integral operators [102] and elementary proofs [53] that the problem

$$\begin{cases} f''' + \frac{1}{2} f f'' = 0, \\ f(0) = a, \quad f'(0) = \zeta, \quad f'(\infty) = 1, \end{cases} \tag{4.1}$$

where $a \in \mathbb{R}$ and $\zeta \in [0, 1)$, has a unique solution. The uniqueness of concave solution is also obtained for the Blasius equation $(4.1)_1$ with the boundary conditions

$$f(0) = a, \quad f'(0) = \zeta, \quad f'(\infty) = \Gamma, \tag{4.2}$$

in [26] for $0 \leqslant \zeta \leqslant \Gamma$.

By the Crocco variables approach Callegari and Nachman [26] established the uniqueness and analyticity of solutions for

$$\begin{cases} f''' + \frac{1}{2} f f'' = 0, \\ f(0) = 0, \quad f'(0) = K, \quad f'(\infty) = 1, \end{cases} \tag{4.3}$$

where $0 < K < 1$ and $1 < K < 6$. Later on, the same authors have showed uniqueness for the problem

$$\begin{cases} (f''^n)' + \frac{1}{n+1} f f'' = 0, \\ f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \end{cases} \tag{4.4}$$

too, where $0 < n < 1$ [80].

We shall extend these result to problem (2.3) and show that problem (2.2) has at most one solution.

Of particular interest will be in finding the (unique) value of γ which ensures the existence of solutions to (2.3). The idea is to introduce a new independent variable χ and a new unknown function $G(\chi)$ which transform the original problem into a singular nonlinear boundary value problem which is easier to analyze. First, note that the ODE in (2.3) (or (2.2)) can be re-written as

$$n|f''|^{n-1} f''^{-1} f''' + \kappa f = 0,$$

in $(0, \eta_{\max})$, $\eta_{\max} \leqslant \infty$, where $|f''| > 0$ on $(0, \eta_{\max})$. Adopting the Crocco variables

$$\chi = f', \qquad h = |f''|^{n-1} f'',$$

leads to the following Emden–Fowler problem, with negative exponent [103]

$$|h|^{1/(n-1)} h h'' + \kappa \chi = 0, \tag{4.5}$$

$$h'(\zeta) = 0, \qquad h(\Gamma) = 0, \tag{4.6}$$

where ζ is not necessarily equal to 0.

From [32,103] one may conclude that the solutions to the above problem are not necessarily unique and the problem may have oscillatory solutions. However, it is shown in [32] that, under some conditions, positive solutions are unique.

If we look for a solution to (4.5) under the form $h(\chi) = w\chi^{\aleph}$, where $n \neq 1$, then we get

$$\aleph = \frac{2n}{n+1}, \qquad |w|^{(n+1)/n} \aleph(\aleph - 1) + \kappa = 0,$$

and it is clear that w only exists when $n < 1$.

As in the preceding analysis, the existence and uniqueness results, for $\zeta = 0$, will be investigated by the shooting method. We shall see that there exists a unique w such that the initial value problem

$$|h|^{1/n-1} h h'' + \kappa \chi = 0, \tag{4.7}$$

$$h'(0) = 0, \qquad h(0) = w, \tag{4.8}$$

has a continuous solution which vanishes for Γ. The uniqueness result for (4.5), (4.6) will be also established. As in [80], employing the following Crocco-like transformation $\chi = f'$, $G = f''$, we arrive at the following problem

$$\begin{cases} |G|^{n-1} G G'' + (n-1)|G|^{n-1}(G')^2 + \frac{\kappa}{n}\chi = 0, \\ G'(\zeta) = 0, \quad G(\Gamma) = 0. \end{cases} \tag{4.9}$$

Clearly, the transformation $h = |G|^{n-1} G$ leads to (4.5) and (4.6). Here, we shall assume that $n > 1$ the case $n < 1$ was treated in [80]. Recall that if f is a solution to (2.3) (or (2.2)) there exist $\eta_c > 0$ and Λ such that $f = \Gamma\eta + \Lambda$ for all $\eta \geqslant \eta_c$, $f'' > 0$ on $[0, \eta_c)$, $0 \leqslant f' \leqslant \Gamma$ and the following ODE

$$\frac{n}{n-1}\left(|f''|^{n-1}\right)' + \kappa f = 0, \quad 0 < \eta < \eta_c \tag{4.10}$$

is satisfied. At this stage it is worth noting that the Crocco-like transformation allows us to replace problem (4.10), where η_c is unknown by a problem posed in (ζ, Γ). Problem (4.9)

will be treated as an independent mathematical object. First, we shall prove that this problem, where $\zeta = 0$, has a unique positive solution on $(0, \Gamma)$, so, problem (2.3) will have a unique solution too.

THEOREM 4.1. *Assume* $n > 1, \zeta = 0$ *and* $\Gamma > 0$. *Problem* (4.9) *has a unique positive solution.*

PROOF. The existence of a solution is obtained in Section 3. Here, we will give a second proof. We adapt some ideas from [80]. We consider the initial value problem

$$\begin{cases} |G|^{n-1}GG'' + (n-1)|G|^{n-1}(G')^2 + \frac{\kappa}{n}\chi = 0, \quad \chi > 0, \\ G(0) = \rho, \quad G'(0) = 0, \end{cases} \tag{4.11}$$

where $\rho > 0$ is the shooting parameter. For any $\rho > 0$ fixed, problem (4.11) has a unique solution, noted G_ρ, defined on some finite interval $(0, \chi_\rho)$. We shall see that there exists a unique $\rho = \rho_c$ such that G_{ρ_c} is positive on $(0, \Gamma_0)$ and vanishes for $\chi = \Gamma_0$. Because $\rho > 0$ we may assume that $G_\rho > 0$ on some $(0, \chi_0), 0 < \chi_0 < \chi_\rho$. Integrating (4.11) from 0 to $\chi < \chi_0$ we have

$$G_\rho^{n-1}G'_\rho(\chi) = -\frac{\kappa}{n}\int_0^\chi \frac{s}{G_\rho(s)}\,\mathrm{d}s. \tag{4.12}$$

We see that G_ρ is monotone decreasing on $(0, \eta_0)$. Next, we shall see that there exists $\Gamma_0 \geqslant \chi_0$ such that $G_\rho(\Gamma_0) = 0$. Assume on the contrary that G_ρ is positive on $(0, \infty)$. Because G is monotone and bounded there exists a real $0 \leqslant l < \rho$ such that $\lim_{\chi\to\infty} G_\rho(\chi) = l$. On the other hand, it follows from the ODE of G_ρ that the function G'_ρ is decreasing, since $n > 1$. Hence $G'_\rho(\chi)$ tends to 0 as $\chi \to \infty$. Together with (4.12) we deduce

$$\int_0^\infty \frac{\chi}{G_\rho(\chi)}\,\mathrm{d}\chi = 0,$$

which is impossible. Consequently, there exists $\Gamma_0 = \Gamma_0(\rho) > 0$ such that $G_\rho(\Gamma_0) = 0$. Next, the new function $G(\chi) = w^{-3/(n+1)}G_\rho(w\chi)$, where $w = \Gamma_0/\Gamma$, satisfies (4.9). Finally the uniqueness is obtained from Theorem 4.2 below. □

THEOREM 4.2. *Assume* $n > 1$ *and* $\Gamma > \zeta \geqslant 0$. *Then problem* (4.9) *has at most one positive solution.*

PROOF. Let G_1, G_2 be positive solutions to (4.9). Assume that $G_1(\zeta) > G_2(\zeta)$.
Define

$$R = \sup\{r \in (0, \Gamma):\ G_1(\chi) > G_2(\chi),\ \text{for all } 0 \leqslant \chi \leqslant r\} \leqslant \Gamma.$$

From (4.12) we have, for all $0 \leqslant \chi \leqslant R$,

$$\begin{aligned} G_1^{n-1} G_1'(\chi) &= -\frac{\kappa}{n} \int_0^{\chi} \frac{s}{G_1(s)} \, \mathrm{d}s \\ &> -\frac{\kappa}{n} \int_0^{\chi} \frac{s}{G_2'(s)} \, \mathrm{d}s \\ &> G_2^{n-1} G_2(\chi). \end{aligned}$$

There follows that the function $G_1^n - G_2^n$ is positive, monotonic (strictly) increasing and vanishes at $\chi = R$. A contradiction. □

CHAPTER 3.3

Exact Solution for the Heat Transfer Past a Vertical Plate with an Applied Magnetic Field

In this chapter, which is a natural continuation of Chapters 3.1 and 3.2, we consider the equation

$$\left(|\theta'|^{n-1}\theta'\right)' + c\theta' + \omega\theta^{2p} = 0, \quad c \geqslant 0,$$

which is, for $p = 1$, a model of equation for a large class of equations arising in many problems of mathematical physics. Under the same discussion as in Chapter 3.1, the above equation appears in the case where the exponent parameter m is equal to $1/(1-2n)$, $n \neq 1/2$. In the present chapter we shall consider a particular case of the magnetohydrodynamic (MHD) flows where an uniform magnetic field is applied. The mathematical investigation deals with the existence, non-existence and blow-up solutions accordingly to the parameter w. A pseudosimilarity solution will be introduced.

1. Introduction

The study of MHD flow and heat transfer problems is of considerable practical interest. Such a system is used in a wide of variety of chemical engineering and metallurgical processes (cooling of continuous strips or filaments, purification of molten metals, ...).

The MHD problem was first studied by Pavlov [86] who investigated the MHD flow over a stretching wall in an electrically conducting fluid, with an uniform magnetic field. Furthermore, this problem was studied by Chakrabarti and Gupta [27], Vajravelu [100], Takhar et al. [98,97], Kumari et al. [64], Andersson et al. [4] and Watanabe and Pop [101]. The possibility of obtaining similarity solutions for the MHD flow over a stretching permeable surface subject to suction or injection was considered by [27,100] for some values of the mass transfer parameter, say, a and by Pop and Na [87], for large values of a and where the stretching velocity varies linearly with the distance and where the suction/injection velocity is constant.

Many problems of MHD flows over a flat plate embedded in a Newtonian fluid have been analyzed and numerical solutions are obtained. However, fewer works are available on the subject of MHD flow of non-Newtonian fluids. The initial aim of the present analysis is to study the effect of the magnetic field on the existence and non-existence of similarity solutions. To clarify the aim of the chapter, we are giving a brief formal derivation of problems which are under consideration. The starting point is the boundary layer system

(see Chapter 3.2)

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial T}{\partial y}\right), \tag{1.1}$$

$$\frac{\partial p}{\partial x}+g\rho+\mu k^{-1}u+\sigma B_0^2 u=0, \qquad \frac{\partial p}{\partial y}+\mu k^{-1}v+\sigma B_0^2 v=0, \tag{1.2}$$

and

$$\rho=\rho_\infty(1-\beta T+\beta T_\infty), \tag{1.3}$$

where $\rho, u, v, p, T, T_\infty, g, \mu, \beta, k, \alpha, n, g$ are given in Chapter 3.2 and σ and B_0 are, respectively, the electric conductivity and applied magnetic field. The wall temperature distribution is assumed to be a power function of the distance from the origin;

$$T_w(x)=T_\infty+Ax^m,$$

where $A>0$ is a constant and m is a real number (m will be specified later) and $n\neq 1/2$. The boundary condition of the above problem are

$$v(x,0)=0, \qquad T(x,0)=T_w(x), \tag{1.4}$$

and

$$\lim_{y\to\infty}u(x,y)=0, \qquad \lim_{y\to\infty}T(x,y)=T_\infty. \tag{1.5}$$

As in preceding chapters, the above model subject to boundary conditions (1.4), (1.5) can be expressed in a simpler form by introducing the stream function ψ and applying boundary approximations. PDEs (1.1)–(1.3) are reduced to

$$\left(\frac{\mu}{k}+\sigma B_0^2\right)\frac{\partial^2\psi}{\partial y^2}=\rho_\infty g\beta\frac{\partial T}{\partial y}, \tag{1.6}$$

$$\alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial T}{\partial y}\right)=\frac{\partial T}{\partial x}\frac{\partial\psi}{\partial y}-\frac{\partial T}{\partial y}\frac{\partial\psi}{\partial x}. \tag{1.7}$$

We then perform the similarity transformations in the usual way,

$$\eta=\{R_{1_{a_x}}\}^{1/(n+1)}\frac{y}{x}, \tag{1.8}$$

$$\begin{aligned}\psi(x,y)&=\alpha^{2n/(n+1)}\{R_{2_{a_x}}\}^{1/(n+1)}f(\eta),\\ T(x,y)&=T_\infty+(T_w-T_\infty)\theta(\eta),\end{aligned} \tag{1.9}$$

where

$$\begin{aligned} R_{1_{a_x}} &= \frac{(\beta g \rho_\infty k)^{2-n}(T_w - T_\infty)^{2-n} x^n}{\alpha \mu^{2-n}}, \\ R_{2_{a_x}} &= \frac{(\beta g \rho_\infty k)^{2n-1}(T_w - T_\infty)^{2n-1} x}{\{\alpha \mu\}^{2n-1}}, \end{aligned} \tag{1.10}$$

are the generalized modified local Rayleigh numbers. Note that for $n = 1$, $R_{1_{a_x}} = R_{2_{a_x}}$ $(= R_{a_x})$, which is the classical modified local Rayleigh number in a porous medium. Equations (1.6) and (1.7) now reduce to

$$f'' = \left(\frac{M^2}{M^2 + N^2}\right)\theta', \tag{1.11}$$

$$\left(|f''|^{n-1}\theta'\right)' + \frac{1 + m(2n-1)}{n+1} f\theta' - mf'\theta = 0, \tag{1.12}$$

where $M^2 = 1/k$ and $N^2 = \sigma B_0^2/\mu$ is the magnetic parameter. The boundary conditions read

$$f(0) = 0, \quad \theta(0) = 1, \quad f'(\infty) = 0, \quad \theta(\infty) = 0.$$

Together with (1.11), we infer

$$f' = \aleph\theta,$$

where $\aleph = M^2/(M^2 + N^2)$, which leads to the familiar equation (2.10) of Chapter 3.1. Solution of this equation were obtained for different values of $m < 0$ such that

$$1 + m(2n-1) > 0.$$

In particular, it is found that there exist (multiple) solutions and the dimensionless temperature θ has exactly one maximum value, bigger than one, at some point $\eta_c > 0$ and decreases as η increases in (η_c, ∞). Here we shall assume that $m = -1/(2n-1)$, where $n \neq 1/2$. In this way, we immediately arrive to the boundary layer problem

$$\begin{cases} (|\theta'|^{n-1}\theta')' + \omega\theta^2 = 0, \\ \theta(0) = 1, \quad \theta(\infty) = 0, \end{cases} \tag{1.13}$$

where $\omega = \aleph^{2-n}/(2n-1) \in \mathbb{R}$. Note that the above problem is completely equivalent to the boundary equation of laminar incompressible flow past a flat plate (see Chapter 3.1).

This chapter is devoted to study the structure of solutions of problem (1.13), where ω can be any real number not equal to 0.

2. The exact analytic form for negative ω

In this section we discuss in detail the eigenvalue problem (1.13) with $\omega < 0$. We are concerned with the question of existence or the non-existence and the uniqueness of solutions to (1.13). We shall assume throughout the present section that $n > 0$.

We routinely consider the general initial value problem

$$\begin{cases} (|\theta'|^{n-1}\theta')' + \omega\theta^{2p} = 0, \\ \theta(0) = 1, \quad \theta'(0) = \gamma, \end{cases} \tag{2.1}$$

where $p > 0$, and obtain, by an explicit construction, γ such that the local solution satisfies the desired condition at infinity. In fact, we are mainly concerned with the structure of the set of solutions of (2.1). Note that any local solution to the initial value problem (2.1) satisfies

$$|\theta'|^{n-1}\theta'(\eta) = |\gamma|^{n-1}\gamma + |\omega| \int_0^\eta \theta(s)^{2p}\, \mathrm{d}s, \tag{2.2}$$

on some $(0, \eta_\gamma)$ and the global structure of solution curves (in the phase plane) are simply given by

$$E(\theta, \theta') = c = const., \tag{2.3}$$

where

$$E(\theta, \theta') = \frac{n}{n+1}|\theta'|^{n+1} + \frac{\omega}{2p+1}\theta^{2p+1},$$

or

$$\theta' = \pm\left[\frac{n+1}{n}\left(c + \frac{|\omega|}{2p+1}\theta^{2p+1}\right)\right]^{1/(n+1)}.$$

On the other hand, since we discuss conditions that guarantee that any solution to (2.1) satisfies $\lim_{\eta\to\infty}\theta(\eta) = 0$, we deduce, by using E, that $\lim_{\eta\to\infty}\theta'(\eta) = 0$, and then

$$\theta' = \pm\left[\frac{n+1}{n}\frac{|\omega|}{2p+1}\theta^{2p+1}\right]^{1/(n+1)}.$$

This leads, in particular, to $\gamma = \pm\gamma_c$, where

$$\gamma_c = \left(\frac{(n+1)|\omega|}{n(2p+1)}\right)^{1/(n+1)}. \tag{2.4}$$

From (2.2) we deduce that γ must be negative.

The meaning of the above analysis is that if (1.13) has a solution, then this solution is unique. To obtain the solution we simply integrate

$$\theta' = -\left[\frac{n+1}{n}\frac{|\omega|}{2p+1}\theta^{2p+1}\right]^{1/(n+1)}.$$

Hence, for $n < 2p$, we see that θ is given by

$$\theta(\eta) = \left[1 + \frac{2p-n}{n+1}\left(\frac{(n+1)|\omega|}{n(2p+1)}\right)^{1/(n+1)}\eta\right]^{(n+1)/(n-2p)}, \tag{2.5}$$

for $n = 2p$, by

$$\theta(\eta) = \mathrm{e}^{-(|\omega|/(2p))^{1/(2p+1)}\eta}, \tag{2.6}$$

and for $n > 2p$, the solution θ is compactly supported and we have

$$\theta(\eta) = \left[1 + \frac{2p-n}{n+1}\left(\frac{(n+1)|\omega|}{n(2p+1)}\right)^{1/(n+1)}\eta\right]_+^{(n+1)/(n-2p)}, \tag{2.7}$$

where $[\cdot]_+ = \max\{0, \cdot\}$.

Returning to our original physical problem ($p = 1$), we may conclude that if $n < 1/2$ and $m = -1/(2n-1)$ the boundary layer problem (1.1)–(1.5) has exactly one similarity solution which is expressed in term of the temperature by

$$T(x,y) = T_\infty + Ax^{-1/(2n-1)}\left[1 + \frac{x}{y}\frac{2-n}{n+1}\left(\frac{(n+1)|\omega|}{3n}\right)^{1/(n+1)} \times \frac{(\beta g \rho_\infty k A)^{2-n} x^{2(n^2-1)/(2n-1)}}{\alpha \mu^{2-n}}\right]^{(n+1)/(n-2)}.$$

Note that $\theta(\eta) \sim \omega_0 \aleph^{-1}\eta^{(n+1)/(n-2)}$, $\omega_0 = \omega_0(n)$, as $\eta \to \infty$, and the rate change of θ increases with increasing value of the magnetic parameter N. However, since $f'(\eta) = \aleph\theta(\eta)$, we have $f'(\eta) \sim \omega_0\eta^{(n+1)/(n-2)}$ as $\eta \to \infty$, and then there is no effect of the magnetic parameter on the boundary layer thickness.

One of the useful features of a solution is that the estimate of the dimensionless characteristic number for (wall) heat transfer; the local Nusselt number, Nu_x. Here, the calculation of the generalized Nusselt number becomes a simple matter. Since, $Nu_x = -qx/(k_m(T_w - T_\infty))$, where q is the heat transfer defined by (1.2) of Chapter 3.2 (at $y = 0$) and k_m is the modified thermal conductivity, the Nusselt number can be expressed as

$$Nu_x = \alpha^{2n(n-1)/(n+1)} x^{2(1-n)} \{R_{2_{ax}}\}^{(n-1)/(n+1)} \{R_{1_{ax}}\}^{(2n-1)/(n+1)}.$$

Hence

$$Nu_x\{R_{1_{a_x}}\}^{(1-2n)/(n+1)} = \aleph^{n-1}\gamma_c^n\alpha^{2n(n-1)/(n+1)}x^{2(1-n)}\{R_{2_{a_x}}\}^{(n-1)/(n+1)},$$

or

$$\begin{aligned} &Nu_x\{R_{1_{a_x}}\}^{(1-2n)/(n+1)} \\ &\quad = c(n)\aleph^{(2n-1)/(n+1)}\alpha^{2n(n-1)/(n+1)}x^{2(1-n)}\{R_{2_{a_x}}\}^{(n-1)/(n+1)}, \\ &\quad c(n) = const. \end{aligned}$$

This shows, in particular, that the Nusselt number decreases with increase in the value $\aleph$.

Let us point out that the boundary condition at infinity in (1.13) is essential for the uniqueness of the solution, since this condition enables us to obtain the unique value of $\gamma = -\gamma_c$ such that the local solution of (2.1) is global, and satisfies (1.13). A natural extension to our analysis is to study the global or non-global character of the solutions of (2.1) where $\gamma \neq -\gamma_c$ or ω is positive. This is the subject of the following section.

3. Blow-up profiles and pseudosimilarity solutions

3.1. *The unbounded solutions*

The large part of this section is concerned with unbounded (global or non-global) solutions to problem (2.1), where $\omega > 0$, for different values of the parameter γ (see Subsection 3.2 below). Actually, any solution can be parameterized by

$$\frac{n}{n+1}|\theta'|^{n+1} + \frac{\omega}{2p+1}\theta^{2p+1} = \frac{n}{n+1}|\gamma|^{n+1} + \frac{\omega}{2p+1}. \tag{3.1}$$

It is argued in Section 2 that, for negative ω, the unique global solution is obtained for $\gamma = -\gamma_c$, where γ_c is given by (2.4). In fact, the necessary and sufficient condition which lead to a bounded global solution to (2.1), is that $\omega < 0$ and $\gamma = -\gamma_c$. To see this, we assume that θ, for fixed γ, is a bounded global solution of (2.1). Since θ is monotonic on (η_0, ∞), for some $\eta_0 > 0$, there exists a real number, l, such that $\lim_{\eta\to\infty}\theta(\eta) = l$, and then $\lim_{\eta\to\infty}\theta'(\eta) = 0$. Next, we use the ODE in (2.1), to deduce that $l = 0$. Hence, it follows from (3.1) that $\omega < 0$ and $\gamma = -\gamma_c$ thanks to Section 2. The meaning of this result is that the ODE in (2.1) with the initial condition $\theta(0) = 1$ has exactly one global bounded solution without any assumption at infinity. The mathematical generalized result of this conclusion is the following.

THEOREM 3.1. *Let $n > 0$, $p > 0$ and $\omega < 0$. For any $\theta_0 > 0$ the problem*

$$\begin{cases} (|\theta'|^{n-1}\theta')' + \omega\theta^{2p} = 0, \\ \theta(0) = \theta_0, \end{cases} \tag{3.2}$$

has a unique global bounded solution.

We now turn to the question of behavior of singular solutions or the large η behavior of global (unbounded) solutions to (2.1), where $\gamma \neq -\gamma_c$.

Let θ be the local solution to (2.1), defined in $(0, \eta_\gamma)$, $\eta_\gamma \leqslant \infty$, with $\gamma \neq -\gamma_c$. We recall that in this case we have unbounded trajectories; $\lim_{\eta\to\eta_\gamma} |\theta(\eta)| = \infty$. Together with (3.1) $\theta(\eta)$ tends to infinity as η approaches η_γ, and θ' is positive on (η_0, η_γ). Using once more (3.1) one can see

$$\int_{\theta(\eta_0)}^{\theta(\eta)} \frac{\mathrm{d}s}{(c + (|\omega|/(2p+1))s^{2p+1})^{1/(n+1)}} = \left(\frac{n+1}{n}\right)^{1/(n+1)} (\eta - \eta_0), \tag{3.3}$$

for all $\eta_0 < \eta < \eta_\gamma$, where $c = (n/(n+1))|\gamma|^{n+1} + \omega/(2p+1)$. Hence, the blow-up condition (at a finite point) is characterized by the following integrability condition:

$$\int_{\theta(\eta_0)}^{\infty} \frac{\mathrm{d}s}{(c + (|\omega|/(2p+1))s^{2p+1})^{1/(n+1)}} < \infty. \tag{3.4}$$

Consequently, we have the following.

THEOREM 3.2. *Let $n > 0$, $p > 0$, $\omega < 0$ and $\gamma \neq -\gamma_c$. Let θ be the local solution to* (2.1).

(i) *If $n \geqslant 2p$ then θ is global and unbounded. Moreover, if $n > 2p$ then*

$$\theta(\eta) \sim \left[\frac{n-2p}{n+1}\left(\frac{(n+1)|\omega|}{n(2p+1)}\right)^{1/(n+1)} \eta\right]^{(n+1)/(n-2p)}, \tag{3.5}$$

and if $n = 2p$, then

$$\theta(\eta) \sim \mathrm{e}^{(|\omega|/(2p))^{1/(2p+1)}\eta}, \tag{3.6}$$

as $\eta \to \infty$.

(ii) *If $n < 2p$, then θ blows up at a finite point η_γ and satisfies*

$$\theta(\eta) \sim \left[\frac{2p-n}{n+1}\left(\frac{(n+1)|\omega|}{n(2p+1)}\right)^{1/(n+1)} (\eta_\gamma - \eta)\right]^{(n+1)/(n-2p)}, \tag{3.7}$$

as $\eta \to \eta_\gamma$.

For the proofs of (3.5)–(3.7) it is sufficient to notice, from (3.1), that

$$\lim_{\eta\to\eta_\gamma} \theta'\theta(\eta)^{-(2p+1)/(n+1)} = \left(\frac{(n+1)|\omega|}{n(2p+1)}\right)^{1/(n+1)}.$$

Note also that, for any $\gamma > 0$, the blowing up point ($n < 2p$) is given explicitly by

$$\eta_\gamma = \left(\frac{n+1}{n}\right)^{-1/(n+1)} \int_1^{\infty} \frac{\mathrm{d}s}{(c + (|\omega|/(2p+1))s^{2p+1})^{1/(n+1)}}.$$

THEOREM 3.3 (Conclusion). *For any* $0 < n < 1/2$ *the physical problem* (1.13) *where* $\omega = \alpha^{2-n}/(2n-1)$, *has a unique global* (*physical*) *solution which is given by* (2.5) *with* $p = 1$. *All other solutions develop a singularity and behave like* (3.7) *where* $p = 1$.

The above arguments can also be used to exhibit the explicit solution in the case of a steady boundary layer flow due to a moving plane surface in a non-Newtonian fluid (see Chapter 3.1). If the surface is moving continuously with the velocity $u(x,0) = u_w x^{-1/(2n-1)}$, $x > 0$, the solutions (stream function) to (1.4), (1.5) of Chapter 3.1 can be described in terms of solutions to

$$\left(|f''|^{n-1} f''\right)' + w f'^2 = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \quad f(0) \in \mathbb{R},$$

where $w = 1/(2n-1)$. Setting $h = f'$ one sees that h satisfies (1.13) and then

$$f(\eta) = c + \frac{n+1}{2-n}\left(\frac{n+1}{3n(1-2n)}\right)^{-1/(n+1)} \times \left[1 + \frac{2-n}{n+1}\left(\frac{n+1}{3n(1-2n)}\right)^{1/(n+1)} \eta\right]^{(n+1)/(n-2)}, \quad c = const.$$

The stream function is given by $\psi(x,y) = A f(\eta)$, A is a constant.

3.2. *The missing solutions*

In the preceding subsection it was shown that if the ordinary differential equation $(1.13)_1$ has a global bounded solution, say θ, then

$$\frac{n}{n+1}\left|\theta'(0)\right|^{n+1} + \frac{\omega}{3}\theta^3(0) = 0,$$

holds. It can be deduced that for $\omega > 0$ the ordinary differential equation is incompatible with the boundary conditions in (1.13). So, no boundary layer motion of the type (1.8)–(1.10) is possible. This means that there is no similarity solution in the sense that the temperature T has the same shape of profile across any transverse section of the layer. The case of the classical Falkner–Skan wedge flows (Newtonian fluids), where the external velocity varies inversely-linear with the distance – divergent channel – is known since a long time [90]. Recently, Magyari, Pop and Keller [70] investigated the free convection boundary layer flow of a Newtonian fluid from a vertical plate with an inverse-linear temperature distribution: $T_w(x) = T_\infty + A/x$, $A > 0$ ($n = 1, m = -1, \omega = 1$). The authors showed that there is no solution in the usual form (see (1.9) in Chapter 3.1)

$$\psi(x,y) = h(x) f(\eta) \tag{3.8}$$

and to obtain exact solutions, a lateral suction has to be applied with a large suction parameter. In such situation the stream function is written under the following form

$$\psi(x, y) = C\big(f(\eta) + \gamma \log(x)\big), \quad \gamma > 0,\ C = const. \tag{3.9}$$

The profile θ satisfies

$$\theta'' + \gamma\theta' + \theta^2 = 0, \quad \theta(0) = 1 \quad \text{and} \quad \theta(\infty) = 0. \tag{3.10}$$

Using a natural point-mechanical analogy they conjectured the existence of $\gamma_{\min} > 0$ such that the above problem has multiple solutions for $\gamma > \gamma_{\min}$, a unique solution for $\gamma = \gamma_{\min}$ and no solution for $\gamma < \gamma_{\min}$. The above predictions were supported by numerical calculations. Note that the stream function ψ defined by (3.9) does not satisfy the invariance property (3.8). This new solution is called pseudosimilarity solution (or missing solution according to [70]), since the boundary layer problem is reduced to an ODE with the appropriate boundary conditions. The pseudosimilarity solution can be regarded as solution with dynamic scaling [61]. The pseudosimilarity solutions of problem (1.1)–(1.5) can also be found for $n > 1/2$, via the transformation (3.9), where $C = \alpha^{1/(n+1)}(\beta g \rho_\infty k A \mu^{-1})^{(2n-1)/(n+1)}$. In such situation the profile θ satisfies the second order equation

$$\big(|\theta'|^{n-1}\theta'\big)' + \gamma\aleph^{1-n}\theta' + \omega\theta^2 = 0, \tag{3.11}$$

with $\omega = \aleph^{n-2}/(2n-1)$, subject to the boundary conditions

$$\theta(0) = 1, \qquad \theta(\infty) = 0. \tag{3.12}$$

Moreover the velocity components may be expressed in terms of θ;

$$u(x, y) = \aleph\beta g \rho_\infty k A \mu^{-1} x^{-1/(2n-1)}\theta(\eta),$$
$$v(x, y) = -\frac{\alpha^{1/(n+1)}(\beta g \rho_\infty k A \mu^{-1})^{(2n-1)/(n+1)}}{x}\left[\gamma - \frac{\aleph}{2n-1}\eta\theta(\eta)\right].$$

The last equation shows that γ plays the role of a non-dimensionless suction or injection velocity, since $v(x, 0) = -(\gamma/x)\alpha^{1/(n+1)}(\beta g \rho_\infty k A \mu^{-1})^{(2n-1)/(n+1)}$. So, the case $\gamma > 0$ corresponds to suction and the case $\gamma < 0$ corresponds to injection.

In what follows, we study the slightly more general ordinary differential equation

$$\big(|\theta'|^{n-1}\theta'\big)' + \gamma\aleph^{1-n}\theta' + \omega\theta^{2p} = 0, \tag{3.13}$$

with boundary condition (3.12), where the real number $p > 1/2$.

Note that this problem appears when looking for the traveling wave solution

$$u(x, t) = \theta(x - \Sigma t), \quad x \in \mathbb{R},\ t > 0,$$

where $\Sigma = \gamma \aleph^{1-n}$, for equation

$$u_t = \left(|u_x|^{n-1}u_x\right)_x + \omega u^{2p}.$$

The most important questions treated in this direction are to determine the minimal speed ($\Sigma^\star$) such that the problem admits a traveling wave solution only for $\Sigma \geqslant \Sigma^\star$ and to investigate the properties of such a traveling wave. The reader is referred to the book of Gilding and Kersner [41] for a detailed discussion of this subject.

For (3.13) a simple inspection reveals that, if $n > 1/2$ (or $\omega > 0$), there is no solution for $\gamma < 0$; that is if a lateral injection is applied.

By a rigorous argument as in the work by Guedda, Hammouch and Kersner [49] we can show that for problem (3.13), (3.12) to possess solutions, for n and p larger than $1/2$, the following estimate

$$\gamma^{(n+1)/n}\aleph^{(1-2n)/n} \geqslant \frac{n}{(2n-1)((2p+1)n-1)} \tag{3.14}$$

must be satisfied. Note for $n = 1$, $p = 1$ and $\aleph = 1$ the above estimate gives $\gamma_{\min} \geqslant 1/\sqrt{2} \simeq 0.707$. In [70] it is found numerically that $\gamma_{\min} = 1.079131$.

Using the Crocco variables

$$\chi = \theta, \qquad h(\chi) = -\theta'(\eta), \tag{3.15}$$

we obtain the following estimate, which is much sharper in some cases.

PROPOSITION 3.1. *Let $n, p > 1/2, \gamma > 0$ and $\omega > 0$. Assume that* (3.11), (3.12) *has a solution. Then*

$$\gamma^2\aleph^{-n} \geqslant \frac{2}{2n-1}\int_0^1 \chi^{2p-1}h^{n-1}\,\mathrm{d}\chi + \aleph^{n-2}h^{2n}(1). \tag{3.16}$$

PROOF. To use (3.15), we first show that $\theta \geqslant 0$. Assume, on the contrary that $\theta(\eta_0)$ is negative for some $\eta_0 > 0$. Because $\theta(0)$ is positive and $\theta(\infty) = 0$ there exists $\eta_1 > 0$ such that $\theta'(\eta_1) = 0$ and $\theta(\eta_1) < 0$. On the other hand, we have from (3.13), that

$$E' = -\gamma\aleph^{1-n}\theta'^2 < 0,$$

where

$$E = \frac{n}{n+1}|\theta'|^{n+1} + \frac{\omega}{2p+1}\theta^{2p+1}.$$

Hence, $E(\eta) \geqslant E(\infty) = 0$ for all $\eta \geqslant 0$. But this inequality is violated at $\eta = \eta_1$. Consequently, θ is non-negative. In fact, we have either θ is positive everywhere or there exists $\eta_c > 0$ such that θ is positive on $(0, \eta_c)$ and $\theta \equiv 0$ on (η_c, ∞).

In the second step we claim that θ has at most one (local) maximum. Otherwise, there exists $\eta_1 < \eta_2$ such that $\theta'(\eta_1) = \theta'(\eta_2) = 0$ and $\theta'(\eta) > 0$ for all $\eta_1 < \eta < \eta_2$. From

(3.13) we deduce that the function $|\theta'|^{n-1}\theta'$ is monotonic decreasing on (η_1, η_2). Therefore, $|\theta'|^{n-1}\theta'(\eta_2) < |\theta'|^{n-1}\theta'(\eta_1)$, which is impossible.

The meaning of the first and second steps is that there exist $0 \leqslant \eta_0 < \eta_c \leqslant \infty$ such that $\theta > 0$ and $\theta' < 0$ on (η_0, η_c) and $\theta(\eta_c) = 0$. Moreover, if $\eta_c < \infty$ $\theta(\eta) = 0$ for all $\eta \geqslant \eta_c$.

We now use the same device as [5,41,49]. Transformation (3.15) leads to the following problem

$$nh^n h' = \gamma \aleph^{1-n} h - \omega \chi^{2p}, \quad 0 < \chi < 1, \tag{3.17}$$

$$h(0) = 0, \qquad h(1) \geqslant 0. \tag{3.18}$$

Next, we multiply (3.17) by $2\chi^{-1}h^{n-1}$ to deduce

$$\left(h^{2n}\chi^{-1}\right)' + \chi^{-2}\left(h^n - \gamma\aleph^{1-n}\chi\right)^2 + 2\omega\chi^{2p-1}h^{n-1} = \gamma^2\aleph^{2(1-n)}.$$

Finally, integrating over (0, 1) we find

$$\gamma^2\aleph^{2(1-n)} \geqslant 2\omega\int_0^1 \chi^{2p-1}h^{n-1}\,\mathrm{d}\chi + h^{2n}(1),$$

provided that $\lim_{\chi\to 0} h^{2n}\chi^{-1} = 0$. To prove this limit, we observe that h can be obtained in implicitly as

$$h^n(\chi) = \gamma\aleph^{1-n}\chi - \omega\int_0^\chi \frac{s^{2p}}{h(s)}\,\mathrm{d}s, \quad 0 < \chi < 1. \tag{3.19}$$

Therefore, $h^n(\chi) \leqslant \gamma\aleph^{1-n}\chi$ and then $h^{2n}\chi^{-1}$ tends to 0 with χ for $n > 1/2$, which ends the proof. □

Note that for $n = 1$ and $p = 1$ estimate (3.16) gives $\gamma\aleph^{-1} \geqslant 1$, which is closer to the numerical results obtained by Magyari et al. for $\aleph = 1$ [70]. Estimate (3.16) has another important consequence. For any γ and $\aleph$ there exists a smallest negative value of the initial $\theta'(0)$ such that solutions are only possible if

$$\theta'(0) \geqslant \theta'_{0,\min}(\gamma, \aleph) \geqslant -\gamma^{1/n}\aleph^{(1-n)/n}.$$

In particular, if $n = 1$,

$$\theta'_{0,\min}(\gamma, \aleph) \geqslant -\left(\gamma^2 - \frac{\aleph}{p}\right)^{1/2}.$$

For $\aleph = 1$, $p = 1$ and $\gamma = 2$, we find that $\theta'(0) \geqslant -\sqrt{3}$ $(= -1.732050808)$. In [70] it is found that $\theta'_{0,\min}(2, 1) = -1.725126$.

In the absence of the magnetic field it is shown in [70] and [49] that the physical problem has solutions only for sufficiently large suction parameter $\gamma \geqslant \gamma_{\min}$. According to the

present analysis we may conclude that, for $n > 1/2$, the physical problem has solutions for any suction parameter provided that the magnetic parameter N is large. For the case $n < 1/2$, the analysis of Section 2 indicates, in particular, that problem (3.11), (3.12) has pseudosimilarity solutions for any $\gamma > 0$ and any $\aleph > 0$.

The remainder of this chapter deals with the asymptotic behavior, for large η, of solutions to (3.13), (3.12). To this end we return to problem (3.17), (3.18). Therefore, the large η-behavior of θ is deduced from an asymptotic behavior of h as $\chi \sim 0$.

Using the works [39,40,49] one sees that the existence of solution to (3.17), (3.18) follows from the existence of solutions to the integral equation (3.19) by

$$\int_{\tau}^{\theta(\eta)} \frac{\mathrm{d}\chi}{h(\chi)} = \eta_0 - \eta,$$

for some $0 \leqslant \tau < 1$ and some real number η_0, and there exists $\Sigma^\star$ (minimal speed according to [40]) such that a solution to (3.19) exists only for $\gamma\aleph^{1-n} \geqslant \Sigma^\star$. From [39, p. 764] we can deduce that if $\gamma\aleph^{1-n} = \Sigma^\star$ there exists a unique solution, $\bar{h}$, which is positive on (0, 1), vanishes at $\chi = 1$ and satisfies $\bar{h}(\chi) \sim C_0\chi^{1/n}$, as $\chi \to 0$, for some $C_0 > 0$, for $2pn \geqslant 1$ and $n > 11$, which may depend on $\Sigma^\star$. This solution is the maximal solution to (3.19). For the classical (Newtonian) case $n = 1$, it is shown in [40] that $\bar{h}$ is such that $\bar{h}(\chi) \sim \Sigma^\star\chi$ as $\chi \to 0$ and for any $\gamma > \Sigma^\star$ any solution h is not equivalent to $\bar{h}$ and satisfies $h(\chi) \sim \frac{\omega}{\gamma}\chi^{2p}$ as χ approaches 0.

The task remaining is therefore to study the asymptotic behavior for $n \neq 1$ and larger than $1/2$. As in [21] we shall see that, in the phase plane (θ, θ'), orbits enter the origin along two directions; *slow orbits* and *fast orbits*.

PROPOSITION 3.2. *Assume* $1 \neq n > 1/2$ *and* $p > 1/2$. *Let* h *be a solution to* (3.17). *Then*

$$\textit{either } \lim_{\chi\to 0} \frac{h(\chi)}{\chi} = 0 \quad \textit{or} \quad \lim_{\chi\to 0} \frac{h(\chi)}{\chi} = \infty.$$

If in addition, $1/2 < n < 1$, $\lim_{\chi\to 0}(h(\chi)/\chi) = 0$.

PROOF. Because, $h^n(\chi) \leqslant \gamma\aleph^{1-n}\chi$, we easily see that $h(\chi)\chi^{-1}$ tends to 0 with χ for $1/2 < n < 1$. Next, we assume $n > 1$. First, we show that $\lim_{\chi\to 0}(h(\chi)/\chi)$ exists in $[0, \infty]$.

Let

$$\lambda_1 = \liminf_{\chi\to 0} h'(\chi), \qquad \lambda_2 = \limsup_{\chi\to 0} h'(\chi).$$

Suppose that $\lambda_1 \neq \lambda_2$ and fix λ such that $\lambda_1 < \lambda < \lambda_2$. Then there exist two sequences (χ_j^1) and (χ_j^2) such that $\chi_j^i \to 0$ as $j \to \infty$, $i = 1, 2$, $h'(\chi_j^i) \to \lambda$ and $(-1)^i h''(\chi_j^i) \geqslant 0$ for all $j \geqslant 0$. On the other hand, differentiating equation (3.17),

$$nh^n h'' = h'\big(\gamma\aleph^{1-n} - n^2 h^{n-1}h'\big) - 2p\omega\chi^{2p-1},$$

substituting $\chi = \chi_j^i$ and passing to the limit as $j \to \infty$, we infer $(-1)^i \lambda \gamma \aleph^{1-n} \geqslant 0$, $i = 1, 2$. Hence $\lambda = 0$. This is impossible since $h' \in C(0, \varepsilon)$, $0 < \varepsilon < 1$, and thus $\lambda_1 = \lambda_2$. Therefore, using the l'Hôpital's rule, we obtain $\lim_{\chi \to 0}(h(\chi)/\chi) = \lambda$ exists in $[0, \infty]$.

Next, let us assume that $\lambda \neq \infty$. From (3.17) we have

$$n \frac{h^n}{\chi} h' = \gamma \aleph^{1-n} \frac{h}{\chi} - w \chi^{2p-1}, \quad 0 < \chi < 1.$$

Passing to the limit $\chi \to 0$ we get $\lambda = 0$, since $p > 1/2$, $n > 1$ and $\lim_{\chi \to 0} h'(\chi) = \lambda$. □

COROLLARY 3.1. *Let θ be a global positive solution to* (3.13). *Then*

$$\textit{either } \lim_{\eta \to \infty} \frac{\theta'(\eta)}{\theta(\eta)} = 0, \quad \textit{or} \quad \lim_{\eta \to \infty} \frac{\theta'(\eta)}{\theta(\eta)} = -\infty.$$

The meaning of this result is that in the phase plane (θ, θ') the orbits are divided into the orbits which approach the origin along the positive θ-axis, called the slow orbits, according to [21], and the orbits which enter the origin along the negative θ'-axis, called the fast orbits. Note that there is no fast orbit for $n \leqslant 1$.

The following results deal with the asymptotic behavior of the fast and the slow orbits at $\eta = \infty$. One has to determine the asymptotic behavior of h at $\chi = 0$.

THEOREM 3.4. *Assume $n > 1$, $p > 1/2$. Let h be a solution to* (3.17), (3.18) *such that $h'(\chi)$ tends to infinity as $\chi \to 0$.*

Then

$$\lim_{\chi \to 0} \chi^{(1-2np)/n} \left[\frac{h(\chi)^n}{\chi} - \gamma \aleph^{1-n} \right] = \frac{n\omega}{1 - n(2p+1)} \left(\gamma \aleph^{1-n} \right)^{-1/n}. \tag{3.20}$$

PROOF. From (3.19) we find

$$\frac{h(\chi)^n}{\chi} = \gamma \aleph^{1-n} - \frac{\omega}{\chi} \int_0^\chi \frac{s^{2p}}{h(s)} \, \mathrm{d}s,$$

for all $0 < \chi < 1$. By l'Hôpital's rule,

$$\lim_{\chi \to 0} \chi^{-1} \int_0^\chi \frac{s^{2p}}{h(s)} \, \mathrm{d}s = \lim_{\chi \to 0} \frac{\chi^{2p}}{h(\chi)} = 0.$$

Hence

$$\lim_{\chi \to 0} \frac{h(\chi)^n}{\chi} = \gamma \aleph^{1-n}. \tag{3.21}$$

Next, we take a real number z and use (3.19) to deduce

$$\chi^z \left[\frac{h(\chi)^n}{\chi} - \gamma \aleph^{1-n} \right] = -\omega \chi^{z-1} \int_0^\chi \frac{s^{2p}}{h(s)} \, \mathrm{d}s.$$

As in the first steep we deduce by means of l'Hôpital's rule

$$\lim_{\chi\to 0}\chi^z\left[\frac{h(\chi)^n}{\chi}-\gamma\aleph^{1-n}\right]=\frac{\omega}{r-1}\lim_{\chi\to 0}\frac{\chi^{2p+z}}{h(\chi)}.$$

By (3.21), we have

$$\lim_{\chi\to 0}\chi^z\left[\frac{h(\chi)^n}{\chi}-\gamma\aleph^{1-n}\right]=\frac{\omega}{r-1}\left(\gamma\aleph^{1-n}\right)^{-1/n},$$

for $z=(1-2np)/n$, which yields (3.20). □

The following result describes the asymptotic behavior of the slow orbit of equation (3.17). Whence, noting that if (3.17) has such a solution for $n\geqslant 1$, there holds $h(\chi)/\chi^{2p}\to\omega\aleph^{n-1}/\gamma$ as $\chi\to 0$. We refine this limit in the next result for $2n>\max\{1,1/p\}$.

THEOREM 3.5. *Assume* $1\neq n>(1/2)\max\{1,1/p\}$ *and* $p>1/2$. *Let* h *be a solution to* (3.17), (3.18) *such that* $h'(\chi)$ *tends to* 0 *with* χ.

Then

$$\lim_{\chi\to 0}\chi^{1-2np}\left[\frac{h(\chi)}{\chi^{2p}}-\frac{\omega}{\gamma\aleph^{1-n}}\right]=\frac{2np\omega^{n+1}}{(\gamma\aleph^{1-n})^{n+2}}. \tag{3.22}$$

PROOF. Let h be a solution to (3.17), (3.18) such that $h(\chi)/\chi\to 0$ as $\chi\to 0$. Relation (3.22) will be obtained by means of the

$$h(\chi)=\chi^{2p}\left(\frac{\omega}{\gamma\aleph^{1-n}}+\varphi(\chi)\right),$$

where $\varphi\in C^1(0,1)$ such that $\varphi(\chi)\to 0$ as $\chi\to 0$.

Using the equation for h, we obtain that φ satisfies the equation

$$\chi\left(\frac{\omega}{\gamma\aleph^{1-n}}+\varphi\right)^n\varphi'+2p\left(\frac{\omega}{\gamma\aleph^{1-n}}+\varphi\right)^{n+1}=\frac{\gamma\aleph^{1-n}}{n}\frac{\varphi}{x^{2pn-1}}. \tag{3.23}$$

Setting

$$H(\chi)=\frac{\varphi(\chi)}{\chi^{2np-1}},\quad \text{in some } (0,\chi_0),\ \chi_0<1, \tag{3.24}$$

we claim that

$$\lim_{\chi\to 0}H(\chi)=2np\frac{\omega^{n+1}}{(\gamma\aleph^{1-n})^{n+2}}, \tag{3.25}$$

which proves (3.22). Assume that H is not monotonic on any interval $(0, \chi)$, $\chi < \chi_0$. Therefore, there exist sequences $(\chi_s^{\pm})_{s\in\mathbb{N}}$ converging to 0 as $s \to \infty$ such that

$$\liminf_{\chi\to 0} H(\chi) = \lim_{s\to\infty} H\big(\chi_s^-\big), \quad \limsup_{\chi\to 0} H(\chi) = \lim_{s\to\infty} H\big(\chi_s^+\big), \quad \text{and}$$
$$H'\big(\chi_s^{\pm}\big) = 0.$$

Because $\chi_s^{\pm}\varphi'(\chi_s^{\pm}) = (2pn-1)\varphi(\chi_s^{\pm})$, we deduce from (3.23)

$$(2pn-1)\left(\frac{\omega}{\gamma\aleph^{1-n}} + \varphi\big(\chi_s^{\pm}\big)\right)^n \varphi\big(\chi_s^{\pm}\big) + 2p\left(\frac{\omega}{\gamma\aleph^{1-n}} + \varphi\big(\chi_s^{\pm}\big)\right)^{n+1} = \frac{\gamma\aleph^{1-n}}{n} H\big(\chi_s^{\pm}\big)$$

and (3.25) holds.

To complete the proof, we have to show (3.25) if H is monotonic in some $(0, \chi_1)$, $\chi_1 < 1$ small. Let $L \in [-\infty, \infty]$ be such that

$$L = \lim_{\chi\to 0} H(\chi).$$

Assume first that L is finite. It follows from (3.23)

$$\left(\frac{\omega}{\gamma\aleph^{1-n}}\right)^n \lim_{\chi\to 0} \chi\varphi'(\chi) + 2p\left(\frac{\omega}{\gamma\aleph^{1-n}}\right)^{n+1} = \frac{\gamma\aleph^{1-n}}{n} L.$$

So, (3.25) is an immediate consequence of the above if $\chi\varphi'(\chi)$ goes to 0 with χ. Assume, on the contrary that $\lim_{\chi\to 0} \chi\varphi'(\chi) = l \neq 0$. In this case we get $\varphi(\chi) \simeq l \ln(\chi)$, as $\chi \to 0$, which is impossible.

Next, assume that $L = \infty$. Hence $H' < 0$ in $(0, \chi_1)$, therefore $\chi\varphi' < (2pn-1)\varphi$ in $(0, \chi_1)$. Together with the equation of φ

$$(2pn-1)\left(\frac{\omega}{\gamma\aleph^{1-n}} + \varphi\right)^n \varphi + 2p\left(\frac{\omega}{\gamma\aleph^{1-n}} + \varphi\right)^{n+1} > \frac{\gamma\aleph^{1-n}}{n} H.$$

The LHS of the latter is bounded, while the RHS is unbounded as χ approaches 0, a contradiction. Similarly, the assumption $L = -\infty$ also leads to a contradiction. The proof is finished. □

The above theorem can also be proved using approximated solutions [49]. Reasoning as in [49], we can show that, if $1 \geqslant n > (1/2)\max\{1, 1/p\}$, a solution h to (3.17), (3.18) satisfies either (3.22) or

$$\lim_{\chi\to 0} \chi^{(1-2np)/n}\big[h(\chi)\chi^{-1/n} - \big(\gamma\aleph^{1-n}\big)^{1/n}\big] = -\frac{\omega}{\gamma\aleph^{1-n}(3n-1)}. \tag{3.26}$$

In the following result we prove the monotonicity of solutions with respect to $h(1)$ and use this result to confirm the uniqueness of the fast solution for $n > 1$ which is the maximal solution.

THEOREM 3.6. *Let $n > 1/2$ and h_1 and h_2 be two solutions to* (3.17), (3.18) *for fixed $\gamma\aleph^{1-n} \geqslant \Sigma^\star$ such that $h_1(1) > h_2(1)$. Then $h_1 > h_2$ on* $(0, 1]$.

PROOF. Define

$$\chi_m = \inf\{\chi\colon\ h_1 > h_2 \text{ on } (\chi, 1)\} < 1.$$

Assume that $\chi_m > 0$. Hence, there exists $0 < \chi_0 \leqslant \chi_m$ such that $h_1 < h_2$ on $(0, \chi_0)$ and $h_1(\chi_0) = h_2(\chi_0)$. Using (3.19) we deduce

$$\int_0^{\chi_0} s^{2p}\left(\frac{h_2(s) - h_1(s)}{h_1(s)h_2(s)}\right) \mathrm{d}s = 0,$$

which is impossible, thus χ_m cannot be positive. □

As in [70], it is expected that there is a parameter family of solutions to (3.17), (3.18) or to the integral equation (3.19) provided that $\gamma\aleph^{1-n}$ is large enough. In fact, using ideas from [39,40], which are based on an approximated or regularized approach, one sees that the integral equation (3.19) has solutions for $\gamma\aleph^{1-n} \geqslant \Sigma^\star$ and for $\gamma\aleph^{1-n} = \Sigma^\star$ there is a solution which is positive on $(0, 1)$ and vanishes in $\chi = 1$. This solution is characterized as the maximal solution which is unique (for any fixed $\aleph$) and satisfies

$$h(\chi) \sim \left[\frac{n+1}{n}\omega(1-\chi)\right]^{1/(n+1)} \quad \text{as } \chi \to 1.$$

The second property of the maximal solution is that, if $n > 1$, the maximal solution is the only solution to the integral equation (3.19) that satisfies $\lim_{\chi\to 0}(h(\chi)/\chi) = \infty$. For any $\gamma\aleph^{1-n} > \Sigma^\star$ there exists multiple ordering solutions with respect to the parameter $h(1)$ in the interval $[0, \gamma^{1/n}\aleph^{(1-n)/n})$. These solutions intersect only at $\chi = 0$ (for fixed γ and $\aleph$, such that $\gamma\aleph^{1-n} > \Sigma^\star$). Arguing as in [40] we can show that any solution h, which is not equivalent to the maximal solution satisfies $h(\chi) \sim (\omega\aleph^{n-1}/\gamma)\chi^{2p}$ as $\chi \to 0$. It follows from the previous asymptotic behaviors (for small χ) that if $\geqslant n > (1/2)\max\{1, 1/p\}$ the maximal solution (which corresponds to the fast orbit if $n > 1$) satisfies (3.20) with $\Sigma^\star$ instead of $\gamma\aleph^{1-n}$. The above observations allow us to prove easily that if $\gamma\aleph^{1-n} > \Sigma^\star$, $\aleph$ fixed, any possible solution to (3.17), (3.18) cannot be maximal. To see this we assume that h_1 is a maximal solution to (3.17), (3.18) for some γ such that $\gamma\aleph^{1-n} > \Sigma^\star$. The function h_1 is positive on $(0, 1)$ and vanishes at $\chi = 1$. As in the preceding proof we get

$$h_1(\chi)^n - h_0(\chi)^n = \left(\gamma\aleph^{1-n} - \Sigma^\star\right)\chi + \omega\int_0^\chi s^{2p}\frac{h_1(s) - h_0(s)}{h_1(s)h_0(s)}\,\mathrm{d}s,$$

where h_0 is the maximal solution given by $\Sigma^\star$. Since $h_1(\chi)^n \sim \gamma\aleph^{1-n}\chi$ as $\chi \to 0$ there exists $\chi_0 > 0$ such that $h_1 > h_0$ on $(0, \chi_0)$. And then $h_1 > h_0$ on $(0, 1]$, a contradiction.

Having obtained the behavior of h at the origin, we are now in a position to get the large-η behavior of θ, which is deduced by a simple integration.

THEOREM 3.7. *Let $\gamma\aleph^{1-n} = \Sigma^\star$, $p > 1/2$. If $n > 1$ then θ is compactly supported from the right. There exists $\eta_0 > 0$ such that*

$$\theta(\eta) = 0, \quad \text{for } \eta \geqslant \eta_0.$$

The behavior of $\theta(\eta)$ at $\eta = \eta_0$ is given by

$$\theta \simeq \left(\frac{n-1}{n}\left[\gamma\aleph^{1-n}\right]^{1/n}(\eta_0 - \eta)_+\right)^{n/(n-1)}.$$

For $n = 1$

$$\theta(\eta) \simeq \theta_0 \mathrm{e}^{-\gamma\aleph^{1-n}\eta}, \quad \theta_0 = \text{const.},$$

as $\eta \to \infty$ and for $(1/2)\max\{1, 1/p\} < n < 1$

$$\theta \simeq \left(\frac{1-n}{n}\left[\gamma\aleph^{1-n}\right]^{1/n}\eta\right)^{n/(n-1)}.$$

THEOREM 3.8. *Assume $\gamma\aleph^{1-n} > \Sigma^\star$, $p > 1/2$ and $n > (1/2)\max\{1, 1/p\}$. Then*

$$\theta(\eta) \simeq \left[\frac{\omega(2p-1)}{\gamma\aleph^{1-n}}\eta\right]^{-1/(2p-1)},$$

as $\eta \to \infty$.

3.3. *Summarizing remarks*

The above discussions lead to the following observations.

(1) The proof of Theorem 3.4 uses the fact that $\chi/h(\chi)$ has a finite limit as χ tends to 0. Consequently if $n = 1$ we have

$$\lim_{\chi\to 0} \chi^{1-2p}\left[\frac{h(\chi)}{\chi} - \gamma\right] = -\frac{\omega\aleph}{2p\gamma},$$

which is equivalent to

$$h(\chi) = \gamma\chi - \frac{\aleph}{2p\gamma}\chi^{2p} + \mathrm{o}\left(\chi^{2p}\right).$$

The last estimate is obtained in [41] and is only satisfied by the unique maximal solution $\bar{h}$ which exists only if $\gamma = \Sigma^\star$.

(2) Up to now we excluded the case $\gamma\aleph^{1-n} < \Sigma^\star$ from our discussion. It is interesting to note that also such a case leads to a solution which vanishes at some point, χ_c, in the interval $(0, 1)$. The real number χ_c satisfies

$$\chi_c^{(1-2np)/n} \geqslant \frac{n}{n(2p+1)-1}\omega\gamma^{-(n+1)/n}.$$

The solution is the maximal solution in the interval $(0, \chi_c)$ and is Hölder continuous with exponent $1/(n+1)$ at $\chi = \chi_c$. Then h' is not continuous and $(0, \chi_c)$ is the maximal interval of the existence, even if $(h^{n+1})'$ is continuous (the flux condition).

(3) The above observations can also be used to obtain the numerical or analytical value of the minimal speed $\Sigma^\star$, if $n > (1/2)\max\{1, 1/p\}$, $p > 1/2$, for any fixed $\aleph$, as follows. First we consider problem (3.19), where $\gamma > 0$ is such that

$$\gamma^{(n+1)/n}\aleph^{(1-2n)/n} < \frac{n}{(2n-1)((2p+1)n-1)},$$

see (3.16). This circumstance shows that $\Sigma < \Sigma^\star$, were $\Sigma = \gamma\aleph^{1-n}$, and, in particular, $\Sigma^{1/n} < \omega/\Sigma$. Define $h_M(\chi) = \Sigma^{1/n}\chi^{1/n}$ and $h_m(\chi) = (\omega/\Sigma)\chi^{2p}$, $0 \leqslant \chi \leqslant 1$. The functions h_M and h_m intersect exactly one time at some point in $(0, 1)$, say χ_Σ. As any local solution to (3.17) satisfies

$$h_m \leqslant h \leqslant h_M$$

on $[0, \chi_s]$, χ_s small, we deduce that h crosses the curve $y = h_m(\chi)$ at some point in $(0, \chi_\Sigma]$, and after this point h is monotonic decreasing. Consequently, there exists $\chi_c < 1$ and a (unique) solution, h_0, on $[0, \chi_c]$, that is positive on $(0, \chi_c)$ and vanishes at χ_c. Next, it can be easily verified that the function $h(\chi) = \chi_c^{-(2p+1)/(n+1)}h_0(\chi\chi_c)$ is positive on $(0, 1)$, vanishes at $\chi = 0$ and at $\chi = 1$ and satisfies (3.17), with $\gamma\aleph^{1-n}\chi_c^{(1-2pn)/(n+1)}$ instead of $\gamma\aleph^{1-n}$. Therefore, $\Sigma^\star = \gamma\aleph^{1-n}\chi_c^{(1-2pn)/(n+1)}$.

(4) The conclusions of items (2) and (3) indicate also the non-global character of the problem

$$nh^n h' = \Sigma h - \omega\chi^{2p}, \quad 0 < \chi < \infty, \quad h(0) = 0.$$

(5) The case $1/2 < n \leqslant 1/2p$ can be handled, since the function

$$h(\chi) = a\chi^{2p}$$

satisfies (3.19), provided $n = 1/2p$ and

$$a^n + \frac{\omega}{a} = \gamma\aleph^{1-n}. \tag{3.27}$$

As a can be any positive real number we deduce that

$$\gamma \aleph^{(1-2n)/(n+1)} \geqslant \frac{n+1}{n} \frac{n^{1/(n+1)}}{(2n-1)^{n/(n+1)}}.$$

Returning to the function θ we find

$$\theta(\eta) = \big(1 + (2p-1)a\eta\big)_+^{1/(1-2p)}.$$

Note that if

$$\gamma \aleph^{(1-2n)/(n+1)} > \frac{n+1}{n} \frac{n^{1/(n+1)}}{(2n-1)^{n/(n+1)}}$$

equation (3.27) has two solutions and then we get two solutions to (3.13).

We close this chapter by the simple observation that the conditions at a boundary play a crucial role in the existence or the non-existence of similarity solutions. For example, if $T_w(x) = T_\infty - Ax^{-1/(2n-1)}$, $A > 0$, which is of physical interest, instead of $T_w(x) = T_\infty + Ax^{-1/(2n-1)}$, $A > 0$, we arrive at (1.13) with $\theta(0) = -1$ instead of $\theta(0) = 1$. Since the new function $\Theta = -\theta$ satisfies,

$$\begin{cases} (|\Theta'|^{n-1}\Theta')' + \overline{\omega}\Theta^2 = 0, \\ \Theta(0) = 1, \quad \Theta(\infty) = 0, \end{cases} \tag{3.28}$$

where $\overline{\omega} = -\aleph^{n-2}/(2n-1)$ we deduce that (1.1)–(1.5) has a unique similarity solution, if $n > 1/2$.

CHAPTER 3.4

Mixed Convection on a Wedge Embedded in a Porous Medium

We report on theoretical investigations for mixed convection on a wedge embedded in a porous medium. Using the non-similarity transformations (or new coordinates) boundary-layer equations, which are partial differential equations, are reduced to two decoupled ordinary differential equations. Attention is focused on how non-similarity solutions exhibit similarity solutions. The method is based on a simple decomposition technique and a modified local similarity method.

A mixed-convection parameter ξ is proposed to replace the non-similarity variable. The resulting ordinary differential equations are studied in detail. It is demonstrated, under favorable conditions, that there exists an infinite number of solutions for both cases of aiding and opposing flow.

The discussion is restricted to mixed convection on a wedge. This restriction, however, is not essential for many problems of boundary-layer theory.

1. Introduction

The prototype of the problem under investigation is

$$\alpha \frac{\partial}{\partial y}\left(\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\frac{\partial^2\psi}{\partial y^2}\right)+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial xy}+\frac{\partial\psi}{\partial y}\frac{\partial u_e}{\partial x}=0, \tag{1.1}$$

with the well known boundary condition

$$\lim_{y\to\infty}\frac{\partial\psi}{\partial y}=u_e(x), \tag{1.2}$$

where the unknown function is the stream function ψ, u_e is the free stream velocity, α and n are the thermal diffusivity and the power law exponent, respectively. A related problem which concerns the mixed convection for the Newtonian case ($n=1$) has been examined extensively in the literature, see for example [1,2,9–11,24,45,65,75]. The major themes of these papers is the study of the numerical or theoretical exact solutions and their properties. Here, by exact solutions we mean solutions which are obtained by a reduction approach (of the Prandtl's equations) which dates back to the pioneering works by Blasius [17] and Falkner–Skan [37] for a steady, two-dimensional, laminar, incompressible flow past a flat

plane or a wedge. The flow, at high Reynolds numbers, is described by the dimensionless non-linear PDE

$$\frac{\partial^3\psi}{\partial y^3} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x y} + u_e\frac{\partial u_e}{\partial x} = 0 \tag{1.3}$$

subject to suitable conditions at $y = 0$, and $y = \infty$ (see Chapter 3.1). As it is well known [7,12,28,48], PDE (1.3) need not be well posed in the class of similarity solutions satisfying ((1.9) of Chapter 3.1)

$$\psi(x, y) = h(x) f(\eta). \tag{1.4}$$

Recently, Guedda and Hammouch showed that, for $u_e(x) = u_\infty/x$, there is no stream function ψ satisfying PDE (1.3) under the similarity form (1.4). In particular, they proved that ψ can be written under the form $\psi(x, y) = f(\eta) + \gamma\log(x)$, $\gamma > 0$, where the function f satisfies an ODE. However, ψ does not satisfy the invariance property (1.9) given in Chapter 3.1. This new solution is called non-similarity solution (or pseudosimilarity solution according to [69,48]). This phenomenon is extended to the non-Newtonian case [49] (cf. Chapter 3.3). Frequently, a class of pseudosimilarity solutions in the boundary layer theory is given by the form

$$\psi(x, y) = h(x) F(\zeta, \eta), \tag{1.5}$$

where η is the similarity variable defined as above and $\zeta = \zeta(x, y)$ is the pseudosimilarity variable which may be taken as independent of y [67]. Here, by pseudosimilarity solution, we mean a solution to (1.1), with appropriate boundary conditions, having the form (1.5), such that in the new coordinates (ζ, η) the PDE satisfied by F is particularly simple. From this it is clear that a similarity solution is also a pseudosimilarity solution, where the variable ζ is constant.

In this work we are interested in pseudosimilarity and similarity solutions to mixed convection on a wedge embedded in a porous medium. The physical model is governed by equation (1.1). In fact, the aim of this paper is twofold: to identify h, ζ, χ and u_e such that the PDE (1.1) may have pseudosimilarity solutions and establish a correspondence between similarity solutions and pseudosimilarity solutions. We require that $\zeta = \zeta(x)$ and $\eta = y\chi(x)$ are such that the Jacobian, J, does not vanish;

$$J \equiv \frac{\partial(\zeta, \eta)}{\partial(x, y)} = \chi\,\partial_x\zeta \neq 0.$$

At first sight, the main idea for determining similarity solutions is to introduce new coordinates in terms of which the original problem has a particularly simple form. However, this requires a careful analysis as it is shown by Kaplun in 1954 [59]. In his Thesis, Kaplun has proved that in the boundary layer theory solutions may be influenced by the choice of coordinates system and have no physical meaning. In particular, different solutions obtained from different system coordinates may lead to different flow fields and hence to

different external flows. This difficulty is connected to the division of the flow into two distinct regions: the boundary layer region where the viscosity must be taken into account and the inviscid flow region. So, to obtain a good picture of the complete flow field, Kaplun proposed to choose the good coordinates system called optimal system or parabolic coordinates (the field flow is the same for all optimal systems). Hence, a solution of the boundary layer problem in the optimal systems contains the extremal flow which depends only on the physical problem. In fact, Kaplun explored the relation between solutions of the boundary layer problem and the given external flow in different system coordinates and showed that an optimal system is obtained accordingly to the extremal flow.

There is another difficulty in which the external flow is under discussion. The classical method for obtaining similarity solutions is the Lie-group method of infinitesimal transformation [18,19,52,83,84,70,71,85], or the dimensional analysis [8]. From mathematical viewpoint, the essence of the method is to seek transformations of each variable of the initial PDE such that this PDE is invariant under these transformations and leads to the determination of the possible symmetries. For (1.3) the external velocity, u_e, was treated as a variable and subject to transformation. However, as it is pointed out by Ovsiannikov [84] and Ludlow et al. [67], the external velocity (or, equivalently the fluid pressure in the boundary layer) is prescribed, in most cases. To overcome this difficulty Ludlow et al. [67] proposed to eliminate the external velocity from (1.3) by taking its y-derivative and to apply the Lie-group method to $(\psi_{yyy} + \psi_x \psi_{yy} - \psi_y \psi_{xy})_y = 0$. It is clear that this idea does not work for eliminating u_e from (1.1).

In the present work we shall combine the methods of Goldstein [42], Mangler [72] and Clarkson and Krustal [31] (see also [90,93]) to obtain exact solutions for mixed convection flow from a wedge embedded in a porous medium. Similarity and pseudosimilarity solutions are considered in the analysis. As a first motivation for the present study, the related problem with $n = 1$ has been examined by Bejan et al. [10]. The authors have obtained approximate non-similarity solutions based on the non-similarity technique or the local non-similarity method (LNSM for short). This method consists of writing solutions under the form (1.5), differentiating the new PDE, satisfied by the profile F, with respect to ζ and neglecting the terms that contain $\partial^2(\cdot)/\partial\zeta^2$. This, frequently, leads to a simplified computational task. This method is also referred as the second level of truncation local non-similarity technique. A detailed discussion on the LNSM can be found in [95]. The local similarity method (LSM) is also given in [95], the strategy of this method is that the first ζ-derivative terms are deleted and the resulting equations are ODEs, where ζ is considered as a parameter. This method is known as the first level of truncation local similarity technique. However, it is expected that the local non-similarity technique should give more accurate results than those from the local similarity method [79,94,73]. In [10] approximate numerical solutions, for the Newtonian case, were obtained for wedge half angles $\phi = 45, 60$ and 90 degrees. It was concluded that the numerical results were in good agreement. Recently, the analogous problems of mixed convection flows over a vertical surface in porous media is considered by Aly et al. [2]. The authors obtained numerical similarity solutions ($\zeta = \zeta_0 = const.$). In particular, they showed the existence of two (similarity) solutions, under favorable conditions. Very recently, the problem of [2] is re-examined in [45]. A theoretical investigation was initiated to determine sufficient conditions for the

existence of multiplicity (similarity) solutions. These have motivated the present investigation.

So, this chapter concerns a mathematical study of similarity and pseudosimilarity solutions to (1.1). Here, by pseudosimilarity solutions we mean solutions obtained by a modified local similarity method. By a simple argument, we shall demonstrate that pseudosimilarity solutions (1.5) are, in some sense, similarity solutions for large choices of ζ. The function F will be represented as the sum a function of η and a function of ζ, i.e. $F(\zeta, \eta) = f(\eta) + g(\zeta)$. Before this we shall see that the mixed convection on a wedge embedded in a porous medium can be modeled by (1.1) and obtain h, ζ, χ and u_e such that (1.1) may have solutions under the form (1.5). These developments are presented in Section 2. Section 3 is devoted to study the ODE satisfied by f. We demonstrate that (1.1) has multiple pseudosimilarity or similarity solutions. The asymptotic solution is given in Sections 4 and 5 deals with pseudosimilarity solutions where the free stream velocity varies as $x^{-1/(2n-1)}$, $n \neq 1/2$.

2. Pseudosimilarity or similarity reductions?

The situation discussed in this work is that of steady mixed convection flow about a wedge, with an inclined angle ϕ, which is embedded in a fluid-saturated porous medium of constant ambient temperature T_∞. The surface of the wedge is subject to a wall temperature denoted by T_w. The materials presented here are based on many references. For example the works [10] by Bejan et al., [104] by Wooding and [35] by Ece and Büyük (see also Chapter 3.2, [29,73,75,11]). Consider the Cartesian coordinates (x, y) where x and y are measured along the wedge from the apex and normal to it, respectively. The free stream velocity (main stream) is given by $u_e = u_e(x)$. Under the Darcy–Boussinesq and boundary-layer approximations the processes is described by the system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u = u_e \pm \frac{g\beta k}{\nu}(T - T_\infty), \tag{2.2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial T}{\partial y}\right), \tag{2.3}$$

where u and v are the velocity components along x and y axes and T is the fluid temperature. The constants g, β, k, ν and α are the gravitational acceleration, coefficient of thermal expansion, permeability of the porous medium, the kinematic viscosity and the effective thermal diffusivity, respectively. The plus sign in equation (2.2) (cf. [10, p. 212], [104, p. 34]) is to designate the flow as an aiding flow when the buoyancy force has a component in the direction of the free stream velocity (such as the case with $T_w > T_\infty$) and the minus as an opposing flow when the buoyancy force component is opposite to the stream free velocity. The boundary conditions are as follows:

$$v = 0, \quad T = T_w = T_\infty + t(x) \quad \text{for } y = 0, \tag{2.4}$$

and

$$u \to u_e, \quad T \to T_\infty \quad \text{as } y \to \infty, \tag{2.5}$$

where the function t will be specified later and u_e will be obtained such that equations (2.1) to (2.3) have exact solutions satisfying the boundary condition (far from the wedge) (2.5). Before this, we show that equations (2.1) to (2.3) can be reduced to (1.1). Thanks to (2.2), the temperature satisfies

$$T = T_\infty \pm \frac{\nu}{g\beta k}(u - u_e),$$

hence, T can be eliminated in (2.3) and we have

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u\frac{\partial u_e}{\partial x} + \alpha\frac{\partial}{\partial y}\left(\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y}\right). \tag{2.6}$$

To the best of the knowledge of the author, there are no numerical or analytical results about existence and qualitative properties of solutions to equation (2.6) for $n \neq 1$. However, several authors studied the Newtonian case [1,2,10,16,24,45]. Here we extend the results of [2,10,45] to the case of arbitrary $n > 0$.

Using the stream function ψ ($u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$) we get equation (1.1). Note that (2.2) can be written as

$$\frac{u}{u_e} = 1 \pm \xi(x)\theta, \tag{2.7}$$

with $\theta = (T - T_\infty)/(T_w - T_\infty)$ is the dimensionless temperature and the so-called mixed convection parameter

$$\xi(x) = \frac{g\beta k}{\nu}\frac{t(x)}{u_e(x)}. \tag{2.8}$$

In view of boundary condition (2.4) u satisfies at $y = 0$

$$\frac{u(x,0)}{u_e(x)} = 1 \pm \xi(x). \tag{2.9}$$

The parameter ξ determines whether the physical phenomenon is pure forced ($\xi = 0$), mixed ($\xi \neq 0$), or pure free convection ($\xi \to \infty$) [1,2,10,76]. Relation (2.9) can also indicate that (1.4) may be violated. In general, the conditions occurring at a boundary may cause the non-similarity solutions (see [95]) and play a part for determining pseudosimilarity variables or material coordinates. In [10] the parameter is given by $\xi = Ra_x/Pe_x$, where Ra_x and Pe_x are the Darcy-modified local Rayleigh number and local Peclet number, respectively. The authors introduced $\zeta = \xi$, as the pseudosimilarity variable. However, the variable ζ is not uniquely determined, as it can be seen from various studies. The

following expressions of ζ are obtained for the Newtonian case. In the case of a wedge with variable permeability and thermal conductivity, the mixed convection parameter can be given by $\zeta = [1 + (Ra_x/Pe_x^{3/2})^{1/3}]^{-1}$ which varies from 0 for free convection to 1 for forced convection [54], while the parameter $\zeta = Pe_x^{1/2}[Pe_x^{1/2} + Gr_x^{1/5}]^{-1}$, with Gr_x denoting the modified Grashof number, has been considered in [57] for the study of the laminar mixed convection flow of a micropolar fluid over a horizontal flat plate. Frequently, the pseudosimilarity variable ζ has the form $\zeta = \zeta_0 x^r$, where ζ_0 and r are prescribed constants [2,69,65,76,88], ...

The expression $\zeta = \zeta_0 x^r$, is also given in the non-Newtonian case, see [51] and [73] for example. In [55] the pseudosimilarity variable is given by (Görtler transformation [43])

$$\zeta = n \int_0^x u_w^{2n-1}(s)\,\mathrm{d}s,$$

where u_w is the surface velocity.

Mathematically, a natural attempt in order to investigate exact solutions is to determine a range of admissible functions, $h, \zeta\ \chi$, and u_e (depending only on x) such that the PDE (1.1) has solutions under the form (1.5). Thus the question of reduction of the PDE (1.1) arises, and together the question of whether a solution of the new problem exists and has a physical meaning or is not distinguished from one being observed in practice.

The interest in this section will be in the answer to the first question related to the similarity and pseudosimilarity reduction of the PDE (1.1). To focus ideas, we consider (1.1) merely supposing that (1.5) holds. A routine computation reveals that F satisfies the following PDE

$$\begin{aligned}
&\alpha|h|^{n-1}h\chi^{2n+1}\frac{\partial}{\partial\eta}\left(\left|\frac{\partial^2 F}{\partial\eta^2}\right|^{n-1}\frac{\partial^2 F}{\partial\eta^2}\right) + \chi^2 h\frac{\mathrm{d}h}{\mathrm{d}x}F\frac{\partial^2 F}{\partial\eta^2}\\
&\quad - h\chi\frac{\mathrm{d}(h\chi)}{\mathrm{d}x}\left(\frac{\partial F}{\partial\eta}\right)^2 + h\chi\frac{\mathrm{d}u_e}{\mathrm{d}x}\frac{\partial F}{\partial\eta}\\
&\quad = h^2\chi^2\frac{\mathrm{d}\zeta}{\mathrm{d}x}\left[\frac{\partial F}{\partial\eta}\frac{\partial^2 F}{\partial\zeta\,\partial\eta} - \frac{\partial F}{\partial\zeta}\frac{\partial^2 F}{\partial\eta^2}\right].
\end{aligned} \tag{2.10}$$

This must be a PDE for F, so that

$$\frac{1}{|h|^{n-1}\chi^{2n-1}}\frac{\mathrm{d}h}{\mathrm{d}x},\quad \frac{1}{|h|^{n-1}\chi^{2n}}\frac{\mathrm{d}(h\chi)}{\mathrm{d}x},\quad \frac{1}{|h|^{n-1}\chi^{2n}}\frac{\mathrm{d}u_e}{\mathrm{d}x},\quad \frac{h}{|h|^{n-1}\chi^{2n-1}}\frac{\mathrm{d}\zeta}{\mathrm{d}x}$$

are constants. In this context (2.10) reads

$$\begin{aligned}
&\frac{\partial}{\partial\eta}\left(\left|\frac{\partial^2 F}{\partial\eta^2}\right|^{n-1}\frac{\partial^2 F}{\partial\eta^2}\right) + aF\frac{\partial^2 F}{\partial\eta^2} - b\left(\frac{\partial F}{\partial\eta}\right)^2 + d\frac{\partial F}{\partial\eta}\\
&\quad = c\left[\frac{\partial F}{\partial\eta}\frac{\partial^2 F}{\partial\zeta\,\partial\eta} - \frac{\partial F}{\partial\zeta}\frac{\partial^2 F}{\partial\eta^2}\right],
\end{aligned} \tag{2.11}$$

where

$$\frac{1}{|h|^{n-1}\chi^{2n-1}}\frac{\mathrm{d}h}{\mathrm{d}x}=\alpha a,\quad \frac{1}{|h|^{n-1}\chi^{2n}}\frac{\mathrm{d}(h\chi)}{\mathrm{d}x}=\alpha b,\quad \frac{1}{|h|^{n-1}\chi^{2n}}\frac{\mathrm{d}u_e}{\mathrm{d}x}=\alpha d \tag{2.12}$$

and

$$\frac{h}{|h|^{n-1}\chi^{2n-1}}\frac{\mathrm{d}\zeta}{\mathrm{d}x}=\alpha c. \tag{2.13}$$

It is easily shown that solutions to equations (2.12) and (2.13) can be expressed, for some $x_0 \geqslant 0$, as

$$h(x)=A|x-x_0|^{a/((n+1)a-b(2n-1))},\quad \chi(x)=B|x-x_0|^{(b-a)/((n+1)a-b(2n-1))}, \tag{2.14}$$

$$u_e(x)=C|x-x_0|^{b/((n+1)a-b(2n-1))} \tag{2.15}$$

and

$$\zeta(x)=D\log\big(|x-x_0|\big), \tag{2.16}$$

if $(n+1)a \neq (2n-1)b$, where A, B, C and D are arbitrary positive constants. For $(n+1)a=(2n-1)b$, we get, for some real number w

$$h(x)=A\mathrm{e}^{wx},\quad \chi(x)=B\mathrm{e}^{wx},\quad u_e(x)=C\mathrm{e}^{2wx} \tag{2.17}$$

and

$$\zeta(x)=D(x-x_0). \tag{2.18}$$

Without loss of generality we may assume that the real number x_0 is zero. In both cases the constants A, B and C are such that $AB=C$ and this leads to $h\chi=u_e$ $(b=d)$ and

$$\frac{\partial F(\zeta,\eta)}{\partial \eta}=1\pm\xi(x)\theta(\zeta,\eta). \tag{2.19}$$

Note that, as indicated in [93, pp. 169–174]), only solution (2.15) is of physical interest, since for a wedge the velocities of outer flows obey the power law

$$u_e(x)=u_\infty x^m \quad (u_\infty>0,\ m\neq 0). \tag{2.20}$$

For the Newtonian case the real number m is given by $m = \beta_0/(2-\beta_0)$, where β_0 is such that $\beta_0\pi/2 = \phi$ is the half wedge angle for wedge flow. We shall see that $m = \beta_0/(n+1-\beta_0(2n-1))$. Combining (2.20), (2.14) and (2.15) we find that

$$AB = u_\infty^{2-n}, \quad u_\infty B^{-(n+1)} = \frac{\alpha b}{m} \quad \text{and} \quad a = b\frac{1+m(2n-1)}{(n+1)m},$$

where b can be any real number. Without any essential physical changes we may take $b = m(n+1)$. Thereafter, the functions h and χ have the expression

$$\begin{aligned} h(x) &= \left(\alpha(n+1)u_\infty^{2n-1}\right)^{1/(n+1)} x^{(1+m(2n-1))/(n+1)}, \\ \chi(x) &= \left(\frac{u_\infty^{2-n}}{\alpha(n+1)}\right)^{1/(n+1)} x^{(m(2-n)-1)/(n+1)}. \end{aligned} \tag{2.21}$$

Note that the power law and the exponential type occur for flow past a flat plate. For a wedge flow the exponential type is the limiting case of $m \to \infty$ ($\beta_0 = 2$ for $n = 1$). In passing, we note that condition (2.18) is used in a large number of papers for the power-law case.

For the sake of completeness, let us now derive a family of coordinates ζ in the case $a \neq 0$ ($m \neq -1/(2n-1)$, $n \neq 1/2$). The case $a = 0$ will be treated in Section 5. Below, ζ will be determined via an arbitrary function. To be more precise, we consider a continuous function Λ and look for ζ satisfying

$$\frac{h}{|h|^{n-1}\chi^{2n-1}}\frac{\mathrm{d}\zeta}{\mathrm{d}x} = \alpha\Lambda(\zeta). \tag{2.22}$$

Together with the first equation in (2.12) we deduce that $\mathrm{d}\zeta/\Lambda = (1/a)\mathrm{d}h/h$, so that the coordinate ζ is given implicitly by

$$H(\zeta) = \frac{1}{a}\log h(x), \tag{2.23}$$

where H is a primitive of $1/\Lambda$. According to (2.14) or (2.21) one obtains

$$H(\zeta) = \frac{1}{(n+1)a - (2n-1)b}\log x. \tag{2.24}$$

With the above in hand, we may carry out a reduction of (2.11) with $\Lambda(\zeta)$ instead of c. For this PDE to be amenable to analysis (from a mathematical point of view) we use a modified of the first level of truncation (modified local similarity technique; MLSM). The main idea is to neglect only the term $\partial^2 F/\partial\zeta\,\partial\eta$. Our expansion is based on written F as

$$F(\zeta,\eta) = f(\eta) + g(\zeta), \tag{2.25}$$

although other forms are possible. This ansatz anticipates that $\partial^2 F/\partial\zeta\,\partial\eta = 0$. It can be verified that

$$\left(|f''|^{n-1}f''\right)' + aff'' + bf'(1-f') + f''\left(ag + \Lambda\frac{\mathrm{d}g}{\mathrm{d}\zeta}\right) = 0,$$

where $a = (1+m(2n-1))/(n+1)$ and $b = m(n+1)$. Thus, the following

$$\left(|f''|^{n-1}f''\right)' + aff'' + bf'(1-f') + \gamma f'' = 0 \tag{2.26}$$

and

$$ag + \Lambda\frac{\mathrm{d}g}{\mathrm{d}\zeta} = \gamma, \tag{2.27}$$

hold, where γ is an undetermined constant. The general solution of (2.27) is

$$g(\zeta) = \frac{\gamma}{a} + \sigma\mathrm{e}^{-aH(\zeta)}, \quad \sigma = const. \tag{2.28}$$

It is therefore surprising to deduce from this and (2.24) that $h(x)g(\zeta)$ does not depend on Λ. In fact we have $h(x)g(\zeta) = \sigma$, so that

$$\begin{aligned}\psi(x,y) &= h(x)\left[f(\eta) + \frac{\gamma}{\alpha}\right] + \sigma,\\ &= \psi_\sigma(x,y).\end{aligned} \tag{2.29}$$

This particular form shows that ψ_σ, which is constant along the streamlines ($d(\psi_\sigma) = 0$), is a similarity solution – up to a translation – of (1.1), whenever f exists. At the first sight, our approach, which has a remarkable degree of simplicity, leads to a wide range of pseudosimilarity variables of the same phenomenon and the associated pseudosimilarity solutions are in fact similarity solutions which have the properties that any solution can be obtained from the other one by a simple translation and all solutions have the same shape. At this point, we would like to stress that $\psi_\sigma, \sigma \neq 0$, is not a similarity solution in the sense of (1.4). However, $\psi_\sigma - \sigma$, satisfies the invariance condition (1.4) and the PDE (1.1) is reduced to a system of two decoupled ordinary differential equations.

Note also that, by using (2.26), the function $\tilde{f} = f + \gamma/a$ satisfies

$$\left(|\tilde{f}''|^{n-1}\tilde{f}''\right)' + a\tilde{f}\tilde{f}'' + b\tilde{f}'(1-\tilde{f}') = 0. \tag{2.30}$$

This ODE is obtained when looking for solutions to (1.1) in the class of similarity solutions ($\psi(x,y) = h(x)\tilde{f}(\eta)$), or when looking for solutions of (2.11) by assuming that the RHS is sufficiently small (the first level of truncation). Equation (2.30) appears also when considering solutions having the Merk–Chao series expansion [73];

$$F(\zeta,\eta) = \tilde{f}(\eta) + \zeta f_1(\eta) + \zeta^2 f_2 + \cdots.$$

For the Newtonian case (2.30) reads

$$\tilde{f}''' + \frac{1+m}{2}\tilde{f}\tilde{f}'' + m\tilde{f}'(1-\tilde{f}') = 0. \tag{2.31}$$

This ODE appears in an approximate solution based on the first level of the truncation [10]. It is also obtained in the study of similarity solutions of mixed convection over a vertical surface [2,45], in mixed convection flow near the stagnation point on a heated vertical flat plate [82] and in unsteady mixed convection on a wedge embedded in a porous medium [16]. In [2] the variable ζ and the mixed convection parameter ξ are such that $\zeta = \xi = const.$, which is the ratio of the Rayleigh to Péclet numbers (R_a/P_e), while in [82,16] the authors have considered, among other things, the case $\zeta = 1$ and $\xi = \lambda$ where $\lambda = R_a/P_e$ is a constant not necessarily equal to 1.

Next, we proceed to determine the temperature distribution of the wedge (T_w) or the mixed convection parameter according to the MLSM. In particular, we will combine the above argument and the boundary condition at the wedge, that allows us to understand the role of T_w on the similarity and pseudosimilarity reduction of the system (2.1) to (2.5). In the above discussion, it is argued that there exists a large choice for the pseudosimilarity variables. For an admissible ζ, it follows from (2.19)

$$f'(\eta) = 1 \pm \xi\theta(\zeta, \eta).$$

Hence

$$f'(0) = 1 \pm \xi, \tag{2.32}$$

from which it follows that the function ξ is constant. Thus, we deduce that θ depends only on η; $\theta(\zeta, \eta) = \theta(\eta)$, and there exists a real number λ such that $t(x) = \lambda u_e(x)$, for all $x > 0$. This means that a necessary condition for the existence of similarity solutions or pseudosimilarity solutions via (1.5) and (2.25) is that the wall temperature is proportional to a power of x. In contrast to the pseudosimilarity variable ζ, the choice of the mixed convection parameter seems to be much restrictive. Thus, it is natural to ask the question in what physical sense an exact solution or approximate solution satisfying (1.5) and (2.25) can reveal if the wall temperature does not obey the power law? To be more precise, if the following problem

$$\begin{cases} (|f''|^{n-1}f'')' + aff'' + bf'(1-f') + \gamma f'' = 0, \\ f(0) = 0, \quad f'(0) = 1 \pm \xi, \quad f'(\infty) = 1, \end{cases} \tag{2.33}$$

is well posed if the quantity ξ varies with x?

To give some light to the above questions, we note first that the ordinary differential equation in (2.33) does not contain the parameter ξ. If ξ varies slowly with distance this quantity may be regarded locally as a constant parameter. Thus local similarity solutions will be obtained by solving problem (2.33) at different locations ξ (see [54,1,78]). Suppose that the free stream satisfies $u_e(x) = u_\infty x^m$ and the wall temperature varies as

$x^m(1+x)^{-1}$. Therefore $\xi(x) = \xi_0(1+x)^{-1}$, $\xi_0 = const.$ Thus, near $x = 0$ solutions correspond to similarity solutions where $\xi = \xi_0$. If x approaches infinity ξ tends to 0 and the motion, in the boundary layer, can be described by a pure forced convection. In between these extreme regions similarity solutions can be obtained at each position ξ_1, where $0 < \xi_1 < \xi_0$ in a continuous manner [7]. It is clear that any solution corresponding to a given ξ is independent of the solution at any other ξ. Note that all solutions, for different ξ, may have the same profile.

We finish this section, by showing that the decomposition (2.25) can also be combined with Görtler Transformation [43,55], [93, p. 182]. If we look for solution to (1.1) under the form

$$\psi(x, y) = \big(\alpha(n+1)\zeta\big)^{1/(n+1)} F(\zeta, \eta), \quad \eta = y\frac{u_e}{(\alpha(n+1)\zeta)^{1/(n+1)}},$$

where $\zeta = \int_0^x u_e(s)^{2n-1}\,\mathrm{d}s$, one sees

$$\frac{\partial}{\partial\eta}\left(\left|\frac{\partial^2 F}{\partial\eta^2}\right|^{n-1}\frac{\partial^2 F}{\partial\eta^2}\right) + F\frac{\partial^2 F}{\partial\eta^2} + \beta_0(\zeta)\frac{\partial F}{\partial\eta}\left(1 - \frac{\partial F}{\partial\eta}\right)$$
$$= (n+1)\zeta\left[\frac{\partial F}{\partial\eta}\frac{\partial^2 F}{\partial\eta\partial\zeta} - \frac{\partial F}{\partial\zeta}\frac{\partial^2 F}{\partial\eta^2}\right], \tag{2.34}$$

where the principal function β_0 is given by

$$\beta_0(\zeta) = \frac{(n+1)\zeta}{u_{\mathrm{e}}^{2n}}\frac{\mathrm{d}u_e}{\mathrm{d}x}. \tag{2.35}$$

The quantity β_0, which can be written as $((n+1)\zeta/u_e)\mathrm{d}u_e/\mathrm{d}\zeta$, is also called the surface velocity parameter. To use the MLSM we assume first that $\beta_0(\zeta)$ is a constant, say β_c. From (2.25) and (2.34) it follows, as above, that

$$g(\zeta) = c\zeta^{-1/(n+1)}, \quad c = const.,$$

and f satisfies (2.33) with one instead of a and β_c instead of b. In this case we deduce that the stream function takes a particular form of (2.29);

$$\psi(x, y) = \big(\alpha(n+1)\zeta\big)^{1/(n+1)} f(\eta) + c, \quad c = const.$$

Let us note that it is not true to conclude that the Görtler Transformation leads to an ODE without any restriction on the free velocity, as it may appear. In fact, from the assumption that β_0 is a constant, it is easily checked that u_e must obey the power law, by using (2.35).

At this point we conclude that the modified local similarity method seems to be advantageous since the PDE (1.1) is reduced to an ordinary differential equation (without any truncation) even if the stream function does not satisfy the invariance property (1.4). However, it should be noted that it is not clear if a solution exists for any ξ or only for some

values of ξ. Using the second level of truncation or the local non-similarity technique Bejan et al. [10] obtained, for the Newtonian case, numerical solutions for $m = 0,\ 1/3,\ 1/2$ and 1, by using the shooting method. It is observed that for $0 \leqslant \xi \leqslant 100$, the convergence was guaranteed, while for large values of ξ the convergence becomes more difficult. In the next section we shall discuss the existence of solutions to (2.33) in the case where the main stream is monotonic decreasing ($m < 0$), completing in this way the results of [10].

3. Eigensolutions

This section concerns a mathematical analysis which can be used to figure out whether or not system (2.1)–(2.5), where all the parameters keep the same meaning as before ($u_e(x) = u_\infty x^m,\ u_\infty > 0,\ m(2n-1)+1 > 0$), admits similarity or pseudosimilarity solutions. So that, according to the above discussion, the analysis is reduced to study the problem (by the transformation

$$\eta \to \big(1 + m(2n-1)\big)^{1/(n+1)} f\left(\frac{\eta}{(1+m(2n-1))^{1/(n+1)}}\right)$$

$$\begin{cases} (|f''|^{n-1}f'')' + ff'' + \beta_0 f'(1-f') + \frac{\gamma}{[1+m(2n-1)]^{n/(n+1)}} f'' = 0, \\ f(0) = 0, \quad f'(0) = 1+\varepsilon, \quad f'(\infty) = 1, \end{cases} \tag{3.1}$$

where $\beta_0 = m(n+1)/(m(2n-1)+1)$ and $\varepsilon = \pm\xi$ are real numbers. This problem is equivalent, by shifting the origin of f by $\gamma/[1+m(2n-1)]^{n/(n+1)}$, to the following eigenvalue problem

$$\begin{cases} (|f''|^{n-1}f'')' + ff'' + \beta_0 f'(1-f') = 0, \\ f(0) = \frac{\gamma}{[1+m(2n-1)]^{n/(n+1)}}, \quad f'(0) = 1+\varepsilon, \quad f'(\infty) = 1, \end{cases} \tag{3.2}$$

where γ is any real number from a mathematical point of view. In fact, in addition to the parameters β_0 and ε, the real number γ is turn out to be an important physical parameter. Problem (3.2) was used to investigate similarity solutions of boundary layer flows over a wedge or a vertical stretching surface [23,73,62]

$$v(x,0) = v_w x^{(m(2n-1)-n)/(n+1)},$$
$$v_w = -\frac{\gamma}{n+1}\big[1+m(2n-1)\big]^{1/(n+1)}\big[\alpha(n+1)u_\infty^{2n-1}\big]^{1/(n+1)}.$$

In this case γ may be positive or negative and is referred to as the suction/injection parameter. The real ε is also referred as the aiding/opposing parameter [2,16]. The case $\varepsilon > 0$ corresponds to the aiding flow and $\varepsilon < 0$ to the opposing flow. The parameter ε can be zero (the forced convection). Problem (3.2) can also be found in the steady Magnetohydrodynamic (MHD) boundary-layer flow where the Magnetic parameter $N = -\beta_0$, see [87,89] and the references therein. The conclusions of the present paper are easily extended to MHD flows for the general case $N \neq -\beta_0$.

Before stating our results, let us recall that for $\beta_0 = 0$ ($m = 0$) the ODE in (3.2) is the famous Generalized Blasius equation, see for example Chapter 3.2, [15] and [17]. For the Newtonian case this equation, with the boundary conditions $f(0) = 0$, $f'(0) \geqslant 0$ and $f'(\infty) = 1$, has been the subject of intensive study. Detailed reviews can be found in Hartmann [53]. For $\varepsilon < -1$, Merkin [74] and Hussaini and Lakin [56] showed that a solution exists only for ε larger than a critical value ε_c. The numerical results indicated that $\varepsilon_c \simeq -1.3541$. In particular, Merkin observed numerically that the problem has two solutions.

Recently, Aly, Elliott and Ingham [2] studied numerical solutions of problem (3.2), where $n = 1$ and $\gamma = 0$, for different values of β_0 and ε. More precisely, the authors studied the effect of m and ε on the existence, non-existence and non-uniqueness of solutions to the problem

$$\begin{cases} f''' + (1+m) f f'' + 2m f'(1 - f') = 0, \\ f(0) = 0, \quad f'(0) = 1 + \varepsilon, \quad f'(\infty) = 1. \end{cases} \tag{3.3}$$

It is found that if m is positive and ε takes place in the rang $[\varepsilon_0, \infty)$, for some negative ε_0, there are two solutions. The case $-1 \leqslant m \leqslant 0$ and $\varepsilon_b \leqslant \varepsilon \leqslant 0.5$, for some $\varepsilon_b < 0$, is also considered in [2]. It is shown that there exists ε_t such that the problem has two numerical solutions for $\varepsilon_b \leqslant \varepsilon \leqslant \varepsilon_t$.

Very recently, Guedda [45] reported on theoretical investigations of the existence and non-existence of solutions to (3.3) where the free stream is being retarded ($m < 0$). It is shown that, for any $-1 < m < 0$ ($\beta_0 < 0$) and $-1 < \varepsilon < 1/2$ fixed, there are multiple eigensolutions parameterized by $E \in (m/3, 0)$, where E is the fundamental "energy" function defined by

$$E = \frac{1}{2} f''^2 - \frac{m}{3} f'^3 + \frac{m}{2} f'^2. \tag{3.4}$$

In addition, it is shown that no eigensolution exists for the limiting case $m = -1$ (except the trivial one $f(\eta) = \eta$ for ε equal to zero) and no positive eigensolution for $m < -1$ and ε is greater than $1/2$. Arguing as in [45] a multiplicity of solutions to (3.2), for $\gamma > 0$, $\beta_0 < 0$, is given by Brighi et al. [23], provided $-1 < \varepsilon < 1/2$. The question of uniqueness of solution, for $m > 0$, was investigated by Brighi and Hoernel [24]. They proved that for $-1 < \varepsilon < 0$ (respectively $\varepsilon > 0$), there exists a unique convex (respectively concave) solution for any real number $f(0)$.

Returning to the original (physical) problem (2.1) to (2.5), the results obtained in [2,45] and [23] (for negative values of m) lead to some restrictive assumptions. To obtain multiple solutions the wall temperature distribution must satisfy

$$-1 < \pm \frac{g\beta k}{\nu} \frac{t(x)}{u_e(x)} < \frac{1}{2}. \tag{3.5}$$

For $t(x) = Bx^r$ [10,73] the latter condition can be written as

$$-1 < \pm \frac{g\beta k B}{\nu u_\infty} x^{r-m} < \frac{1}{2}, \tag{3.6}$$

which is not valid for all $x > 0$ if $r \neq m$. Obviously, for an aiding flow and $r > m$, condition (3.6) is satisfied for $x < l$, where the real $l = (\nu u_\infty/(2g\beta k B))^{1/(r-m)}$ can be considered as the characteristic length. Unfortunately, if $r < m$ condition (3.6) is violated for $x < l$.

The study of the present section, therefore, is to obtain the range of β_0 (the specter) for which problem (3.2) has multiple eigensolutions without any condition on ε, and to explore the effect of γ on the existence of solutions. The section is organized into a discussion of solutions for $-1 < \beta_0 < 0$ (Section 3.1) and for $-2 < \beta_0 < -1$ (Section 3.2). In our analysis the real number γ takes place in the positive real axes.

3.1. *Eigensolutions for* $-1 \leqslant \beta_0 < 0$

As usual we establish the existence of eigensolutions by mean of the so-called shooting method. Hence the condition $f'(\infty) = 1$ has to be replaced by $f''(0) = \tau$, where the real number τ is the shooting parameter. Since γ is an arbitrary real number we shall obtain the existence solutions to problem (3.2) with γ instead of $\gamma[1 + m(2n-1)]^{-n/(n+1)}$;

$$\begin{cases} (|f''|^{n-1}f'')' + ff'' + \beta_0 f'(1-f') = 0, \\ f(0) = \frac{\gamma}{[1+m(2n-1)]^{n/(n+1)}}, \quad f'(0) = 1+\varepsilon, \quad f'(\infty) = 1. \end{cases} \tag{3.7}$$

So, we consider the initial value problem

$$\begin{cases} (|f''|^{n-1}f'')' + ff'' + \beta_0 f'(1-f') = 0, \\ f(0) = \gamma, \quad f'(0) = 1+\varepsilon, \quad f''(0) = \tau. \end{cases} \tag{3.8}$$

It is important to observe that after the substitution of F (under the form (2.25)) into (2.19) the parameter τ has a physical meaning. This parameter originates from the local Nusselt number, Nu_x, and the local Peclet number via [35, p. 3], [10, p. 213],

$$Nu_x = \pm Pe_x^{1/(n+1)} \frac{|\tau|^{n-1}\tau}{\xi}.$$

So, a natural desire would be to understand how the global existence depends on τ. In fact, the real number τ will be selected such that any local solution to (3.8) is global and satisfies the boundary condition at infinity ($f'(\infty) = 1$).

According to classical theory, problem (3.8) has a unique local (maximal) solution for every τ. We denote this solution by f_τ, and its maximal interval of existence by $(0, \eta_\tau)$, $\eta_\tau \leqslant \infty$. Integrating the ordinary differential equation in (3.8), over $(0, \eta)$, $\eta < \eta_\tau$, yields to the following equality that will be useful later on:

$$\begin{aligned} &|f_\tau''|^{n-1}f_\tau''(\eta) + f_\tau(\eta)\big(f_\tau'(\eta) + \beta_0\big) \\ &\quad = |\tau|^{n-1}\tau + \gamma(1+\varepsilon+\beta_0) + (1+\beta_0)\int_0^\eta f_\tau'(s)^2\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

Note that, following the work of Coppel [33, p. 112], if η_τ is finite f_τ, f'_τ and f''_τ are unbounded as η approaches η_τ from below. In order to examine the circumstances which force f_τ to be an eigenfunction, it is convenient to separate the cases $\varepsilon \geqslant -1$ and $\varepsilon < -1$.

3.1.1. *Multiple monotonic eigensolutions.* The objective of this subsection is to present multiple solutions to (3.7) for $\varepsilon \geqslant -1$ and $\gamma \geqslant 0$. In passing, we note that it was already seen that if the parameter ε takes place in the interval $(-1, 1/2)$, there is an infinite number of monotonic solutions [45,23]. This class of solutions was obtained by showing that any local solution to (3.8) such that

$$\varepsilon \geqslant -1, \quad \frac{n}{n+1}|\tau|^{n+1} \leqslant \frac{|\beta_0|}{6}(1+\varepsilon)^2(1-2\varepsilon) \tag{3.10}$$

or, for some $\eta_0 \geqslant 0$,

$$\begin{aligned} &f_\tau(\eta_0) > 0, \quad 0 < f'_\tau(\eta_0) < \frac{3}{2}, \\ &\frac{n}{n+1}|f''_\tau|^{n+1} \leqslant \frac{|\beta_0|}{6} f'^2_\tau(\eta_0)\big[3 - 2f'_\tau(\eta_0)\big], \end{aligned} \tag{3.11}$$

is global and satisfies the boundary condition at infinity.

There is a certain analogy with the Generalized Falkner–Skan Problem. Close to the state $f = 1$ an asymptotic form of the ordinary differential equation in (3.8) is

$$\big(|f''|^{n-1}f''\big)' + ff'' + \frac{\beta_0}{2}\big(1 - f'^2\big) = 0,$$

which, for $n = 1$, is the Falkner–Skan (FS) equation with $\beta_0/2$ instead of β_0. In [96], it is indicated that in 1937 Hartree considered the (FS) equation with the boundary condition $f(0) = f'(0) = 0$ and $f'(\infty) = 1$. Hartree found, among other results, that if $-0.1988 < \beta_0 < 0$ the problem has solutions for an infinite number of values of $f''(0)$.

In [33] it is obtained bounds of $f''(0)$ such that the (FS) equation with the appropriate boundary conditions has a solution for $\beta_0 > 0$. The lower bound is similar to (3.10) with $n = 1$ and $>$ instead of $\leqslant$ (the opposite direction). This estimate is obtained by using a comparison principle and the explicit solution to $f''' + \beta_0(1 - f'^2) = 0$. Unfortunately, this method does not work here, since the equation $f''' + \beta_0 f'(1 - f') = 0$ has no non-trivial solution such that $f'(\infty) = 1$ if $\beta_0 < 0$ (see Section 5).

Here, we present a different sufficient condition which guarantees the existence of solutions for any $\varepsilon \geqslant -1$. It is clear that condition (3.10) leads to reasonable restrictions on τ and ε ($\varepsilon < 1/2$). In this work we remove restriction (3.10) and investigate monotonic solutions, including solutions which exhibit overshoot, i.e. $f' > 1$ for some $\eta > 0$. One of the possible conditions in the existence result that we state below requires that f_τ and f'_τ still positive. To see this, we take τ such that

$$|\tau|^{n-1}\tau + \gamma(1 + \varepsilon + \beta_0) > 0, \tag{3.12}$$

and use identity (3.9) to observe that f'_τ and f_τ are always positive except at the origin if $\varepsilon = -1$ and $\gamma = 0$. This is evident when one notes that $f_\tau(0) \geqslant 0$, $f'_\tau(0) \geqslant 0$ and $|f''_\tau|^{n-1} f''_\tau(\eta) + f_\tau(\eta)(f'_\tau(\eta) + \beta_0) > 0$.

Using the "energy" function

$$E = \frac{n}{n+1}|f''_\tau|^{n+1} - \frac{\beta_0}{3} f'^3_\tau + \frac{\beta_0}{2} f'^2_\tau,$$

we deduce that f'_τ and f''_τ are bounded, since G is bounded from bellow by $\beta_0/6$ and satisfies

$$E'(\eta) = -f_\tau(\eta) f''_\tau(\eta)^2 \quad \text{for } 0 \leqslant \eta < \eta_\tau.$$

Consequently, f_τ must be global and goes to infinity as $\eta \to \infty$. Otherwise, $f_\tau(\eta) \to l$, $0 < l < \infty$, $f'_\tau(\eta) \to 0$ as $\eta \to \infty$ and $f''_\tau(\eta_j) \to 0$ for some consequence η_j converging to infinity with j. In view of (3.9) one sees

$$0 = -l\beta_0 + |\tau|^{n-1}\tau + \gamma(1 + \varepsilon + \beta_0) + (1 + \beta_0) \int_0^\infty f'_\tau(\eta)^2 \, \mathrm{d}\eta,$$

which is impossible. Proceeding in the familiar way [45], we can check that $f''_\tau(\eta) \to 0$ as $\eta \to \infty$. First, assume that $|f''_\tau|^{n-1} f''_\tau$ is monotone on some interval $[\eta_0, \infty)$. Since f''_τ and f'_τ are bounded, we immediately get that $f''_\tau(\eta) \to 0$ as $\eta \to \infty$. Next, let us assume that $|f''_\tau|^{n-1} f''_\tau$ is not monotone on any interval $[\eta_0, \infty)$. Then, there exists an increasing sequence (η_r) going to infinity, such that $(|f''_\tau|^{n-1} f''_\tau)'(t_r) = 0$, $|f''_\tau|^{n-1} f''_\tau(\eta_{2r})$ and $|f''_\tau|^{n-1} f''_\tau(\eta_{2r+1})$ are a local minimum and a local maximum for $|f''_\tau|^{n-1} f''_\tau(\eta)$, respectively. On the other hand, we deduce from the ODE satisfied by f_τ that

$$f''_\tau(\eta_r) = -\frac{\beta_0}{f_\tau(\eta_r)}\big(1 - f'_\tau(\eta_r)\big) f'_\tau(\eta_r),$$

which implies $|f''_\tau|^{n-1} f''_\tau(\eta_r) \to 0$ as $r \to \infty$, since f'_τ is bounded and $f_\tau(\eta) \to \infty$ with η. Therefore, $f''_\tau(\eta) \to 0$ as $\eta \to \infty$, and then there exists a real number $L \geqslant 0$ such that $f'_\tau(\eta) \to L$ as $\eta \to \infty$. We once again make recourse to (3.9) to deduce

$$|f''_\tau|^{n-1} f''_\tau(\eta) = \beta_0 L(L-1)\eta + \mathrm{o}(1),$$

as $\eta \to \infty$. This leaves only the possibility that L is either 1 or 0. To show that $L = 1$, we assume, on the contrary, that L is zero. In view of the equation of f_τ

$$\big(f''_\tau \mathrm{e}^P\big)' = -\beta_0(1 - f'_\tau) f'_\tau |f''_\tau|^{1-n} \mathrm{e}^P, \tag{3.13}$$

where P is a primitive of $\frac{1}{n} f_\tau |f''_\tau|^{1-n}$. Therefore, f'_τ is monotonic decreasing in some (η_0, ∞), η_0 large and f'''_τ is non-negative in (η_0, ∞). From (3.8) one sees

$$\int_{\eta_0}^{\eta} \frac{n(f''_\tau)^{n-1} f'''_\tau}{f_\tau(1 - f'_\tau)} - \log\left(\frac{1 - f'_\tau(\eta)}{1 - f'_\tau(\eta_0)}\right) = -\beta_0 \log\left(\frac{f_\tau(\eta)}{f_\tau(\eta_0)}\right).$$

Since the integral on the LHS is well defined for $\eta \to \infty$, we occasionally find that f_τ has a finite limit at infinity, a contradiction. Therefore, $f'_\tau(\infty) = 1$. Hence, the result found for problem (3.8) shows that problem (3.7) has an infinite number of monotonic eigensolutions for any (fixed) real β_0 which is ranged from -1 to 0 ($-1 \leqslant \beta_0 < 0$) if $\varepsilon \geqslant -1$.

3.1.2. *Multiple eigensolutions for $\varepsilon < -1$ and $n \geqslant 1$.* In this subsection we report on the existence result concerning reverse flow solutions. Here, by a reverse flow solution of the problem (3.7) we mean an eigensolution f such that there exists a real $\eta_0 > 0$ such that $f'(\eta_0) < 0$. Reverse solutions were first obtained by Stewartson [96] for the Falkner–Skan equation (see (1.11) of Chapter 3.1), subject to the initial conditions $f(0) = f'(0) = 0$, where $-0.1988 < \beta_0 < 0$. The case $\beta_0 < -0.1988$ was considered by Libby and Liu [66] (numerical solutions). Recently, Aly et al. [2] have numerically investigated a region of reversed flow for problem (3.3). The variation of $f''(0)$ as a function of ε has been obtained for ε in some range $\varepsilon_b \leqslant \varepsilon \leqslant 0.5$, where ε_b is a negative real number which depends on m. The numerical analysis showed that $f''(0)$ is positive for $\varepsilon < 0$.

In this subsection, we assume that $n \geqslant 1$ and ε takes place in the interval $(-\infty, -1)$. We shall see that there exists a real number $\eta_2 > 0$ such that $f' < 0$ on $(0, \eta_2)$ and $f' > 0$ on (η_2, ∞). First, observe that with the help of (3.13) the boundary condition at infinity in (3.7) is violated for $\tau \leqslant 0$. Therefore, we assume $\tau > 0$. More precisely, we take τ such that

$$\tau^n > \frac{(1+\varepsilon)^2}{2\gamma}\tau^{n-1} - \gamma\left(1 + \varepsilon + \frac{\beta_0}{2}\right), \tag{3.14}$$

where $\gamma > 0$ (since it is required to find positive eigensolutions the real number γ is positive). To solve (3.7) we set, as in Chapter 3.1,

$$\Gamma = \left\{\eta \geqslant 0:\ f_\tau, -f'_\tau \text{ and } f''_\tau \text{ are positive on } (0, \eta)\right\},$$
$$H = f_\tau |f''_\tau|^{n-1} f''_\tau - \frac{1}{2} f'^2_\tau |f''_\tau|^{n-1} + f^2_\tau f'_\tau + \frac{\beta_0}{2} f^2_\tau$$

and let

$$\eta_0 = \sup \Gamma.$$

From the initial conditions, η_0 is positive, may be infinite, and $H(0) > 0$. Moreover, the following

$$H' = \left(\beta_0 + 2 + \frac{n-1}{2n}\right) f_\tau f'^2_\tau + \frac{n-1}{2n}\beta_0 f'^3_\tau [1 - f'_\tau] f''^{-1}_\tau,$$

holds on $(0, \eta_0)$ (recall that f_τ is the local solution of (3.8)). We shall see that f_τ is positive, has only one minimum, is monotonic increasing after this minimum, goes to infinity at infinity and its first derivative goes to unity as η approaches infinity.

Assume that $\eta_0 = \infty$. Then

$$f_\tau(\eta) > 0, \quad f'_\tau(\eta) < 0, \quad f''_\tau(\eta) > 0 \quad \text{and} \quad H'(\eta) > 0 \quad \forall \eta > 0. \tag{3.15}$$

Hence, f_τ has a finite limit at infinity, say $0 \leqslant l < \infty$, $f'_\tau(\eta)$ tends to 0 as $\eta \to \infty$, and there exists a sequence $(s_r)_r$ converging to infinity with r such that $f''_\tau(s_r)$ tends to 0 as $r \to \infty$). Because $H(0) \leqslant H(s_r)$, we obtain a contradiction by taking the limit as r approaches infinity. Hence only property $\eta_0 < \infty$ is acceptable. Note that in this case we arrive again at a contradiction if we assume that $f_\tau(\eta_0) = 0$ or $f''_\tau(\eta_0) = 0$. The logic of this argument is that $f'_\tau(\eta_0) = 0$ and $f_\tau(\eta_0)$ is a positive local minimum of $f_\tau(\eta)$. Suppose that there exists $\eta_1 > \eta_0$ such that $f''_\tau > 0$ on $[\eta_0, \eta_1)$, $f''_\tau(\eta_1) = 0$ and $f'_\tau(\eta_1) > 0$ on $(\eta_0, \eta_1]$. If $f'_\tau(\eta_1) < 3/2$, we deduce that (3.10) holds at the point $\eta = \eta_1$ and then f_τ is a solution to (3.7) which is monotonic increasing after η_0. Next, assume that $f'_\tau(\eta_1) \geqslant 3/2$. We use identity (3.9) to deduce

$$|f''_\tau|^{n-1} f''_\tau(\eta) + f_\tau(\eta)\big(f'_\tau(\eta) + \beta_0\big) = C + (1 + \beta_0) \int_{\eta_1}^{\eta} \big(f'_\tau(s)\big)^2 \, \mathrm{d}s,$$

where

$$C = |f''_\tau|^{n-1} f''_\tau(\eta_1) + f_\tau(\eta_1)\big(f'_\tau(\eta_1) + \beta_0\big),$$

which is positive, since $\beta_0 + 1 \geqslant 0$. Hence, arguing as in Subsection 3.1.1 we get a global solution to our problem. Note that we arrive at the same conclusion if we assume that the second derivative of f_τ never vanishes.

4. Asymptotic solution ($\varepsilon \gg 1$)

This section presents a method used in [2] for constructing asymptotic solutions as the parameter ε tends to infinity, since solutions to (3.7) constructed in the previous sections exist for any ε. As in [2], we assume that the asymptotic solution has the form

$$f(\eta) = \eta + \varepsilon^a \Theta(s), \quad s = \varepsilon^a \eta, \ \varepsilon > 0,$$

where a and b are positive constants to be determined. Substituting the above expression into (3.7) leads to the new problem for Θ,

$$\begin{cases} \varepsilon^{a(n-2)+b(2n-1)}(|\Theta''|^{n-1}\Theta'')' + \Theta\Theta'' - \beta_0(\Theta')^2 + \eta\varepsilon^{-a}\Theta'' \\ \quad - \beta_0\varepsilon^{-(a+b)}\Theta' = 0, \\ \Theta(0) = \gamma\varepsilon^{-a}, \quad \Theta'(0) = \varepsilon^{1-a-b}, \quad \Theta'(\infty) = 0. \end{cases} \tag{4.1}$$

To eliminate the parameter ε it is convenient to take $a = (2n-1)/(n+1)$, $b = (2-n)/(n+1)$, where $n > 1/2$. As $\varepsilon \to \infty$, we obtain, formally, the following problem, in variable s,

$$\big(|\Theta''|^{n-1}\Theta''\big)' + \Theta\Theta'' - \beta_0\Theta'^2 = 0, \tag{4.2}$$

and

$$\Theta(0)=0, \quad \Theta'(0)=1, \quad \Theta'(\infty)=0. \tag{4.3}$$

Problem (4.2), (4.3) occurs in modeling the free convection along a vertical flat plate embedded in a porous medium (Chapter 3.1), see for example the references [30,12,46,47,55]. This problem is in fact a particular situation of system (2.1) to (2.3) when the free stream velocity is zero ($u_e \equiv 0$). For β_0 equal to -1 and $n=1$ ($m=-1/3$) the problem has an explicit solution given by $\Theta(s)=\sqrt{2}\tanh(s/\sqrt{2})$ [6]. In [12] the authors constructed a bounded solution for any $\beta_0 > -1$ and $n=1$. They showed that there exists $a^\star < 0$ (which is independent of ε) such that the local solution given by $\Theta''(0)=a^\star$ is global and satisfies $0 \leqslant \Theta(s) < \sqrt{2(2-\beta_0)}$. The study of unbounded solutions to (4.2), (4.3), where $n=1$, has been made by Guedda [46] for the range $-1 \leqslant \beta_0 < 0$. It is proved that any local solution such that $\Theta''(0) > 0$ is global and satisfies

$$\Theta(s) \sim s^{1/(1-\beta_0)} \quad \text{and} \quad \Theta'(s) \sim s^{\beta_0/(1-\beta_0)},$$

for large s. The general case $n \neq 1$ is studied in Chapter 3.1 (see also [47]). Consequently, from the relation $f''(0)=\varepsilon^{3/2}\Theta''(0)$, $n=1$, it is expected that the local solution $f_{\tau^\star}$, where $\tau^\star = \varepsilon^{3/2}a^\star$ is global for $\varepsilon \gg 1$ and the function $\eta \to f_{\tau^\star}(\eta)-\eta$ is bounded for $-1 \leqslant \beta_0 < 0$. The relation between $f''(0)$ and $\Theta''(0)$ may also lead to a large η behavior of $f_\tau(\eta)$, where $n > 1/2$, for any positive τ and for $\varepsilon \gg 1$:

$$f_\tau(\eta) \sim \eta\big(1+\varepsilon^{(2n-1)(2-\beta_0)/((n+1)(1-\beta_0))}\eta^{\beta_0/(1-\beta_0)}\big).$$

5. The limit case $\beta_0 = -\infty$

In this section we study system (2.1)–(2.5), where the free stream velocity is assumed to be given by

$$u_e(x) = u_\infty x^{-1/(2n-1)}, \quad n > \frac{1}{2}. \tag{5.1}$$

Using steps analogous to the above, we find that $h \equiv const.$ ($a=0$) and the function g must vary as $\log(x)$. So, substituting

$$\begin{aligned} &\psi(x,y) = \big(\alpha(n+1)u_\infty^{2n-1}\big)^{1/(n+1)}\big(f(\eta)+\gamma\log(x)\big), \\ &\eta = \left(\frac{u_\infty^{2-n}}{\alpha(n+1)}\right)^{1/(n+1)} y x^{-1/(2n-1)}, \end{aligned} \tag{5.2}$$

into (1.1) leads, obviously, to

$$\big(|f''|^{n-1}f''\big)' + \gamma f'' + \omega f'(1-f') = 0, \quad \omega = -\frac{1}{2n-1}, \tag{5.3}$$

with boundary conditions

$$f'(0) = 1 + \varepsilon, \quad f'(\infty) = 1. \tag{5.4}$$

Of course, $f(0)$ is any real number and γ is connected to the injection/suction parameter v_w ($v(x,0) = v_w x^{-1}$) by the relation

$$\left(\alpha(n+1)u_\infty^{2n-1}\right)^{1/(n+1)}\gamma + v_w = 0. \tag{5.5}$$

The case of the impermeable surface ($\gamma = v_w = 0$) is easy to analyze. Any solution to equation (5.3) satisfies

$$f'' = \pm\left[\frac{n+1}{n}\left(c + \frac{\omega}{3}f'^3 - \frac{\omega}{2}f'^2\right)\right]^{1/(n+1)}, \quad c = const.$$

Hence, problem (5.3), (5.4) has no solution if $\gamma = 0$, except the trivial one, $f(\eta) = \eta$, if $\varepsilon = 0$.

We note that, for $1/2 < n < 2$, there is no global solution if $\varepsilon > 1/2$ and there is a unique – up to a translation – global non-trivial solution to (5.3) with the boundary condition $f'(0) = 1 + \varepsilon$. If $n = 1$ this solution is given explicitly by

$$f'(\eta) = \begin{cases} \frac{3}{2}[1 - \mathrm{th}^2\{\eta/2 + \mathrm{arcth}(\sqrt{(1-2\varepsilon)/3})\}], & \text{for } -1 < \varepsilon < \frac{1}{2}, \\ \frac{3}{2}[1 - \coth^2\{\eta/2 + \mathrm{arcoth}(\sqrt{(1-2\varepsilon)/3})\}], & \text{for } \varepsilon < -1. \end{cases}$$

Our aim is to obtain $\gamma_{\min}$ (the spectrum) such that the problem has multiple solutions for any $\gamma > \gamma_{\min}$. To this end we study the problem satisfied by $Z = f'$;

$$\begin{cases} (|Z'|^{n-1}Z')' + \gamma Z' + \omega Z(1-Z) = 0, \\ Z(0) = 1 + \varepsilon, \quad Z(\infty) = 1. \end{cases} \tag{5.6}$$

We stress that there are also some connections between (5.6) and the question of finding the traveling wave (TW) solution $U(x,t) = Z(x - \gamma t)$ with $Z(\infty) = 1$, for (see Chapter 3.3)

$$U_t = \left(|U_x|^{n-1}U_x\right)_x + \omega U(1-U), \tag{5.7}$$

which is nothing but the famous Generalized F–KPP (Fischer–Kolmogorov Petrovskii Piskunov) equation if we put $V = 1 - U$; that is [38,63]

$$V_t = \left(|V_x|^{n-1}V_x\right)_x + \omega V(1-V). \tag{5.8}$$

For $n = 1$ this PDE reads

$$V_t = V_{xx} + V(1-V). \tag{5.9}$$

It turns out that there exists the minimal speed, $\gamma^\star = 2$, such that equation (5.9) admits a unique monotonic TW, $\Gamma_\gamma \in [0, 1]$, connecting 1 and 0 for any $\gamma \geqslant \gamma^\star$ and no such TW for $\gamma < \gamma^\star$ [63,99,39–41]. In fact, if we are looking for an approximate solution to (5.9)

$$Z(\eta) \sim 1 - \mathrm{e}^{-r\eta}, \quad r > 0, \text{ for large } \eta,$$

it is easy to see that

$$\gamma(r) = \frac{1}{r} + r.$$

As r is arbitrary, one sees that $\gamma(r) \geqslant 2$ and the minimal speed $\gamma^\star = 2$ is reached for $r = 1$. Consequently, for any $\gamma \geqslant 2$ there exists $-1 < \varepsilon < 0$ depending on γ such that (5.9) has a solution given by $Z(\eta) = 1 - \Gamma_\gamma(\eta)$, for all $\eta > 0$. It is worth mentioning here that (5.6) with $n = 1$ admits the exact solution [34, p. 37]

$$Z(\eta) = 1 - \left(1 + \mathrm{e}^{\eta/\sqrt{6}}\right)^{-2}, \tag{5.10}$$

where $\varepsilon = -1/4$ and $\gamma = 5/\sqrt{6} > \gamma^\star = 2$. The corresponding velocity components and the fluid temperature are then given by

$$u(x, y) = \frac{u_\infty}{x}\left[1 - \left(1 + \mathrm{e}^{\eta/\sqrt{6}}\right)^{-2}\right], \quad \eta = \sqrt{\frac{u_\infty}{2\alpha}}\, y x^{-1},$$

$$v(x, y) = -\frac{5}{\sqrt{6}}\frac{\sqrt{\alpha u_\infty}}{x} + \frac{y}{x^2}\left[1 - \left(1 + \mathrm{e}^{\eta/\sqrt{6}}\right)^{-2}\right],$$

$$T(x, y) = T_\infty + 4(T_w - T_\infty)\left(1 + \mathrm{e}^{\eta/\sqrt{6}}\right)^{-2}.$$

Very recently, the existence of oscillatory TW solutions has been considered by Sánchez-Valdés and Hernández-Bermejo [92]. The authors proved that if $0 < \gamma < 2$ there exists a TW solution to (5.9) (connecting 1 and 0). This indicates that $\gamma_{\min} = 0$. Moreover, the TW solution oscillates around the limit value 1 with strictly decreasing amplitude.

A global view of the nature and multiplicity of solutions to (5.6), where $\gamma \neq 0$, can be obtained by a natural point-mechanical analogy [69]. Problem (5.6) describes, formally, the motion of a classical particle of coordinate Z, time η and potential energy $W = \omega Z^2/3 - \omega Z^3/3$ in the presence of the friction coefficient γ. Here, we continue with our analytical approach. The construction of multiple solutions to problem (5.6) relies on a construction of trajectories in a phase space (Z, Z'). Close to the equilibrium point $(1, 0)$, we get by linearization, for $n = 1$,

$$Z'' + \gamma Z' + Z = 0,$$

which is solved explicitly. The general solution is given by

$$Z(\eta) = \begin{cases} c_1 e^{\lambda_- \eta} + c_2 e^{\lambda_+ \eta} & \text{for } |\gamma| > 2, \\ (c_3 \eta + c_4) e^{-(\gamma/2)\eta} & \text{for } |\gamma| = 2, \\ c_5 e^{-(\gamma/2)\eta} \cos\left(\frac{\sqrt{4-\gamma^2}}{2}(\eta - c_6)\right) & \text{for } |\gamma| < 2, \end{cases} \tag{5.11}$$

where c_j, $j = 1, \ldots, 6$, are constants and

$$\lambda_\pm = \begin{cases} \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2} & \text{for } |\gamma| > 2, \\ -\frac{\gamma}{2} & \text{for } |\gamma| = 2. \end{cases}$$

Hence, the boundary condition $Z(\infty) = 1$ requires that $\gamma > 0$ (suction velocity). In what follows, we use the "energy method" to obtain a sufficient condition on (ε, τ) such that any local solution to the differential equation in (5.6) such that $Z(0) = 1 + \varepsilon$, $Z'(0) = \tau$, is global and its first derivative tends to 1 at infinity. In particular, a Domain of Attraction for $(1, 0)$ will be determined.

In order to obtain global solutions, it is convenient to make the transformation $Z = 1 + \varphi$. Then φ satisfies

$$\left(|\varphi'|^{n-1}\varphi'\right)' + \gamma\varphi' + |\omega|(\varphi + 1)\varphi = 0, \tag{5.12}$$

and

$$\varphi(0) = \varepsilon, \qquad \varphi(\infty) = 0. \tag{5.13}$$

Define

$$\mathcal{P} = \left\{ (\varepsilon, \tau);\ -1 < \varepsilon,\ \frac{n}{n+1}|\tau|^{n+1} + \frac{|\omega|}{2}\varepsilon^2 + \frac{|\omega|}{3}\varepsilon^3 < \frac{|\omega|}{6} \right\}.$$

Let us consider a one-parameter of family of curves defined by

$$E(\varphi, \varphi') = C, \tag{5.14}$$

where

$$E(\varphi, \varphi') = \frac{n}{n+1}|\varphi'|^{n+1} + \frac{|\omega|}{2}\varphi^2 + \frac{|\omega|}{3}\varphi^3$$

is the "energy" function for (5.12). We observe that $\mathcal{P}$ can be defined by $E(\varepsilon, \tau) = C$, $\varepsilon > -1$, for all $0 \leqslant C < |\omega|/6$ and (5.14) represents the trajectories of solutions of (5.12) where $\gamma = 0$. When $C = |\omega|/6$ we obtain an homoclinic orbit, $\mathcal{H}$, for (5.12), where $\gamma = 0$, with $\lim_{\eta \to \pm\infty}(\varphi, \varphi') = (-1, 0)$ and define a separatrix cycle; the bounded open domain with the boundary $\mathcal{H}$ is the set $\mathcal{P}$. We shall see that $\mathcal{P}$ is an attractor set for (5.12) for

$\gamma > 0$. Suppose we start with an initial condition $(\varepsilon, \tau) \in \mathcal{P}$ in the non-linear ordinary equation (5.12). As $\mathrm{d}E/\mathrm{d}\eta \leqslant 0$ the local solution cannot leave $\mathcal{P}$. By LaSalle invariance principle we deduce that the w-limit set, $\Gamma^+(\varepsilon, \tau)$ is a non-empty, connected subset of $\mathcal{P} \cap \{\varphi' = 0\}$ (see [3, p. 234]). However, if $\varphi \neq 0$, $\varphi' = 0$ is a transversal of the phase-flow, so the w-limit set is $\{(0, 0)\}$. This means that $\mathcal{P}$ is a domain of attraction for the equilibrium point $(0, 0)$.

Clearly, for any $\gamma > 0$ and any $-1 < \varepsilon < 1/2$, problem (5.6) has multiple solutions (including monotonic solutions and solutions which oscillate an infinite number of times through $Z = 1$) and reach $Z = 1$ asymptotically for $\eta \to \infty$ as (5.11), if $n = 1$.

Acknowledgements

The author would like to thank the referee(s) for the careful reading of the original manuscript and for making constructive suggestions, which have improved the presentation of this work. To the memory of Professor Bernard Risbourg and Maurice Avantin.

References

[1] M. Ali and F. Al-Yousef, *Laminar mixed convection boundary-layers induced by a linearly stretching permeable surface*, Inter. J. Heat Mass Transfer **45** (2002), 4241–4250.
[2] E.H. Aly, L. Elliott and D.B. Ingham, *Mixed convection boundary-layer flow over a vertical surface embedded in a porous medium*, Eur. J. Mech. B Fluids **22** (2003), 529–543.
[3] H. Amann, *Ordinary Differential Equation, An Introduction to Nonlinear Analysis*, De Gruyter Stud. Math., vol. 13 (1990).
[4] H.I. Andersson, *MHD flow of a viscous fluid past a stretching surface*, Acta Mech. **95** (1992), 227–230.
[5] C. Atkinson, G.E.H. Reuter and C.J. Ridler-Rowe, *Traveling wave solutions for some nonlinear diffusion equations*, SIAM J. Math. Anal. **12** (1981), 880–892.
[6] W.H.H. Banks, *Similarity solutions of the boundary layer equations for a stretching wall*, J. Mécan. Théor. Appl. **2** (1983), 375–392.
[7] W.H.H. Banks and M.B. Zaturska, *Eigensolutions in boundary-lary flow adjacent to a stretching wall*, IMA J. Appl. Math. **36** (1986), 375–392.
[8] G.I. Barenblatt, *Scaling, Selfsimilar, and Intermediate Asymptotic*, Cambridge Texts Appl. Math., vol. 14 (1996).
[9] A. Bejan, *Convection Heat Transfer*, Wiley, New York (1984) (Chapter 10).
[10] A. Bejan, T.A. Laursen and J.V.C. Vargas, *Nonsimilar solutions for mixed convection on a wedge embedded in a porous medium*, Int. J. Heat Fluid Flow **16** (1995), 211–216.
[11] A. Bejan and D.A. Nield, *Convection in Porous Media*, second ed., Springer, New York (1999).
[12] Z. Belhachmi, B. Brighi and K. Taous, *On a family of differential equation for boundary layer approximations in porous media*, Eur. J. Appl. Math. **12** (2001), 513–528.
[13] Z. Belhachmi, B. Brighi and K. Taous, *Solutions similaires pour un problème de couches limites en milieu poreux*, C. R. Acad. Paris, Ser. IIb **328** (2000), 407–410.
[14] Z. Belhachmi, B. Brighi and K. Taous, *On the concave solutions of the Blasius equation*, Acta Math. Univ. Comenian. (N.S.) **69** (2) (2000), 199–214.
[15] M. Benlahsen, M. Guedda and R. Kersner, *On the generalized Blasius equation*, in preparation.
[16] S. Bhattacharyya, A. Pal and I. Pop, *Unsteady mixed convection on a wedge in a porous medium*, Int. Comm. Heat Mass Transfer **25** (1998), 743–752.
[17] H. Blasius, *Grenzschichten in Flüssigkeiten mit kleiner Reibung*, Z. Math. Phys. **56** (1908), 1–37.

[18] G.W. Bluman and J.D. Cole, *Similarity Methods for Differential Equations*, Appl. Math. Sci., vol. 13, Springer, New York (1974).
[19] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Appl. Math. Sci., vol. 81, Springer, New York (1989).
[20] H. Bohme, *Non-Newtonian Fluid Mechanics*, North-Holland Ser. Appl. Math. Mech. (1987).
[21] H. Brezis, L.A. Peletier and D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Ration. Mech. Anal. **95** (1986), 185–209.
[22] B. Brighi, *On a similarity boundary layer equation*, Z. Anal. Anwend. **21** (4) (2002), 931–948.
[23] B. Brighi, M. Benlahsen, M. Guedda and S. Peponas, *Mixed convection on a wedge embedded in a porous medium*, in preparation.
[24] B. Brighi and D. Hoernel, *On the concave and convex solution of mixed convection boundary layer approximation in a porous medium*, Appl. Math. Lett. **19** (2006), 69–74.
[25] B. Brighi and T. Sari, *Blowing up coordinates for a similarity boundary layer equation*, Discrete Contin. Dynam. Systems Ser. A **12** (5) (2005), 929–948.
[26] A.J. Callegari and A. Nachman, *Some singular nonlinear differential equations arising in boundary layer theory*, J. Math. Anal. Appl. **64** (1978), 96–105.
[27] A. Chakrabarti and A.S. Gupta, *Hydromagnetic flow and heat transfer over a stretching sheet*, Quart. Appl. Math. **37** (1979), 73–78.
[28] M.A. Chaudhary, J.H. Merkin and I. Pop, *Similarity solutions in free convection boundary-layer flows adjacent to vertical permeable surfaces in porous media. I. Prescribed surface temperature*, Eur. J. Mech. B Fluids **14** (2) (1995), 217–237.
[29] P. Cheng, *Combined free and forced convection flow about inclined surfaces in porous media*, Int. J. Heat Mass Transfer **20** (1977), 807–814.
[30] P. Cheng and W.J. Minkowycz, *Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike*, J. Geophys. Res. **82** (14) (1977), 2040–2044.
[31] P.A. Clarkson and M.D. Kurscal, *New similarity reductions of the Boussinesq equation*, J. Math. Phys. **30** (1989), 2201–2213.
[32] C.V. Coffman, *On the positive solutions of boundary value problems for a class of nonlinear differential equations*, J. Differential Equations **3** (1967), 92–111.
[33] W.A. Coppel, *On a differential equation of boundary layer theory*, Philos. Trans. Roy. Soc. London Ser. A **253** (1960), 101–136.
[34] V.G. Danilov, V.P. Maslov and K.A. Volosov, *Mathematical Modeling of Heat and Mass Transfer Processes*, Math. Appl., Kluwer Academic Publishers (1995).
[35] M.C. Ece and E. Büyük, *Similarity solutions for free convection to power-law fluids from a heated vertical plate*, Appl. Math. Lett. **15** (2002), 1–5.
[36] L.E. Ericson, L.T. Fan and V.G. Fox, *Heat and mass tranfer on a moving continuous flat plate with suction or injection*, Ind. Eng. Chem. **5** (1966), 19–25.
[37] V.M. Falkner and S.W. Skan, *Solutions of the boundary layer equations*, Philos. Mag. **12** (1931), 865–896.
[38] A.A. Fisher, *The wave of advance of advantageous genes*, Ann. Eugenics **7** (1937), 355–369.
[39] B.H. Gilding and R. Kersner, *A necessary and sufficient condition for finite speed of propagation in the theory of doubly nonlinear degenerate parabolic equations*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 739–767.
[40] B.H. Gilding and R. Kersner, *A Fisher/KPP-type equation with density-dependent diffusion and convection: Travelling-wave solutions*, J. Phys. A: Math. Gen. **38** (2005), 3367–3379.
[41] B.H. Gilding and R. Kersner, *Travelling waves in nonlinear diffusion–convection reaction*, Progr. Nonlinear Differential Equations Appl. **60** (2005).
[42] S. Goldstein, *A note on the boundary layer equations*, Proc. Cambridge Philos. Soc. **35** (1939), 338–340.
[43] H. Görtler, *Eine neue Reihenentwicklung für laminare Grenzschichten*, ZAMM Z. Angew. Math. Mech. **32** (1952), 270–271.
[44] M. Guedda, *Nonuniqueness of solutions to differential equations for a boundary layer approximations in porous media*, C. R. Mecanique **300** (2002), 279–283.
[45] M. Guedda, *Multiple solutions of mixed convection boundary-layer approximations in a porous medium*, Appl. Math. Lett. **19** (2006), 63–68.

[46] M. Guedda, *Similarity solutions to differential equations for boundary-layer approximations in porous media*, ZAMP J. Appl. Math. Phys. **56** (2005), 749–762.
[47] M. Guedda and Z. Hammouch, *Self-similar solutions of a power-law non-Newtonian fluid flows*, in preparation.
[48] M. Guedda and Z. Hammouch, *On similarity and pseudo-similarity solutions of Falkner–Skan problems*, Fluid Dynamics Res. **38** (2006), 211–223.
[49] M. Guedda, Z. Hammouch and R. Kersner, *On a spectral problem in laminar boundary-layer theory*, in preparation.
[50] M. Guedda and L. Veron, *Bifurcation phenomena associated to the p-Laplace operator*, Trans. Amer. Math. Soc. **310** (1) (1988), 419–431.
[51] F.M. Hady, *Mixed convection boundary-layer flow of non-Newtonian fluids on a horizontal plate*, Appl. Math. Comp. **68** (1995), 105–112.
[52] B.K. Harrison and F.B. Estabrook, *Geometric approach to invariance groups ans solution of partial differential systems*, J. Math. Phys. **12** (1971), 653–670.
[53] P. Hartmann, *Ordinary Differential Equations*, John Willey & Sons, New York (1964).
[54] I.A. Hassanien, *Variable permeability effects on mixed convection along a vertical wedge embedded in a porous medium with variable surface heat flux*, Appl. Math. Comp. **138** (2003), 41–59.
[55] T.G. Howell, D.R. Jeng and K.J. De Witt, *Momentum and heat transfer on a continuous moving surface in a power-law fluid*, Int. J. Heat Mass Transfer **40** (8) (1997), 1853–1861.
[56] M.Y. Hussaini and W.D. Lakin, *Existence and nonuniqueness of similarity solutions of a boundary-layer problem*, Quart J. Mech. Appl. **39** (1986), 15–23.
[57] F.S. Ibrahim and I.A. Hassanien, *Local nonsimilarity solutions for mixed convection boundary layer flow of a micropolar fluid on horizontal flat plates with variable surface temperature*, Appl. Math. Comp. **122** (2001), 133–153.
[58] D.B. Ingham and S.N. Brown, *Flow past a suddenly heated vertical plate in a porous medium*, J. Proc. Roy. Soc. London Ser. A **403** (1986), 51–80.
[59] S. Kaplun, *The role of coordinate systems in boundary layer theory*, Ph.D. Thesis, California Institute of Technology (1954) and J. Appl. Math. Phys. **5** (1954), 111–135.
[60] J.N. Kapur and R.C. Srivastava, Z. Angew. Math. Phys. **14** (1963), 383–388.
[61] R. Kersner and M. Vicsek, *Travelling waves and dynamic scaling in a singular interface equation: analytical results*, J. Phys. A: Math. Gen. **30** (1997), 2457–2465.
[62] H.H. Kim and A.H. Eraslan, *Flow of a non-Newtonian fluid past wedges with wall mass injection*, J. Hydronaut. **3** (1) (1969), 57–59.
[63] A. Kolmogorov, I. Petrovskii and I. Piskunov, *Study of the diffusion equation with growth of the quantity of matter and its application to a biology problem*, Bull. l'Université d'état à Moscou, Ser. Int., Section A, vol. 1 (1937).
[64] M. Kumari, H.S. Takhar and G. Nath, *MHD flow and heat transfer over a stretching surface with prescribed wall temperature or heat flux*, Warm und Stoffubert **25** (1990), 331–336.
[65] M. Kumari, H.S. Takhar and G. Nath, *Mixed convection flow over a vertical wedge embedded in a highly porous medium*, Heat Mass Transfer **37** (2000), 139–146.
[66] P.A. Libby and T. Liu, *Further solutions of the Falkner–Skan equation*, AIAA J. **5** (1967), 1040–1042.
[67] D.K. Ludlow, P.A. Clarkson and A.P. Bassom, *New similarity solutions of the unsteady incompressible boundary-layer equations*, Quart J. Mech. Appl. Math. **53** (2) (2000), 175–206.
[68] E. Magyari and B. Keller, *Exact solutions for self-similar boundary-layer flows induced by permeable stretching walls*, Eur. J. Mech. B Fluids **19** (1) (2000), 109–122.
[69] E. Magyari, B. Keller and I. Pop, *The "missing" self-similar free convection boundary-layer flow over a vertical permeable surface in a porous medium*, Transport Porous Media **46** (1) (2002), 91–102.
[70] E. Magyari, I. Pop and B. Keller, *The "missing" similarity boundary-layer flow over a moving plane surface*, Z. Angew. Math. Phys. **53** (2002), 782–793.
[71] J. Malek, K.R. Rajagopal and M. Růžička, *Existence and regularity of solutions and the stability of the rest state for fluids with shear dependent viscosity*, Math. Models Methods Appl. Sci. **5** (1995), 789–897.
[72] W. Mangler, *Die"ahnlichen" Losungen der Prandtlsschen Grenzschichtgleichungen*, ZAMM Z. Angew. Math. Mech. **23** (1943), 241–251.

[73] M. Massoudi, *Local non-similarity solutions for the flow of a non-Newtonian fluid over a wedge*, Int. J. Non-Linear Mech. **36** (2001), 961–976.
[74] J.H. Merkin, *Mixed convection boundary-layer flow on a vertical surface in a saturated porous medium*, J. Engrg. Math. **14** (1980), 301–313.
[75] J.H. Merkin and I. Pop, *Mixed convection boundary-layer on a vertical cylinder embedded in a saturated porous medium*, Acta Mech. **66** (1987), 251–262.
[76] J.H. Merkin and I. Pop, *Mixed convection along a vertical surface: Similarity solutions for uniform flow*, Fluid Dyn. Res. **30** (2002), 233–250.
[77] H.K. Moffatt, *High frequency excitation of liquid metal systems*, IUTAM Symposium: Metallurgical Application of Magnetohydrodynamics, Cambridge (1982).
[78] A. Mucoglu and T.S. Chen, *Mixed convection on inclined surfaces*, ASME J. Heat Transfer **101** (1979), 422–426.
[79] T.Y. Na, *Computational Methods in Engineering Boundary Value Problems*, Academic Press, New York (1979).
[80] A. Nachman and A. Callegari, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math. **38** (2) (1980), 275–281.
[81] A. Nachman and S. Taliafero, *Mass transfer in boundary layers for power fluids*, Proc. Roy. Soc. London Ser. A **365** (1722) (1979), 313–326.
[82] R. Nazar, N. Amin and I. Pop, *Unsteady mixed convection boundary-layer flow near the stagnation point over a vertical surface in a porous medium*, Int. J. Heat Mass Transfer **47** (2004), 2681–2688.
[83] P.J. Olver, *Application of Lie Groups to Differential Eequations*, second ed., Graduate Texts in Math., vol. 107, Springer, New York (1993).
[84] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).
[85] M. Pakdemirli, M. Yürüsoy and A. Küçükbursa, *Symmetry groups of boundary layer equations of a class of non-Newtonian fluids*, Int. J. Non-Linear Mech. **31** (3) (1996), 267–276.
[86] K.B. Pavlov, *Magnetohydrodynamic flow of an incompressible viscous fluid caused by deformation of a surface*, Magnitnaya Gidrodinamika **4** (1974), 146–147.
[87] I. Pop and T.Y. Na, *A note on MHD flow over a stretching permeable surface*, Mech. Res. Comm. **25** (3) (1998), 263–269.
[88] I. Pop and T.Y. Na, *Darcian mixed convection along slender vertical cylinders with variable surface heat flux embedded in a porous medium*, Int. Comm. Heat Mass Transfer **25** (2) (1998), 251–260.
[89] A. Raptis, C. Perdikis and H.S. Takhar, *Effect of thermal radiation on MDH flow*, Appl. Math. Comp. **153** (2004), 645–649.
[90] L. Rosenhead (Ed.), *Laminar Boundary Layers*, Calderon Press, Oxford (1963).
[91] B.C. Sakiadis, *Boundary-layer behavior on continuous solid surface, I. Boundary-layer equations for two-dimensional and axisymmetric flow*, AIChE J. **7** (1961), 26–28.
[92] A. Sánchez-Valdés and B. Hernández-Bermejo, *New travelling wave solutions for the Fisher–KPP equation with general exponents*, Appl. Math. Lett. **18** (2005), 1281–1285.
[93] H. Schlichting and K. Gersten, *Boundary Layer Theory*, 8th revised and enlarged ed., Springer-Verlag Berlin (2000).
[94] R. Seshadri and T.Y. Na, *Group Invariance in Engineering Boundary Value Problems*, Springer, New York (1985).
[95] E.M. Sparrow, H. Quack and J. Boerner, *Local nonsimilarity boundary solutions*, AIAA **8** (1970), 1936–1942.
[96] K. Stewartson, *Further solutions of the Falkner–Skan equation*, Proc. Cambridge Philos. Soc. **50** (1954), 454–465.
[97] H.S. Takhar, M.A. Ali and A.S. Gupta, *Stability of magnetohydrodynamic flow over a stretching sheet*, Lielpeteris and Moreau, eds, Liquid Metal Hydrodynamics, Kluwer Academic Publishers, Dordrecht (1989), 465–471.
[98] H.S. Takhar, A. Raptis and C. Perdikis, *MHD asymmetric flow past a semi-infinite moving plate*, Acta Mech. **65** (1987), 287–290.
[99] K. Ushiyama, *The behavior of solutions of some nonlinear diffusion equations for large time*, J. Math. Kioto Univ. **18** (3) (1978), 453–511.

[100] K. Vajravelu, *Hydromagnetic flow and heat transfer over a continuous moving, porous surface*, Acta Mech. **64** (1986), 179–185.

[101] T. Watanabe and I. Pop, *Hall effects on magnetohydrodynamic boundary layer flow over a continuous moving flat plate*, Acta Mech. **108** (1995), 35–47.

[102] H. Weyl, *On the differential equations of the simplest boundary-layer problems*, Ann. Math. **43** (1942), 381–407.

[103] J.S.W. Wong, *On the generalized Emden–Fowler equations*, SIAM Rev. **17** (1975), 339–360.

[104] R.A. Wooding, *Convection in a saturated porous medium at large Rayleigh number or Péclet number*, J. Fluid Mech. **15** (1963), 527–545.

[105] L.C. Zheng and X.X. Zhang, *Skin friction and heat tranfer in power-law fluid laminar boundary layer along a moving surface*, Int. J. Heat Mass Transfer **45** (2002), 2667–2672.

CHAPTER 4

Monotonicity and Compactness Methods for Nonlinear Variational Inequalities

N. Kenmochi

Department of Mathematics, Chiba University, 1-33 Yayoi-cho, Inage-Ku, Chiba, 263-8222 Japan
E-mail: kenmochi@faculty.chiba-u.jp

Contents

1. Preliminaries . . . 205
2. Mappings of monotone type and basic properties . . . 215
3. Fixed-point results . . . 223
4. Maximal monotonicity of mappings . . . 227
5. Perturbations of monotone type . . . 234
6. Examples of subdifferentials . . . 243
7. Convergence of maximal monotone mappings . . . 256
8. Convergence of convex functions . . . 265
9. Variational inequalities . . . 277
10. Quasi-variational inequalities . . . 285
References . . . 297

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 4
Edited by M. Chipot
© 2007 Elsevier B.V. All rights reserved

This page intentionally left blank

In this chapter we study nonlinear variational inequalities of elliptic type, which have arisen in various mathematical modelings of nonlinear phenomena, such as reaction-diffusion problems, free boundary problems and finance problems. Our approach to these problems is based on the theory of nonlinear mappings of the monotone type.

Firstly we shall introduce the concepts of maximal monotonicity, semi-monotonicity, pseudo-monotonicity and of type M in abstract settings, and give some existence results for abstract functional equations of the form:

$$f \in Au + Bu, \tag{1}$$

where A and B are multivalued (nonlinear) mappings from a real reflexive Banach space X into its dual space X^*. The solvability of (1) is reduced to investigation of the range of $A + B$, and the regular approximation theory is especially important in the procedure of solving many concrete problems. As to this point we study the convergence of monotone mappings in the graph sense and convex functions in the sense of Mosco.

Furthermore, we study abstract variational inequalities of the form:

$$u \in K; \quad \langle Au - f, u - v\rangle \leqslant 0, \quad \forall v \in K, \tag{2}$$

where K is a closed convex set in X and $\langle\cdot,\cdot\rangle$ stands for the duality between X^* and X. This is a special case of (1) when B is the subdifferential of the indicator function $I_K(\cdot)$ of the convex set K

The goal of this chapter is to apply these abstract results to the boundary-value problems for elliptic partial differential equations or systems, and to variational inequalities. Finally we shall try to solve the so-called quasi-variational inequalities of the form:

$$u \in K(u); \quad \langle Au - f, u - v\rangle \leqslant 0, \quad \forall v \in K(u), \tag{3}$$

in which the constraint set $K = K(u)$ depends upon the unknown u. It seems that many of stationary free boundary problems have such sorts of nonlinear structures as (3).

1. Preliminaries

In this section we introduce some general concepts for multivalued mappings, and mention briefly some fixed-point theorems, the mini-max lemma, the lemma of choice to extract a convergent sequence from a given convergent filter and function spaces of the Sobolev type with generalized Green's formula which will be used in this chapter.

(1) *Multivalued mappings*

In general, for two real linear Hausdorff topological spaces Y, Z and a subset Y_0 of Y we mean by $A : Y_0 \to Z$ a multivalued mapping A from Y_0 into Z, namely, A assigns a subset Au of Z, which may be empty, to each u in Y_0. We use the following notation:

$$D(A) = \{u \in Y_0;\ Au \neq \emptyset\},$$

$$R(A) = \bigcup_{u \in Y_0} Au,$$

$$G(A) = \big\{[u, u^*] \in Y_0 \times Z;\ u^* \in Au\big\},$$

which are respectively called the *domain, range* and *graph* of A. Of course, $A = B$ if and only if $G(A) = G(B)$. Also, the *inverse* of A, denoted by A^{-1}, is defined by

$$A^{-1}(u^*) = \begin{cases} \{u \in Y_0;\ u^* \in Au\} & \text{if } u^* \in R(A), \\ \emptyset & \text{if } u^* \notin R(A); \end{cases}$$

note here that for any mapping from Y_0 into Z its inverse is always defined, since the concept of inverse is considered in the multivalued sense. For two mappings A and B from Y_0 into Z, the sum $A + B$ is a mapping from Y_0 into Z which is given by

$$D(A + B) = D(A) \cap D(B)$$

and

$$(A + B)u = \begin{cases} \{a^* + b^*;\ a^* \in Au,\ b^* \in Bu\}, & \forall u \in D(A) \cap D(B), \\ \emptyset, & \text{otherwise.} \end{cases}$$

The concept of "boundedness" is used in the multivalued sense, namely, a mapping $A : Y_0 \to Z$ is called bounded, if the image $\bigcup_{u \in E} Au$ is bounded in Z for each bounded subset E of Y_0. Also, a mapping $A : Y_0 \to Z$ is called *upper semicontinuous*, if for each $u \in D(A)$ and each neighborhood W of Au there is a neighborhood U of u such that

$$Av \subset W, \quad \forall v \in U \cap D(A).$$

For singlevalued mappings the upper semicontinuity means the usual continuity.

For a general real reflexive Banach space Y we denote by Y^* its dual space and by $\langle u^*, u\rangle_{Y^*,Y}$ (or simply $\langle u^*, u\rangle$, if no confusion) the duality pairing between $u^* \in Y^*$ and $u \in Y$, and by $|u|_Y$ and $|u^*|_{Y^*}$ the norms of $u \in Y$ and $u^* \in Y^*$, respectively; unless otherwise stated, $|\cdot|_{Y^*}$ denotes the dual norm of $|\cdot|_Y$. For simplicity, the space Y (resp. Y^*) equipped with its weak topology is denoted by Y_w (resp. Y^*_w).

For any sequence $\{v_n\}$ in Y we indicate by "$v_n \to v$ in Y (as $n \to +\infty$)" its strong convergence, and by "$v_n \to v$ weakly in Y" its weak convergence. A subset E of $Y \times Y^*$ is called *demiclosed*, if E is closed in $Y \times Y^*_w$ and $Y_w \times Y^*$.

For multivalued mappings from a subset Y_0 of a real reflexive Banach space Y into Y^*, the upper semicontinuity is often considered for those from Y_0 into Y^*_w. When singlevalued mappings are continuous from Y_0 into Y^*_w, they are called *demicontinuous* from Y_0 into Y^*.

(2) *Mini-max lemma*

We prove the so-called "Mini-Max lemma".

MIN–MAX LEMMA. *Let E_1 and E_2 be two non-empty, bounded, closed and convex subsets of $\mathbf{R}^N$, and let $H(\cdot,\cdot)$ be a real-valued continuous function on $E_1 \times E_2$ such that*

- *for each $y \in E_2$ the function $x \to H(x, y)$ is convex on E_1,*
- *for each $x \in E_1$ the function $y \to H(x, y)$ is concave (i.e. $y \to -H(x, y)$ is convex) on E_2.*

Then

$$\min_{x\in E_1}\max_{y\in E_2} H(x, y) = \max_{y\in E_2}\min_{x\in E_1} H(x, y). \tag{1.1}$$

Moreover, (1.1) *is equivalent to the fact that there are $x_0 \in E_1$ and $y_0 \in E_2$ such that*

$$H(x_0, y) \leqslant H(x_0, y_0) \leqslant H(x, y_0), \quad \forall x \in E_1,\ \forall y \in E_2. \tag{1.2}$$

PROOF. We simply denote by $|\cdot|$ the usual euclidean norm $|\cdot|_{\mathbf{R}^N}$. It is enough to prove that

$$\min_{x\in E_1}\max_{y\in E_2} H(x, y) \leqslant \max_{y\in E_2}\min_{x\in E_1} H(x, y), \tag{1.3}$$

since the opposite inequality is easily seen.

Fixing any small $\varepsilon > 0$, we put

$$H_\varepsilon(x, y) := H(x, y) + \varepsilon|x|^2, \quad \forall x \in E_1,\ \forall y \in E_2.$$

Clearly $H_\varepsilon(x, y)$ has the same properties as $H(x, y)$ and is strictly convex in x. By the strictly convexity of $H_\varepsilon(\cdot, y)$ on E_1 for each $y \in E_2$ there is a unique point $x(y) \in E_1$ at which $H_\varepsilon(\cdot, y)$ attains the minimum on E_1. Now we define $m_\varepsilon : E_2 \to E_1$ by putting $m_\varepsilon(y) = x(y)$, namely

$$H_\varepsilon\big(m_\varepsilon(y), y\big) = \min_{x\in E_1} H_\varepsilon(x, y) \quad \text{for each } y \in E_2.$$

We show that m_ε is continuous from E_2 into E_1. In fact, assume that $y_n \in E_2$ and $y_n \to y$ in $\mathbf{R}^N$. Let z be any cluster point of $\{m_\varepsilon(y_n)\}$ with $m_\varepsilon(y_{n_k}) \to z$ for a subsequence $\{n_k\}$ of $\{n\}$. Then we have

$$H_\varepsilon\big(m_\varepsilon(y_{n_k}), y_{n_k}\big) \leqslant H_\varepsilon(x, y_{n_k}), \quad \forall x \in E_1.$$

Letting $k \to +\infty$ yields

$$H_\varepsilon(z, y) \leqslant H_\varepsilon(x, y), \quad \forall x \in E_1,$$

which shows that $z = m_\varepsilon(y)$. It results that $m_\varepsilon(y_n) \to m_\varepsilon(y)$. Thus m_ε is continuous from E_2 into E_1.

Next, putting $h_\varepsilon(y) = H_\varepsilon(m_\varepsilon(y), y)$ for $y \in E_2$, we observe that h_ε is clearly continuous on E_2. Also, h_ε is concave on E_2, since

$$h_\varepsilon\big((1-t)y_1 + ty_2\big) = \min_{x\in E_1} H_\varepsilon\big(x, (1-t)y_1 + y_2\big)$$

$$\begin{aligned}
&\geqslant \min_{x\in E_1}\big\{(1-t)H_\varepsilon(x,y_1)+tH_\varepsilon(x,y_2)\big\}\\
&\geqslant (1-t)\min_{x\in E_1}H_\varepsilon(x,y_1)+t\min_{x\in E_1}H_\varepsilon(x,y_2)\\
&=(1-t)h_\varepsilon(y_1)+th_\varepsilon(y_2)
\end{aligned}$$

for any $y_1, y_2 \in E_2$ and any $t \in [0,1]$. Therefore, h_ε attains the maximum on E_2 at a certain point $y_\varepsilon \in E_2$, and it holds that

$$h_\varepsilon(y_\varepsilon)\ (=\min_{x\in E_1}H_\varepsilon(x,y_\varepsilon))=\max_{y\in E_2}h_\varepsilon(y)=\max_{y\in E_2}\min_{x\in E_1}H_\varepsilon(x,y) \tag{1.4}$$

Now we show with the help of (1.4) that for any $y \in E_2$ and any $t \in [0,1]$

$$h_\varepsilon(y_\varepsilon)\geqslant H_\varepsilon\big(m_\varepsilon\big((1-t)y_\varepsilon+ty\big),y\big). \tag{1.5}$$

In fact, for any $x \in E_1$, $y \in E_2$ and any $t \in [0,1]$ we have by (1.4)

$$\begin{aligned}
H_\varepsilon\big(x,(1-t)y_\varepsilon+ty\big)&\geqslant (1-t)H_\varepsilon(x,y_\varepsilon)+tH_\varepsilon(x,y)\\
&\geqslant (1-t)h_\varepsilon(y_\varepsilon)+tH_\varepsilon(x,y).
\end{aligned}$$

In particular, take $x=m_\varepsilon((1-t)y_\varepsilon+ty)$ to obtain

$$\begin{aligned}
h_\varepsilon(y_\varepsilon)&\geqslant h_\varepsilon\big((1-t)y_\varepsilon+ty\big)\\
&\geqslant (1-t)h_\varepsilon(y_\varepsilon)+tH_\varepsilon\big(m_\varepsilon\big((1-t)y_\varepsilon+ty\big),y\big)
\end{aligned}$$

from which we infer (1.5). By passing to the limit in (1.5) as $t \downarrow 0$, we arrive at

$$h_\varepsilon(y_\varepsilon)\geqslant H_\varepsilon\big(m_\varepsilon(y_\varepsilon),y\big),\quad \forall y\in E_2.$$

From the definition of h_ε it follows that $h_\varepsilon(y_\varepsilon)\leqslant H_\varepsilon(x,y_\varepsilon)$ for all $x \in E_1$. Consequently it results by putting $x_{0\varepsilon}:=m_\varepsilon(y_\varepsilon)$ and $y_{0\varepsilon}:=y_\varepsilon$ that

$$H_\varepsilon(x_{0\varepsilon},y)\leqslant H_\varepsilon(x_{0\varepsilon},y_{0\varepsilon})\leqslant H_\varepsilon(x,y_{0\varepsilon}),\quad \forall x\in E_1,\ \forall y\in E_2.$$

Hence

$$\min_{x\in E_1}\max_{y\in E_2}H(x,y)\leqslant \max_{x\in E_1}\min_{x\in E_2}H(x,y)+\varepsilon\max_{x\in E_1}|x|^2.$$

By the arbitrariness of $\varepsilon > 0$, (1.3) holds and hence (1.1) is obtained. The equivalence of (1.1) and (1.2) is easily checked. □

(3) *A lemma of choice*

Let Y be a real reflexive Banach space and Y^* be its dual space. We denote by B_M the closed ball of radius $M > 0$ about the origin in Y and by B_M^* the closed ball of radius $M > 0$ about the origin in Y^*. For a fixed finite dimensional subspace F_0, we denote by $\mathcal{F}$ the family of all finite dimensional subspaces F, including F_0, of Y. We are given a multivalued mapping $\Lambda : \mathcal{F} \to B_M$ such that for every $F \in \mathcal{F}$ the set $\Lambda(F)$ is a non-empty subset of B_M, and put

$$U(F) = \bigcup_{F' \supset F} \Lambda\big(F'\big) \quad \text{for each } F \in \mathcal{F}.$$

We denote by $\widetilde{U}(F)$ the weak-closure of $U(F)$ in Y for each $F \in \mathcal{F}$. We note that $\widetilde{U}(F) \subset B_M$ for every $F \in \mathcal{F}$. Then, since the family $\{U(F)\}_{F \in \mathcal{F}}$ has the finite intersection property and B_M is weakly compact in Y, we have

$$\bigcap_{F \in \mathcal{F}} \widetilde{U}(F) \neq \emptyset.$$

LEMMA OF CHOICE. *Let $u_0 \in \bigcap_{F \in \mathcal{F}} \widetilde{U}(F)$. Then, for each $F_0' \in \mathcal{F}$, there exists an increasing sequence $\{F_n\}$ in $\mathcal{F}$ with $F_n \supset F_0'$ and a sequence $\{u_n\}$ with $u_n \in \Lambda(F_n)$, $n = 1, 2, \ldots$, such that $u_n \to u_0$ weakly in Y as $n \to +\infty$.*

We prepare a lemma for the proof of our lemma of choice; the idea for the proof is due to F. Browder and P. Hess [7].

LEMMA 1.1. *Let F_0' be any finite dimensional subspace in $\mathcal{F}$, δ be any positive number and ℓ be any positive integer. Then, there is a finite collection $S := \{F_1, F_2, \ldots, F_{r_0}\}$ from $\mathcal{F}$ with $F_0' \subset F_k$, $1 \leqslant k \leqslant r_0$, and $Z := \{u_1, u_2, \ldots, u_{r_0}\}$ with $u_k \in \Lambda(F_k)$, $1 \leqslant k \leqslant r_0$, for a positive integer $r_0 := r_0(\ell, \delta)$ depending on ℓ and δ such that for any elements $y_i^* \in B_1^*$, $i = 1, 2, \ldots, \ell$, there is $u_k \in Z$ satisfying that*

$$\sum_{i=1}^{\ell} \big|\langle y_i^*, u_k - u_0 \rangle\big| < \delta. \tag{1.6}$$

PROOF. Let y_i^*, $i = 1, 2, \ldots, \ell$, be any collection from the unit ball B_1^*. Then, since $u_0 \in \widetilde{U}(F_0')$, there is F, including F_0', and $u \in \Lambda(F)$ such that

$$\sum_{i=1}^{\ell} \big|\langle y_i^*, u - u_0 \rangle\big| < \delta.$$

Now, for each pair $[F, u]$ with $u \in \Lambda(F)$ and $F \supset F_0'$ we put

$$Q(F, u) := \left\{ [y_1^*, y_2^*, \ldots, y_\ell^*];\ y_k^* \in Y^*,\ 1 \leqslant k \leqslant \ell,\ \sum_{i=1}^{\ell} \big|\langle y_i^*, u - u_0 \rangle\big| < \delta \right\},$$

and note that the product set $B_1^{*\ell}$ of ℓ copies of B_1^* is weakly compact in the product space $Y^{*\ell}$. As observed above, the family $\{Q(F,u);\ F \in \mathcal{F}, F_0' \subset F, u \in \Lambda(F)\}$ is an open covering of $B_1^{*\ell}$ with respect to the weak topology of $Y^{*\ell}$. Therefore, there are a finite collection $S := \{F_1, F_2, \ldots, F_{r_0}\}$, $F_k \supset F_0'$ for all $1 \leqslant k \leqslant r_0$, from $\mathcal{F}$ and a finite set $\{u_k\}_{1 \leqslant k \leqslant r_0}$ with $u_k \in \Lambda(F_k)$ for all $1 \leqslant k \leqslant r_0$ such that

$$B_1^{*\ell} \subset \bigcup_{i=k}^{r_0} Q(F_k, u_k).$$

This shows that $S = \{F_1, F_2, \ldots, F_{r_0}\}$ and $Z = \{u_1, u_2, \ldots, u_{r_0}\}$ are desired ones. □

LEMMA 1.2. *Let F_0' be any finite dimensional subspace belonging to $\mathcal{F}$. Then there are sequences $\{S_n\}_{n=1}^{\infty}$ with $S_n := \{F_{n1}, F_{n2}, \ldots, F_{nr_n}\} \subset \mathcal{F}$, r_n being a positive integer depending on n, and $\{Z_n\}_{n=1}^{\infty}$ with $Z_n := \{u_{n1}, u_{n2}, \ldots, u_{nr_n}\} \subset Y$ such that*

(1) *$F_0' \subset F_{nk}$ and $u_{nk} \in \Lambda(F_{nk})$, $1 \leqslant k \leqslant r_n$, $\forall n \in \mathbf{N}$.*
(2) *If $n, m \in \mathbf{N}$ and $n > m$, then $F_{nk} \supset F_{mj}$ for all $F_{nk} \in S_n$ and $F_{mj} \in S_m$.*
(3) *For any n elements $y_i^* \in B_1^*$, $1 \leqslant i \leqslant n$, there is $u_{nk} \in Z_n$ such that*

$$\sum_{i=1}^{n} \left| \langle y_i^*, u_{nk} - u_0 \rangle \right| < \frac{1}{n}. \tag{1.7}$$

PROOF. We take as $S_1 := \{F_{11}, F_{12}, \ldots, F_{1r_1}\}$ and $Z_1 := \{u_{11}, u_{12}, \ldots, u_{1r_1}\}$ the families $S := \{F_1, F_2, \ldots, F_{r_0}\}$ and $Z := \{u_1, u_2, \ldots, u_{r_0}\}$, respectively, obtained by Lemma 1.1 with $\delta = 1$. By recursion, we define the desired families $\{S_n\}$ and $\{Z_n\}$. To this end, assume that S_j and Z_j are defined for all $1 \leqslant j \leqslant n$ with $n > 1$ and properties (1) and (3) are satisfied as well as (2) for any $1 \leqslant m < n$. We are going to define $S_{n+1} \subset \mathcal{F}$ and $Z_{n+1} \subset Y$. Putting

$$F_n' := \bigcup_{k=1}^{r_n} F_{nk}, \tag{1.8}$$

we consider S and Z which are constructed by Lemma 1.1 for F_0' replaced by F_n', $\ell = n+1$ and $\delta = \frac{1}{n+1}$ and take them as $S_{n+1} := \{F_{n+11}, F_{n+12}, \ldots, F_{n+1r_{n+1}}\}$ and $Z_{n+1} := \{u_{n+11}, u_{n+12}, \ldots, u_{n+1r_{n+1}}\} \subset Y$, where r_{n+1} is an positive integer depending on n. In this way two families $\{S_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ are defined, and it is easy to check by (1.8) and Lemma 1.1 that (1), (2) and (3) hold. □

PROOF OF THE LEMMA OF CHOICE. Let $\{S_n\}$ and $\{Z_n\}$ be as constructed by Lemma 1.2. We denote by Y_0 the (strong) closure of the linear subspace spanned by $\bigcup_{n=1}^{\infty} Z_n$. Clearly Y_0 is a closed separable subspace of Y. Also, on account of (3) of Lemma 1.2, u_0 lies in the weak closure of $\bigcup_{n=1}^{\infty} Z_n$, which shows $u_0 \in Y_0$. Now, let π be the natural injection from Y_0 into Y and by π^* be its adjoint from Y^* onto Y_0^*, where Y_0^* be the dual space of Y_0. Note that Y_0^* is separable, since Y_0 is separable, and hence $\{\pi^* B_1^*\}$ is separable. Choose a

countable set $\{\pi^* y_i^*\}_{i=1}^{\infty}$, with $y_i^* \in B_1^*$ for all $i \in \mathbf{N}$, which is dense in $\{\pi^* B_1^*\}$. Then, we see from Lemma 1.2 that for each $n \in \mathbf{N}$ there is $u_{nk} \in Z_n$ such that

$$\sum_{i=1}^{n} \left|\left\langle \pi^* y_i^*, \pi(u_{nk} - u_0)\right\rangle\right| = \sum_{i=1}^{n} \left|\langle y_i^*, u_{nk} - u_0\rangle\right| < \frac{1}{n}. \tag{1.9}$$

We simply denote this u_{nk} by u_n as well as $F_{nk} \in S_n$ by F_n. The sequences $\{F_n\}$ and $\{u_n\}$ are desired ones. In fact, by properties (1) and (2) of Lemma 1.2 we see that $F_0' \subset F_n$, F_n is increasing in n and $u_n \in \Lambda(F_n)$. Moreover, from (1.9) it follows that

$$\left\langle \pi^* y_i^*, \pi(u_n - u_0)\right\rangle \to 0 \quad \text{as } n \to +\infty,\ \forall i \in \mathbf{N}. \tag{1.10}$$

Let y^* be any element in B_1^*. Given $\varepsilon > 0$, by the density of $\{\pi^* y_i^*\}$ in $\pi^* B_1^*$ there is y_i^* such that $|\pi^*(y^* - y_i^*)|_{Y_0^*} < \varepsilon$. Therefore, by (1.10), we have

$$\begin{aligned}
&\limsup_{n\to+\infty} \left|\left\langle \pi^* y^*, \pi(u_n - u_0)\right\rangle\right| \\
&\quad \leqslant \limsup_{n\to+\infty} \left|\left\langle \pi^*(y^* - y_i^*), \pi(u_n - u_0)\right\rangle\right| + \limsup_{n\to+\infty} \left|\left\langle \pi^* y_i^*, \pi(u_n - u_0)\right\rangle\right| \\
&\quad \leqslant 2M\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the above inequalities imply that

$$\langle y^*, u_n - u_0\rangle = \left\langle \pi^* y^*, \pi(u_n - u_0)\right\rangle \to 0 \quad \text{as } n \to +\infty,\ \forall y^* \in B_1^*,$$

namely $u_n \to u_0$ weakly in Y as $n \to +\infty$. □

The above lemma of choice is extensively used in order to select a weakly convergent sequence from a given weakly convergent filter, having the same weak limit, in the following setting. Let X and X^i, $i = 1, 2, \ldots, N$, be real reflexive Banach spaces; for convenience of notation, we put $X^0 = X$. We denote by B_M^i the closed ball of radius $M > 0$ about the origin in X^i, $i = 0, 1, 2, \ldots, N$. For a fixed finite-dimensional subspace F^0 of X, we denote by $\mathcal{F}$ the family of all finite dimensional subspaces F, including F^0, of X. We are given a multivalued mapping $\Lambda : \mathcal{F} \to \prod_{i=0}^{N} B_M^i$, $\prod_{i=0}^{N} B_M^i$ being the product set of B_M^i, such that for every $F \in \mathcal{F}$ the set $\Lambda(F)$ is a non-empty subset of $\prod_{i=0}^{N} B_M^i$, and put

$$U(F) = \bigcup_{F' \supset F} \Lambda\left(F'\right) \quad \text{for each } F \in \mathcal{F}.$$

we denote by $\widetilde{U}(F)$ the weak-closure of $U(F)$ in $\prod_{i=0}^{N} X^i$ for each $F \in \mathcal{F}$, where $\prod_{i=0}^{N} X^i$ is the product space of X^i. We note that $\widetilde{U}(F) \subset \prod_{i=0}^{N} B_M^i$ for every $F \in \mathcal{F}$, and

$$\bigcap_{F \in \mathcal{F}} \widetilde{U}(F) \neq \emptyset.$$

COROLLARY. *Let $F_0' \in \mathcal{F}$ and $\tilde{u}_0 := [u^0, u^1, u^2, \ldots, u^N] \in \bigcap_{F \in \mathcal{F}} \widetilde{U}(F)$. Then there exists an increasing sequence $\{F_k\}$ in $\mathcal{F}$ with $F_k \supset F_0'$ and a sequence*

$$\{\tilde{u}_k := [u_k^0, u_k^1, u_k^2, \ldots, u_k^N]\}$$

with $[u_k^0, u_k^1, u_k^2, \ldots, u_k^N] \in \Lambda(F_k)$, $k = 1, 2, \ldots$, such that $\tilde{u}_k \to \tilde{u}_0$ weakly in $\prod_{i=0}^N X^i$, namely $u_k^i \to u^i$ weakly in X^i for each $i = 0, 1, 2, \ldots, N$, as $k \to +\infty$.

PROOF. We denote by $\widehat{\mathcal{F}}$ the family of all finite dimensional subspaces $\widehat{F}$ in the product space $Y := \prod_{i=0}^N X^i$ such that $\mathrm{Pr}^0(\widehat{F})$, the projections of $\widehat{F}$ onto $X \times \prod_{i=1}^N \{0\}$, include $\widehat{F}^0 := (F^0, \{0\}, \ldots, \{0\})$. For simplicity, we identify the space $X \times \prod_{i=1}^N \{0\}$ with X and hence $\widehat{F} = F \times \prod_{i=1}^N \{0\}$ with F, namely $F = \mathrm{Pr}^0(\widehat{F})$. Also we define a multivalued mapping $\hat{\Lambda} : \widehat{\mathcal{F}} \to \prod_{i=0}^N B_M^i$ by

$$\hat{\Lambda}\big(\widehat{F}\big) = \Lambda(F), \quad \forall \widehat{F} \in \widehat{\mathcal{F}} \text{ with } F = \mathrm{Pr}^0\big(\widehat{F}\big).$$

Choose a large number $R > 0$ so that $\prod_{i=0}^N B_M^i \subset B_R^Y$, where B_R^Y is the closed ball with radius R about the origin of Y . Moreover, we set

$$\widehat{U}\big(\widehat{F}\big) = \bigcup_{\widehat{F}' \supset \widehat{F}} \hat{\Lambda}\big(\widehat{F}'\big) \quad \text{for each } \widehat{F} \in \widehat{\mathcal{F}},$$

and denote by $\widehat{U}(\widehat{F})^{\sim}$ the weak-closure of $\widehat{U}(\widehat{F})$ in Y. Since $\{\widehat{U}(\widehat{F})\}_{\widehat{F} \in \widehat{\mathcal{F}}}$ has the finite intersection property and B_R^Y is a weakly compact in Y, it follows that

$$B_R^Y \supset \bigcap_{\widehat{F} \in \widehat{\mathcal{F}}} \widehat{U}\big(\widehat{F}\big)^{\sim} \neq \emptyset.$$

Let $\tilde{u}_0 := [u_0^0, u_0^1, u_0^2, \ldots, u_0^N] \in \bigcap_{\widehat{F} \in \widehat{\mathcal{F}}} \widehat{U}(\widehat{F})^{\sim}$. Here, apply our lemma of choice for a given $\widehat{F}_0' \in \widehat{\mathcal{F}}$ to find an increasing sequence $\{\widehat{F}_n\} \subset \widehat{\mathcal{F}}$, $\widehat{F}_n \supset \widehat{F}_0'$, and $\tilde{u}_n := [u_n^0, u_n^1, \ldots, u_n^N] \in \hat{\Lambda}(\widehat{F}_n) = \Lambda(F_n)$, $F_n := \mathrm{Pr}^0(\widehat{F}_n)$, $n = 1, 2, \ldots$, such that $u_n^i \to u_0^i$ weakly in X^i as $n \to \infty$ for every $i = 0, 1, 2, \ldots, N$. This is nothing but the assertion in the corollary. □

(4) *Function spaces*

Let $\mathbf{R}^N$, $1 \leqslant N < \infty$, be the N-dimensional euclidean space. We denote by $x := (x_1, x_2, \ldots, x_N)$ a generic point in $\mathbf{R}^N$. Given a domain Ω in $\mathbf{R}^N$, with smooth boundary $\Gamma := \partial\Omega$, we use the following usual notation:

$C_0^n(\Omega)$: the space of all continuous functions, together with all the partial derivatives up to n-th order, on Ω with compact supports in Ω, equipped with the maximum-norm; put $C_0^\infty(\Omega) := \bigcup_{n=1}^\infty C_0^n(\Omega)$.

$C^n(\overline{\Omega})$: the space of all continuous functions together with all the partial derivatives up to n-th order, on $\overline{\Omega}$, equipped with the maximum-norm; $C^\infty(\overline{\Omega}) := \bigcup_{n=1}^\infty C^n(\overline{\Omega})$.

$L^p(\Omega)$: $1 \leqslant p \leqslant \infty$: the space of all measurable functions on Ω, equipped with the usual L^p-norm, denoted by $|\cdot|_p$.

$W^{k,p}(\Omega)$: $k \in \mathbf{N}$, $1 \leqslant p \leqslant \infty$, the Sobolev space of all functions $u \in L^p(\Omega)$ such that all the partial derivatives $D^\alpha u$, with $|\alpha| \leqslant k$, are in $L^p(\Omega)$, where $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is a multi-index and $|\alpha| := \sum_{i=1}^N |\alpha_i|$ and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_N^{\alpha_N}},$$

equipped with norm

$$|u|_{k,p} := \begin{cases} \{|u|_p^p + \sum_{1 \leqslant |\alpha| \leqslant k} |D^\alpha u|_p^p\}^{1/p} & \text{for } 1 \leqslant p < \infty, \\ |u|_\infty + \sum_{1 \leqslant |\alpha| \leqslant k} |D^\alpha u|_\infty & \text{for } p = \infty; \end{cases}$$

when $p = 2$, $W^{k,p}(\Omega)$ is often denoted by $H^k(\Omega)$.

$W_0^{k,p}(\Omega)$: $k \in \mathbf{N}$, $1 \leqslant p \leqslant \infty$, the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$, equipped with norm $|\cdot|_{k,p}$; when $p = 2$, $W_0^{k,p}(\Omega)$ is often denoted by $H_0^k(\Omega)$.

$W^{-k,q}(\Omega)$: $k \in \mathbf{N}$, $1 \leqslant p \leqslant \infty$, $\frac{1}{p} + \frac{1}{q}$, the dual space of $W_0^{k,p}(\Omega)$ equipped with the norm $|\cdot|_{-k,q}$ which is the dual norm of $|\cdot|_{k,p}$.

$W^{1/q,p}(\Gamma)$: $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, the space of all traces $\hat{u}$ (on Γ) of functions u in $W^{1,p}(\Omega)$; denote by γ the trace operator, namely $\hat{u} = \gamma u$ for $u \in W^{1,p}(\Omega)$, equipped with norm

$$|\hat{u}|_{\frac{1}{q},p} := \left\{ \int_\Gamma |\hat{u}|^p \, \mathrm{d}\Gamma + \iint\limits_{\Gamma \times \Gamma} \frac{|\hat{u}(x) - \hat{u}(y)|^p}{|x - y|^{p+N-2}} \, \mathrm{d}\Gamma(x) \, \mathrm{d}\Gamma(y) \right\}^{1/p};$$

note that $\gamma u = u_{|\Gamma}$ if u is smooth on $\overline{\Omega}$ and γ is continuous and linear from $W^{1,p}(\Omega)$ onto $W^{1/q,p}(\Gamma)$; when $p = 2$, $W^{1/q,p}(\Omega)$ is often denoted by $H^{1/2}(\Gamma)$.

$W^{-1/q,q}(\Gamma)$: $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, the dual space of $W^{1/q,p}(\Gamma)$ equipped with the norm $|\cdot|_{-1/q,q}$ which is the dual norm of $|\cdot|_{1/p,p}$.

For functional spaces of the Sobolev type and related basic properties, see [1,16,18,19].

(5) *A generalized Green's formula*

Let Ω be a bounded domain, with smooth boundary Γ, in $\mathbf{R}^N$, $1 \leqslant N < \infty$. For $1 < p < +\infty$ and $1 < q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we introduce a reflexive Banach space

$$\mathcal{E}^q(\Omega) = \left\{ \mathbf{v} = (v_1, v_2, \ldots, v_N) \in L^q(\Omega)^N; \ \operatorname{div} \mathbf{v} \in L^q(\Omega) \right\}$$

equipped with norm

$$|\mathbf{v}|_{\mathcal{E}^q} = \left\{ \sum_{i=1}^N |v_i|_q^q + |\operatorname{div} \mathbf{v}|_q^q \right\}^{1/q}.$$

It is proved that $C^\infty(\overline{\Omega})^N$ is dense in $\mathcal{E}^q(\Omega)$. Now, thanks to the divergence theorem, we have

$$\int_\Omega (\operatorname{div}\mathbf{v})w\,\mathrm{d}x + \int_\Omega \mathbf{v}\cdot\nabla w\,\mathrm{d}x = \int_\Gamma (\mathbf{v}\cdot\mathbf{n})\widehat{w}\,\mathrm{d}\Gamma \tag{1.11}$$

for every $\mathbf{v}\in C^\infty(\overline{\Omega})^N$ and every $\widehat{w}\in W^{1/q,p}(\Gamma)$, w being any function in $W^{1,p}(\Omega)$ such that the trace of w coincides with $\widehat{w}$ on Γ, where $\mathbf{n}=\mathbf{n}(x)$ denotes the unit outward normal on Γ at $x\in\Gamma$. This shows that the functional $\widehat{w}\to\int_\Gamma(\mathbf{v}\cdot\mathbf{n})\widehat{w}\,\mathrm{d}\Gamma$ is linear and continuous on $W^{1/q,p}(\Gamma)$. Therefore each $\mathbf{v}\in C^\infty(\overline{\Omega})^N$ determines a unique element in $W^{-1/q,q}(\Gamma)$, denoted by $\mathbf{Bv}$, so that

$$\langle\mathbf{Bv},\widehat{w}\rangle_\Gamma = \int_\Gamma(\mathbf{v}\cdot\mathbf{n})\widehat{w}\,\mathrm{d}\Gamma,\quad \forall\widehat{w}\in W^{1/q,p}(\Gamma), \tag{1.12}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the duality between $W^{-1/q,q}(\Gamma)$ and $W^{1/q,p}(\Gamma)$. Moreover, from (1.11) and (1.12) it follows that

$$\begin{aligned}\left|\langle\mathbf{Bv},\widehat{w}\rangle_\Gamma\right| &\leqslant \inf_{w\in W^{1,p}(\Omega),\ w=\widehat{w}\text{ a.e. on }\Gamma}\left\{|\operatorname{div}\mathbf{v}|_q|w|_p+\sum_{i=1}^N|v_i|_q|w_{x_i}|_p\right\},\\ &\leqslant b_0|\mathbf{v}|_{\mathcal{E}^q}|\widehat{w}|_{1/q,p},\quad \forall\mathbf{v}\in C^\infty\big(\overline{\Omega}\big)^N,\ \forall\widehat{w}\in W^{1/q,p}(\Gamma),\end{aligned}$$

where b_0 is a positive constant; we used above the fact that for some positive constants b_1, b_2,

$$|\widehat{w}|_{1/q,p}\leqslant b_1\inf_{w\in W^{1,p}(\Omega),\ w=\widehat{w}\text{ a.e. on }\Gamma}|w|_{1,p}\leqslant b_2|\widehat{w}|_{1/q,p},\quad\forall\widehat{w}\in W^{1/q,p}(\Gamma).$$

Accordingly, $\mathbf{B}$ is a linear mapping from $C^\infty(\overline{\Omega})^N$ into $W^{-1/q,q}(\Gamma)$ and satisfies

$$|\mathbf{Bv}|_{-1/q,q}\leqslant b_0|\mathbf{v}|_{\mathcal{E}^q}. \tag{1.13}$$

Hence $\mathbf{B}$ can be uniquely extended as a linear continuous mapping from $\mathcal{E}^q(\Omega)$ into $W^{-1/q,q}(\Gamma)$ such that

$$\mathbf{Bv}=\mathbf{v}\cdot\mathbf{n}\quad\text{if }\mathbf{v}\in C^\infty\big(\overline{\Omega}\big)^N.$$

Consequently, we have:

GENERALIZED GREEN'S FORMULA. *The following formula holds*:

$$\int_\Omega(\operatorname{div}\mathbf{v})w\,\mathrm{d}x+\int_\Omega\mathbf{v}\cdot\nabla w\,\mathrm{d}x=\langle\mathbf{Bv},w\rangle_\Gamma,\quad\forall\mathbf{v}\in\mathcal{E}^q(\Omega),\ \forall w\in W^{1,p}(\Omega).$$

We refer to [13] for the detailed proof.

2. Mappings of monotone type and basic properties

We begin our study with introducing an important class of mappings in Banach spaces into their dual spaces. Throughout this section, let X be a real reflexive Banach space.

DEFINITION 2.1.

(i) Let A be a (multivalued) mapping from X into X^*. If

$$\langle u^* - v^*, u - v\rangle \geqslant 0, \quad \forall [u, u^*], [v, v^*] \in G(A), \tag{2.1}$$

then A is called *monotone*.

(ii) Let $A : X \to X^*$ be a monotone mapping. If A has no proper monotone extension from X into X^*, then A is called *maximal monotone*; in other words, A is maximal monotone if and only if it satisfies that

$$B : X \to X^* \text{ is monotone and } G(A) \subset G(B) \quad \Longrightarrow \quad A = B. \tag{2.2}$$

Now we consider an important monotone mapping, which is called the duality mapping. To each $u \in X$ we assign the subset

$$Ju := \left\{u^* \in X^*;\ \langle u^*, u\rangle = |u|_X^2 = |u^*|_{X^*}^2\right\} \tag{2.3}$$

of X^*. Then it is easy to see from the Hahn–Banach theorem that $Ju \neq \emptyset$ for every $u \in X$. Thus J is a mapping from X into X^*, satisfying $D(J) = X$. Also, J is monotone, namely (2.1) holds, since

$$\langle u^* - v^*, u - v\rangle \geqslant \left(|u|_X - |v|_X\right)^2 \geqslant 0, \quad \forall [u, u^*], [v^*, v] \in G(J).$$

Moreover, we have:

PROPOSITION 2.1. *The duality mapping J from X into X^* satisfies that*

(i) *J is bounded from X into X^*.*

(ii) *Ju is a convex closed subset of X^* for each $u \in X$.*

(iii) *$G(J)$ is demiclosed in $X \times X^*$.*

(iv) *J is a maximal monotone mapping from X into X^*.*

PROOF. (i) and (ii) are immediately obtained from the definition of duality mapping. We now prove (iii). Let $\{[u_n, u_n^*]\}$ be a sequence in $G(J)$ such that $[u_n, u_n^*] \to [u, u^*]$ in the strong-weak topology of $X \times X^*$. Then,

$$\langle u^*, u\rangle = \lim_{n\to+\infty} \langle u_n^*, u_n\rangle = \lim_{n\to+\infty} |u_n|_X^2 = |u|_X^2. \tag{2.4}$$

By (2.4), we have $|u^*|_X \geqslant |u|_X$. On the other hand,

$$|u^*|_{X^*} \leqslant \liminf_{n\to+\infty} |u_n^*|_{X^*} = \lim_{n\to+\infty} |u_n|_X = |u|_X,$$

so that $\langle u^*, u\rangle = |u|_X^2 = |u^*|_{X^*}^2$, i.e., $u^* \in Ju$. In the case when $[u_n, u_n^*] \to [u, u^*]$ in the weak-strong topology of $X \times X^*$, we have the same conclusion. Thus $G(A)$ is demiclosed in $X \times X^*$.

Next we show (iv). Let $\tilde{J}$ be a monotone mapping from X into X^* such that $G(J) \subset G(\tilde{J})$. Let $[u, f]$ be any element of $G(\tilde{J})$ and fix it. Then, by the monotonicity of $\tilde{J}$, we have

$$\langle f - z^*, u - z\rangle \geqslant 0, \quad \forall [z, z^*] \in G(J). \tag{2.5}$$

Now, for any $v \in X$ and a sequence $\varepsilon_n > 0$ with $\varepsilon_n \downarrow 0$ (as $n \to +\infty$), we take $z_n = u - \varepsilon_n v$ and any $z_n^* \in J(z_n)$ as z and $z^* \in Jz$ in (2.5) to get

$$\langle f - z_n^*, u - z_n\rangle = \varepsilon_n \langle f - z_n^*, v\rangle \geqslant 0. \tag{2.6}$$

By (i) and (iii), we may assume that $\{z_n^*\}$ weakly converges to some z^* in X^* and $z^* \in Ju$. Dividing (2.6) by ε_n and letting $n \to +\infty$ in (2.6) yield $\langle f - z^*, v\rangle \geqslant 0$. Thus we have obtained the following statement:

$$\begin{cases} \text{for each } v \in X \text{ there exists an element } z^* \in Ju \text{ such that} \\ \langle f - z^*, v\rangle \geqslant 0. \end{cases} \tag{2.7}$$

We note that $f - Ju$ is a weakly compact convex subset of X^*. Therefore, if $f \notin Ju$, then by the Hahn–Banach theorem there is an element $v_0 \in X$ such that $\langle f - z^*, v_0\rangle < 0$ for all $z^* \in Ju$, which contradicts (2.7). Thus we see $f \in Ju$, which shows that $J = \tilde{J}$. Hence J is maximal monotone. □

It is easy to check that the inverse J^{-1} of J is the duality mapping from X^* into X and $D(J^{-1}) = R(J) = X^*$. Also, if X^* (resp. X) is strictly convex, then J (resp. J^{-1}) is singlevalued. In fact, assuming that X^* is strictly convex, let $u \in X$ and u_i^*, $i = 1, 2$, be any elements in Ju. Then, putting $u^* := ru_1^* + (1 - r)u_2^*$ for any $r \in (0, 1)$, we see from the definition of J that

$$\langle u_i^*, u\rangle = |u|_X^2 = |u_i^*|_{X^*}^2, \quad i = 1, 2,$$

which implies that

$$\langle u^*, u\rangle = r\langle u_1^*, u\rangle + (1 - r)\langle u_2^*, u\rangle = |u|_X^2,$$

and

$$|u^*|_{X^*} \leqslant r|u_1^*|_{X^*} + (1 - r)|u_2^*|_{X^*} \leqslant |u|_X.$$

Moreover, since $|u^*|_{X^*} \geqslant \langle u^*, u\rangle / |u|_X = |u|_X$, we have together with the above inequality that $|u^*|_{X^*} = |u|_X$. Therefore by the strict convexity of X^* it follows that $u_1^* = u_2^*$ and thus J is singlevalued. Similarly, the strict convexity of X implies that J^{-1} is singlevalued. As

is easily seen, the above argument enables us to relax the definition (2.3) of the duality mapping $J : X \to X^*$ to the following one:

$$Ju := \{u^* \in X^*;\ \langle u^*, u\rangle = |u|_X^2,\ |u^*|_{X^*} \leqslant |u|_X\}. \tag{2.3′}$$

As a typical example of monotone mappings, there is the class of subdifferentials of convex functions, which is precisely defined below.

Let $\psi(\cdot)$ be a function defined on X such that $-\infty < \psi(u) \leqslant +\infty$ for any $u \in X$ and ψ is not identically ∞ on X; such a function ψ is called *proper* on X. If ψ satisfies that

$$\psi\big(\lambda u + (1-\lambda)v\big) \leqslant \lambda\psi(u) + (1-\lambda)\psi(v), \quad \forall u, v \in X,\ \forall \lambda \in [0,1], \tag{2.8}$$

then ψ is called *convex* on X. We call $D(\psi) := \{u \in X;\ \psi(u) < +\infty\}$ the effective domain of ψ.

Also, for a proper and convex function ψ, if the strict inequality holds in (2.8) for any $u, v \in D(\psi)$, $u \neq v$, and any $\lambda \in (0,1)$, then it is called *strictly convex* on X.

DEFINITION 2.2. Suppose that ψ is proper and convex on X. For $u \in D(\psi)$, an element $u^* \in X^*$ is called a *subgradient* of ψ at u, if

$$\langle u^*, v-u\rangle \leqslant \psi(v) - \psi(u), \quad \forall v \in X. \tag{2.9}$$

We denote by $\partial\psi(u)$ the set of all subgradients of ψ at $u \in D(\psi)$ and put $\partial\psi(u) = \emptyset$ for any $u \notin D(\psi)$. In this way $\partial\psi$ is a mapping from X into X^*, and it is called the *subdifferential* of ψ.

For any proper and convex function ψ on X, it is seen from the definition of subdifferential that $u^* \in \partial\psi(u)$ if and only if the function $v \to \psi(v) - \langle u^*, v\rangle$ attains the minimum on X at $v = u$.

We note here that $\partial\psi(u)$ may be empty, even if $u \in D(\psi)$. Also, the subdifferential $\partial\psi$ is monotone, because

$$\begin{aligned}\langle u^* - v^*, u - v\rangle &= \langle u^*, u - v\rangle + \langle v^*, v - u\rangle \\ &\geqslant \psi(u) - \psi(v) + \psi(v) - \psi(u) = 0\end{aligned}$$

for all $[u, u^*], [v, v^*] \in G(\partial\psi)$. In the next section we shall show that subdifferentials of proper, lower semicontinuous (l.s.c.) convex functions on X are maximal monotone.

EXAMPLE 2.1. The subdifferential of the continuous convex function $\psi(u) := \frac{1}{2}|u|_X^2$ on X is the duality mapping J. In fact, for any $u, v \in X$ and any $u^* \in Ju$ we have by definition

$$\langle u^*, v-u\rangle \leqslant |u^*|_{X^*}|v|_X - |u|_X^2 \leqslant \psi(v) - \psi(u).$$

This implies that $G(J) \subset G(\partial\psi)$, so that $J = \partial\psi$ by the maximal monotonicity of J.

We prove some fundamental properties of maximal monotone mappings.

PROPOSITION 2.2. *Let A be a maximal monotone mapping from X into X^*. Then we have*:

(a) *For each $u \in D(A)$, Au is closed and convex in X^*.*

(b) *Suppose that $\{[u_n, u_n^*]\}$ is a sequence in $G(A)$ such that $u_n \to u$ weakly in X, $u_n^* \to u^*$ weakly in X^* and*

$$\limsup_{n\to+\infty} \langle u_n^*, u_n\rangle \leqslant \langle u^*, u\rangle. \tag{2.10}$$

Then $[u, u^*] \in G(A)$, namely $u \in D(A)$ and $u^* \in Au$, and

$$\lim_{n\to+\infty} \langle u_n^*, u_n\rangle = \langle u^*, u\rangle.$$

Therefore, $G(A)$ is demiclosed in $X \times X^*$.

(c) *A is locally bounded at any interior point of $D(A)$.*

PROOF. (a) is easily seen from the maximal monotonicity of A. We now show (b). Let $\{[u_n, u_n^*]\}$ be a sequence in $G(A)$ which converges weakly to $[u, u^*]$ in $X \times X^*$ and for which (2.10) holds. For any $[v, v^*] \in G(A)$, we have by monotonicity

$$0 \leqslant \langle v^* - u_n^*, v - u_n\rangle = \langle v^*, v - u_n\rangle - \langle u_n^*, v\rangle + \langle u_n^*, u_n\rangle.$$

Pass to the limit in n and use (2.10) to get

$$0 \leqslant \langle v^*, v - u\rangle - \langle u^*, v\rangle + \limsup_{n\to+\infty} \langle u_n^*, u_n\rangle \leqslant \langle v^* - u^*, v - u\rangle.$$

This implies $[u, u^*] \in G(A)$ by the maximal monotonicity of A. Furthermore, since $\langle u_n^* - u^*, u_n - u\rangle \geqslant 0$, it follows that $\liminf_{n\to\infty} \langle u_n^*, u_n\rangle \geqslant \langle u^*, u\rangle$ and hence $\lim_{n\to\infty} \langle u_n^*, u_n\rangle = \langle u^*, u\rangle$.

Next we show (c), assuming that the interior of $D(A)$, denoted by $Int\, D(A)$, is nonempty. Given a positive number r, we denote by B_r (resp. B_r^*) the closed ball about the origin of X (resp. X^*) with radius r. By considering a mapping $u \to A(u + a)$ for a point $a \in Int\, D(A)$, we may assume without loss of generality that $0 \in Int\, D(A)$ and B_R is included in $Int\, D(A)$ for a number $R > 0$. Now, for each positive integer n we put

$$S_n := \{u \in B_R;\ Au \cap B_n^* \neq \emptyset\}.$$

By (b), each S_n is closed in X and $B_R = \bigcup_{n=1}^{\infty} S_n$. Hence from Baire's second category theorem it follows that S_{n_0} contains at least one interior point u_0 for a certain positive integer n_0. We choose a small number $r_0 > 0$ so that $B_{r_0} + u_0 \subset B_R (\subset Int\, D(A))$. Put $S := (B_{r_0} + u_0) \cup \{-u_0\}$. Then

$$B_{r_0/2} \subset \left\{ \frac{-u_0 + u}{2};\ u \in S \right\} \subset \mathrm{co}(S),$$

and hence

$$B_{r_0/4} \subset \bigcap_{u \in B_{r_0/4}} \operatorname{co}(S-u), \tag{2.11}$$

where $\operatorname{co}(\cdot)$ denotes the closed convex hull of $(\cdot)$. Moreover we choose an element $u_0^* \in A(-u_0)$ and put $R_1 := n_0 + |u_0^*|_{X^*}$. We are going to show that $A(B_{r_0/4})$ is bounded in X^*. In fact, let $u \in B_{r_0/4}$ and $u^* \in Au$. Then, for any $[v, v^*] \in G(A)$ with $v \in S$ and $|v^*|_{X^*} \leqslant n_0$ or $v^* = u_0^* \in A(-u_0)$ it follows from the monotonicity of A that $\langle v^* - u^*, v - u\rangle \geqslant 0$, and hence

$$\langle u^*, v-u\rangle \leqslant |v^*|_{X^*}|v-u|_X \leqslant R_1\left(\frac{r_0}{4}+R\right), \quad \forall v \in S.$$

Therefore

$$\langle u^*, w\rangle \leqslant R_1\left(\frac{r_0}{4}+R\right), \quad \forall w \in \bigcap_{u \in B_{r_0/4}} \operatorname{co}(S-u).$$

This implies by (2.11) that

$$\langle u^*, w\rangle \leqslant R_1\left(\frac{r_0}{4}+R\right), \quad \forall w \in B_{r_0/4},$$

so that $|u^*|_{X^*} \leqslant R_1(r_0/4+R)(4/r_0) := R_2$. Thus $A(B_{r_0/4}) \subset B_{R_2}^*$. □

COROLLARY 2.1. *Let A be a maximal monotone mapping from X into X^*. Assume that A is singlevalued. Then A is demicontinuous in the interior of $D(A)$.*

PROOF. Let $\{u_n\}$ be any sequence in $Int\, D(A)$ and $u \in Int\, D(A)$ such that $u_n \to u$ in X (as $n \to \infty$). Since A is locally bounded at u, we see that $\{Au_n\}$ is bounded in X^*. Therefore there exists a subsequence $\{u_{n_k}\}$ such that $Au_{n_k} \to u^*$ weakly in X^* (as $k \to \infty$). On account of the demiclosedness of $G(A)$ we have $u^* = Au$. This shows that A is demicontinuous on $Int\, D(A)$. □

The theory of monotone mappings is quite powerful in solving some typical classes of nonlinear elliptic partial differential equations. But the monotonicity assumption is often too restrictive, when the theory is applied to many real world problems. Therefore we introduce some wide classes of mappings, taking into account perturbations to maximal monotone mappings.

DEFINITION 2.3. A mapping $\tilde{A}(\cdot,\cdot): X \times X \to X^*$ is called semi-monotone, if $D(\tilde{A}) = X \times X$ and the following conditions (SM1) and (SM2) are satisfied:

(SM1) For any fixed $v \in X$ the mapping $\tilde{A}(v, \cdot)$ is maximal monotone from

$$D\big(\tilde{A}(v, \cdot)\big) = X$$

into X^*.

(SM2) Let u be any element of X and $\{v_n\}$ be any sequence in X such that $v_n \to v$ weakly in X. Then, for every $u^* \in \tilde{A}(v, u)$ there exists a sequence $\{u_n^*\}$ in X^* such that $u_n^* \in \tilde{A}(v_n, u)$ and $u_n^* \to u^*$ in X^* as $n \to \infty$.

REMARK 2.1. For any semi-monotone mapping $\tilde{A}(\cdot, \cdot) : X \times X \to X^*$, from conditions (SM1) and (SM2) the following condition (SM3) is derived:

(SM3) Let u be any element of X and $\{v_n\}$ be any weakly convergent sequence, with the weak limit v, in X. Then, if $u_n^* \in \tilde{A}(v_n, u)$, $n = 1, 2, \ldots$, and $u_n^* \to u^*$ weakly in X^*, we have $u^* \in \tilde{A}(v, u)$.

In fact, let $\{v_n\}$ and $\{u_n^*\}$ be as in (SM3). Let w be any element of X as well as z^* be any element of $\tilde{A}(v, w)$. Then, by (SM2), there is a sequence $\{z_n^*\}$ with $z_n^* \in \tilde{A}(v_n, w)$ such that $z_n^* \to z^*$ in X^*. Moreover, by (SM1), $\langle u_n^* - z_n^*, u - w \rangle \geqslant 0$, so that letting $n \to \infty$ yields that $\langle u^* - z^*, u - w \rangle \geqslant 0$. Hence the maximal monotonicity of $\tilde{A}(v, \cdot)$ implies $u^* \in \tilde{A}(v, u)$.

Furthermore we introduce two wider classes of nonlinear mappings of monotone type, including semi-monotone mappings.

DEFINITION 2.4. Let A be a mapping from X into X^*. Then:

(i) A is called *pseudo-monotone*, if the following conditions (PM1) and (PM2) are satisfied:

(PM1) For each $u \in X$, Au is a non-empty, closed, bounded and convex subset of X^*; hence $D(A) = X$.

(PM2) For each weakly convergent sequence $\{u_n\}$ in X, denoted by u the weak limit, if $u_n^* \in Au_n$, $u_n^* \to g$ weakly in X^* and $\limsup_{n\to\infty} \langle u_n^*, u_n \rangle \leqslant \langle g, u \rangle$, then $g \in Au$ and $\lim_{n\to\infty} \langle u_n^*, u_n \rangle = \langle g, u \rangle$.

(ii) A is called *of type M*, if the following conditions (TM1) and (TM2) are satisfied:

(TM1) For each $u \in X$, Au is a non-empty, closed, bounded and convex subset of X^*; hence $D(A) = X$.

(TM2) For each weakly convergent sequence $\{u_n\}$ in X, denoted by u the weak limit, if $u_n^* \in Au_n$, $u_n^* \to g$ weakly in X^* and $\limsup_{n\to\infty} \langle u_n^*, u_n \rangle \leqslant \langle g, u \rangle$, then $g \in Au$.

From the above definitions we immediately observe that the graphs of mappings of type M are demiclosed, and pseudo-monotone mappings are of type M. In particular, when X is a finite dimensional Banach space and A is a bounded mapping satisfying property (TM1) (= (PM1)), the property of type M and pseudo-monotonicity coincide and they are equivalent to the upper semicontinuity from X into X^*. Moreover, in order to illustrate the structure of pseudo-monotonicity we give its simple examples and the relationship between semi-monotonicity and pseudo-monotonicity.

PROPOSITION 2.3.

(1) *Any singlevalued compact mapping A from $D(A) = X$ into X^* is pseudo-monotone. Also, any singlevalued weakly continuous mapping A from $D(A) = X$ into X^*, namely continuous from X_w into X_w^*, is of Type M.*

(2) *Any maximal monotone mapping A from $D(A) = X$ into X^* is pseudo-monotone.*

(3) *Let $\tilde{A} : D(\tilde{A}) = X \times X \to X^*$ be a semi-monotone mapping; for simplicity, denote by $A : X \to X^*$ the mapping given by $Au := \tilde{A}(u, u)$ for every $u \in X$. Then, A is pseudo-monotone from X into X^*.*

PROOF. *Proof of* (1). By the compactness of A, "$u_n \to u$ weakly in X" implies that $Au_n \to Au$ in X^* and hence $\lim_{n\to+\infty}\langle Au_n, u_n\rangle = \langle Au, u\rangle$. It is obvious that any weakly continuous mapping is of type M.

Proof of (2). Suppose that $[u_n, u_n^*] \in G(A)$, $u_n \to u$ weakly in X, $u_n^* \to g$ weakly in X^* and $\limsup_{n\to\infty}\langle u_n^*, u_n\rangle \leqslant \langle g, u\rangle$. Then, Proposition 2.2(b), we have $g \in Au$ and $\lim_{n\to\infty}\langle u_n^*, u_n\rangle = \langle g, u\rangle$.

Proof of (3). Assume that $\{u_n\}$ and $\{u_n^*\}$ are sequences in X and X^*, respectively, such that $u_n^* \in Au_n = \tilde{A}(u_n, u_n)$, $u_n \to u$ weakly in X, $u_n^* \to g$ weakly in X^* and $\limsup_{n\to+\infty}\langle u_n^*, u_n\rangle \leqslant \langle g, u\rangle$. Now we note from (SM1) that

$$\langle u_n^* - v_n^*, u_n - v\rangle \geqslant 0, \quad \forall v \in X,\ \forall v_n^* \in \tilde{A}(u_n, v). \tag{2.12}$$

By (SM2), for each $\tilde{u}^* \in \tilde{A}(u, u)$ choose a sequence $\{\tilde{u}_n^*\}$ with $\tilde{u}_n^* \in \tilde{A}(u_n, u)$ and $\tilde{u}_n^* \to \tilde{u}^*$ in X^* and take $[u, \tilde{u}_n^*]$ as $[v.v_n^*]$ in (2.12). Then we have

$$0 \geqslant \limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle \geqslant \liminf_{n\to\infty}\langle u_n^*, u_n - u\rangle \geqslant \liminf_{n\to\infty}\langle \tilde{u}_n^*, u_n - u\rangle = 0,$$

namely, $\lim_{n\to\infty}\langle u_n^*, u_n - u\rangle = 0$. Therefore,

$$\lim_{n\to\infty}\langle u_n^*, u_n\rangle = \langle g, u\rangle. \tag{2.13}$$

Next, we see by (2.13) and (SM2) again that

$$\langle g - v^*, u - v\rangle \geqslant 0, \quad \forall v \in X,\ \forall v^* \in \tilde{A}(u, v).$$

This implies that $g \in \tilde{A}(u, u) = Au$, because $\tilde{A}(u, \cdot)$ is maximal monotone from X into X^*. Thus we have seen that A is pseudo-monotone from X into X^*. □

PROPOSITION 2.4. *Suppose that A and B are pseudo-monotone from X into X^* and A or B is bounded. Then the sum $A + B$ is pseudo-monotone from X into X^*.*

PROOF. By assumptions $D(A + B) = X$ and $A + B$ is a mapping from X into X^*. Let $[u_n, u_{1n}^*] \in G(A)$ and $[u_n, u_{2n}^*] \in G(B)$ such that $u_n \to u$ weakly in X, $u_{1n}^* + u_{2n}^* \to h$ weakly in X^* and

$$\limsup_{n\to\infty}\langle u_{1n}^* + u_{2n}^*, u_n\rangle \leqslant \langle h, u\rangle. \tag{2.14}$$

Note by our assumptions that $\{u_{1n}^*\}$ or $\{u_{2n}^*\}$ is bounded in X^*. We may assume, by extracting a subsequence from $\{u_n\}$ if necessary, that $u_{1n}^* \to f$ and $u_{2n}^* \to g := h - f$ weakly in X^*, and

$$\lim_{n\to\infty} \langle u_{1n}^* + u_{2n}^*, u_n \rangle = \lim_{n\to\infty} \langle u_{1n}^*, u_n \rangle + \lim_{n\to\infty} \langle u_{2n}^*, u_n \rangle.$$

In this case, it follows from (2.14) that

$$\lim_{n\to\infty} \langle u_{1n}^*, u_n \rangle \leqslant \langle f, u \rangle \tag{2.15}$$

or

$$\lim_{n\to\infty} \langle u_{2n}^*, u_n \rangle \leqslant \langle g, u \rangle. \tag{2.16}$$

In case (2.15) holds, by the pseudo-monotonicity of A we have $f \in Au$ and

$$\lim_{n\to\infty} \langle u_{1n}^*, u_n \rangle = \langle f, u \rangle. \tag{2.17}$$

Hence, it follows from (2.17) with (2.14) that

$$\lim_{n\to\infty} \langle u_{2n}^*, u_n \rangle \leqslant \langle g, u \rangle.$$

By the pseudo-monotonicity of B again, this implies that $g \in Bu$ and

$$\lim_{n\to\infty} \langle u_{2n}^*, u_n \rangle = \langle g, u \rangle. \tag{2.18}$$

Consequently we have $h = f + g \in Au + Bu$ and see from (2.17) and (2.18) that

$$\lim_{n\to\infty} \langle u_{1n}^* + u_{2n}^*, u_n \rangle = \langle h, u \rangle.$$

Thus $A + B$ is pseudo-monotone. In case (2.16) holds, we can similarly prove the pseudo-monotonicity of $A + B$. □

COROLLARY 2.2. *Let A be a maximal monotone mapping from $D(A) = X$ into X^* and B be a pseudo-monotone mapping from X into X^*. Suppose that A or B is bounded. Then the sum $A + B$ is pseudo-monotone from X into X^*.*

PROOF. By Proposition 2.3(2), A is pseudo-monotone from X into X^*. Hence the corollary is obtained as a consequence of Proposition 2.4. □

REMARK 2.2. The concepts of type M and of pseudo-monotonicity were originally introduced for singlevalued mappings from reflexive Banach spaces into the dual spaces by H. Brézis [5] and their definitions are given by using generalized sequences or filters in

stead of usual sequences. This was necessary in the construction of a solution of nonlinear functional equations formulated in nonseparable Banach spaces; in fact, that was essentially used in order to discuss the convergence of the Galerkin approximate solutions. Subsequently, an idea dispensing with generalized sequences was proposed by F. Browder and P. Hess [7] and the theory of mappings of monotone type has been generalized to multivalued cases together with existence results for variational inequalities (cf. [7,12]).

3. Fixed-point results

In this section we recall some fixed-point theorems, which are used in this chapter, with their brief proofs. The starting point of the fixed-point property for nonlinear mappings is the so-called retraction principle.

Let Y be a Hausdorff topological space, and Y_0 and K be subsets of Y such that $K \subset Y_0$. Then a singlevalued mapping $R_0 : Y_0 \to K$ is called a retraction from Y_0 onto K if R_0 is continuous in Y_0 and $R_0(w) = w$ for all $w \in K$. For instance, when Y is a real Banach space and K is a closed ball about $v_0 \in Y$ with radius $r > 0$, the function

$$R_0(v) := \begin{cases} v & \text{if } |v - v_0|_Y \leqslant r, \\ \frac{r}{|v-v_0|_Y}(v - v_0) + v_0 & \text{if } |v - v_0|_Y > r. \end{cases}$$

is a retraction from Y onto K.

We try to prove some fixed-point theorems, which are applied later in order to evolve our nonlinear theory of monotone mappings, starting from the retraction principles stated below. We use them without proof.

RETRACTION PRINCIPLES. *Let* $Y := \mathbf{R}^N$, $1 \leqslant N < \infty$. *Then*:

(R1) *For every bounded, closed and convex subset* K *of* Y, *there exists a retraction from* Y *onto* K.

(R2) *For the closed ball* K *about* $x_0 \in Y$ *with radius* $r > 0$, *there is no retraction from* K *onto its boundary* $\partial K := \{x \in Y;\ |x - x_0|_Y = r\}$.

It is easy to check the retraction principle (R1), but not (R2) any longer. We know several ways to prove principle (R2), for instance, it is proved at the beginning of the degree theory for mappings (cf. K. Deimling [8], E. Zeidler [18]). First we derive Brouwer's fixed point theorem from the retraction principles.

THEOREM 3.1 (Brouwer's Fixed-Point Theorem). *Let* $Y := \mathbf{R}^N$, $1 \leqslant N < \infty$, *and* $B_r(x_0)$ *be the closed ball about* $x_0 \in Y$ *with radius* $r > 0$. *Let* f *be a singlevalued continuous mapping from* $B_r(x_0)$ *into itself. Then* f *has at least one fixed-point in* $B_r(x_0)$.

PROOF. For a contradiction suppose that f has no fixed points in $B_r(x_0)$, namely $f(x) \neq x$ for all $x \in B_r(x_0)$. In this case, for each $x \in B_r(x_0)$ the directed half line $\ell(x)$ starting from $f(x)$ to x meets with the sphere $\partial B_r(x_0)$. We denote the meeting point by $g(x)$ for each $x \in B_r(x_0)$. It is clear that g is a continuous mapping from $B_r(x_0)$ onto the sphere $\partial B_r(x_0)$

and $g(x) = x$ for all $x \in \partial B_r(x_0)$, which means that there is a retraction from $B_r(x_0)$ onto the sphere $\partial B_r(x_0)$. This contradicts (R2). □

COROLLARY 3.1. *Let K be any closed, bounded and convex subset of $Y := \mathbf{R}^N$, $1 \leqslant N < \infty$, and f be a singlevalued continuous mapping from K into itself. Then f has at least one fixed-point in K.*

PROOF. Choose a sufficient large number $r > 0$ so that $K \subset B_r$, where B_r is the closed ball about the origin in Y with radius r. By principle (R1), there is a retraction R_K from Y onto K.Then the composition $\tilde{f} := f(R_K(\cdot))$ is a continuous mapping from B_r into itself. Therefore, by Theorem 3.1, $\tilde{f}$ has a fixed-point x in B_r, and $\tilde{f}(x) = x \in K$. Since $R_K(x) = x$, we have $f(x) = x$. □

THEOREM 3.2 (Variational Inequality). *Let Y be a finite dimensional real Banach space, K be any non-empty, compact and convex subset of Y, and A be an upper semicontinuous mapping from K into the dual space Y^* of Y such that Au is a bounded, closed and convex set in Y^*. Then there is a point $u_0 \in Y$ such that*

$$\begin{cases} u_0 \in K, \quad u_0^* \in Au_0, \\ \langle u_0^*, u_0 - v \rangle \leqslant 0, \quad \forall v \in K, \end{cases} \tag{3.1}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality between Y^* and Y.

As to the variational inequality (3.1), we have an equivalent form given in the following lemma.

LEMMA 3.1. *Under the same assumptions as in Theorem* 3.2, *the variational inequality* (3.1) *is equivalent to*

$$\begin{cases} u_0 \in K, \\ \forall v \in K, \ \exists u^*(v) \in Au_0 \ s.t. \ \langle u^*(v), u_0 - v \rangle \leqslant 0. \end{cases} \tag{3.1$'$}$$

PROOF. It is enough to derive (3.1) from (3.1′). Assume that $u_0 \in K$ satisfies the inequality in (3.1′). Putting $E_1 := Au_0$ and $E_2 := \{w := u_0 - v; \ v \in K\}$, we see that both of E_1 and E_2 are bounded, closed and convex subsets in Y^* and Y, respectively, and the function

$$H(u^*, w) := \langle u^*, w \rangle, \quad \forall u^* \in E_1, \ \forall w \in E_2,$$

is continuous on $E_1 \times E_2$ and affine in u^* and in w. Therefore, by the mini-max lemma,

$$\min_{u^* \in E_1} \max_{w \in E_2} H(u^*, w) = \max_{w \in E_2} \min_{u^* \in E_1} H(u^*, w),$$

namely

$$\min_{u^* \in Au_0} \max_{v \in K} \langle u^*, u_0 - v \rangle = \max_{v \in K} \min_{u^* \in Au_0} \langle u^*, u_0 - v \rangle =: \alpha. \tag{3.2}$$

Now we observe from (3.1′) that

$$\min_{u^* \in Au_0} \langle u^*, u_0 - v \rangle \leqslant 0, \quad \forall v \in K,$$

which implies $\alpha \leqslant 0$. Putting $P(u^*) = \max_{v \in K} \langle u^*, u_0 - v \rangle$ for each $u^* \in Au_0$, we have $\alpha = \min_{u^* \in Au_0} P(u^*) \leqslant 0$, so that there is $u_0^* \in Au_0$ satisfying $P(u_0^*) = \alpha \leqslant 0$, i.e. $\langle u_0^*, u_0 - v \rangle \leqslant 0$ for all $v \in K$. Thus u_0 satisfies (3.1). □

PROOF OF THEOREM 3.2. For a contradiction suppose that there is no point $u_0 \in Y$ satisfying (3.1). Then, by Lemma 3.1, for each $u \in K$ there is a point $v \in K$ such that $\langle u^*, u - v \rangle > 0$ for all $u^* \in Au$. Now, for each $v \in K$, set

$$V(v) := \{u \in K;\ \langle u^*, u - v \rangle > 0,\ \forall u^* \in Au\}. \tag{3.3}$$

Then, $V(v)$ is relatively open in K by the upper semicontinuity of A, and $K = \bigcup_{v \in K} V(v)$. Since K is compact in Y, we can find a finite number of $V(v_i)$, $i = 1, 2, \ldots, N$, such that $K = \bigcup_{i=1}^N V(v_i)$. Taking a partition $\{a_i(\cdot)\}_{i=1}^N$ of unity associated with the family $\{V(v_i)\}_{i=1}^N$, we define a singlevalued mapping $p : K_0 \to K_0$ by

$$p(v) = \sum_{i=1}^N a_i(v) v_i, \quad \forall v \in K_0,$$

where K_0 is the closed convex hull including $v_1, v_2, \ldots, v_N$, and we note that $\operatorname{supp}(a_i) \subset V(v_i)$, $a_i(v) \geqslant 0$ and $\sum_{i=1}^N a_i(v) = 1$ for all $v \in K_0$. Clearly K_0 is compact and convex in Y and p is continuous in K_0. Furthermore, we see from Brouwer's fixed-point theorem that p has a fixed-point $u_0 \in K_0$, namely $p(u_0) = u_0$. At this fixed point u_0 we have $\langle Au_0, u_0 - p(u_0) \rangle = 0$. On the other hand, we observe that $a_{i_0}(u_0) > 0$ for a certain i_0, $u_0 \in V(v_{i_0})$ and $\langle Au_0, u_0 - v_{i_0} \rangle > 0$ (cf. (3.3)). Hence

$$\langle Au_0, u_0 - p(u_0) \rangle = \sum_{i=1}^N a_i(u_0) \langle Au_0, u_0 - v_i \rangle \geqslant \langle Au_0, u_0 - v_{i_0} \rangle > 0.$$

This is a contradiction. □

We are now in a position to give a fixed-point theorem for multivalued mappings as an application of the results obtained above. So far as the fixed-point property for multivalued mappings is concerned, Kakutani [11] gave an important result in finite dimensional spaces in 1941 and subsequently its various generalizations were given in infinite dimensional spaces for instance by K. Fan [9], I. Glicksberg [10], F. Browder [6] and others.

THEOREM 3.3 (Fixed-Point Theorem for Multivalued Mappings). *Let Y be a locally convex, Hausdorff and real linear topological space, and K be a non-empty, compact and convex subset of Y. Let f be a multivalued mapping from K into itself such that*

(i) *$f(v)$ is a non-empty, closed and convex subset of K for every $v \in K$.*
(ii) *f is upper semicontinuous from K into itself.*
Then f has at least one fixed point in K.

PROOF. For a contradiction suppose that f has no fixed-points in K. We note that $v - f(v)$ is a non-empty and compact convex set in Y. Then, for every $v \in K$, $0 \notin v - f(v)$ and by the Hahn–Banach separation theorem there is an $v^*(v)$ in the dual space Y^* such that

$$\langle v^*(v), v - w\rangle > 0, \quad \forall w \in f(v). \tag{3.4}$$

Now, fix a family $\{v^*(v)\}_{v\in K}$ satisfying (3.4). For each $v \in K$ we set

$$V(v) := \{z \in K;\ \langle v^*(v), z - w\rangle > 0,\ \forall w \in f(z)\}.$$

By the upper semicontinuity of f we see that $V(v)$ is relatively open in K, and $K = \bigcup_{v\in K} V(v)$ because of $v \in V(v)$. Since K is compact in Y, we can find a finite number of $v_i \in K$, $i = 1, 2, \ldots, N$, such that

$$K = \bigcup_{i=1}^{N} V(v_i).$$

Now, take a partition of unity, $\{a_i(\cdot)\}_{i=1}^N$, associated with the family $\{V(v_i)\}_{i=1}^N$; note that $\operatorname{supp}(a_i) \subset V(v_i)$, $a_i(v) \geqslant 0$ and $\sum_{i=1}^N a_i(v) = 1$ for all $v \in K$. Further, define a mapping $q : K \to Y^*$ by

$$q(v) := \sum_{i=1}^{N} a_i(v) v^*(v_i), \quad \forall v \in K.$$

Since q is a singlevalued continuous mapping from K into Y^*. Therefore, on account of Theorem 3.2, there exists $v_0 \in K$ such that

$$\langle q(v_0), v_0 - w\rangle \leqslant 0, \quad \forall w \in K. \tag{3.5}$$

On the other hand, since $v_0 \in K$, it follows that $a_{i_0}(v_0) > 0$ for a certain i_0, $v_0 \in V(v_{i_0})$ and

$$\langle v^*(v_{i_0}), v_0 - w\rangle > 0, \quad \forall w \in f(v_0),$$

whence

$$\langle q(v_0), v_0 - w\rangle = \sum_{i=1}^{N} a_i(v_0)\langle v^*(v_i), v_0 - w\rangle > 0, \quad w \in f(v_0).$$

This is a contradiction to (3.5). □

The above fixed-point theorem provides us various special cases of the fixed-point property for nonlinear mappings.

COROLLARY 3.2. *Let Y be a locally convex, Hausdorff and real linear topological space, and K be a non-empty, compact and convex subset. Further let f be a singlevalued continuous mapping from K into itself. Then f has at least one fixed-point in K.*

COROLLARY 3.3. *Let X be a real reflexive Banach space, and K be a non-empty, closed, bounded and convex subset of X. Then we have*:

(i) *If f is a singlevalued weakly continuous mapping from K into itself, namely f is continuous in K with respect to the weak topology of X, then f has at least one fixed point in K.*

(ii) *Let f be a multivalued mapping from K into itself such that $f(v)$ is non-empty, bounded, closed and convex for every $v \in K$ and f is weakly upper semicontinuous in K. Then f has at least one fixed point in K.*

Corollary 3.2 is a special case of Theorem 3.2, when f is singlevalued. Also, in case $Y := X_w$ (the linear space X with the weak topology of X), a bounded closed and convex subset K of X is compact and convex in X_w and two assertions of Corollary 3.3 follow directly from Corollary 3.2 and Theorem 3.2, respectively.

4. Maximal monotonicity of mappings

Let X be a real reflexive Banach space and X^* be its dual space. Throughout this section, in order to avoid some irrelevant arguments we assume that X and X^* are strictly convex, so that the duality mapping $J : X \to X^*$ and its inverse J^{-1} are singlevalued.

The next theorem gives an useful characterization of "maximal monotonicity" of monotone mappings.

THEOREM 4.1. *Let $A : X \to X^*$ be a monotone mapping. Then, A is maximal monotone if and only if $R(A + J) = X^*$, or equivalently $R(A + \lambda J) = X^*$ for all positive number λ.*

Prior to the proof of Theorem 4.1 we prepare two lemmas.

LEMMA 4.1. *Let $A : X \to X^*$ be a monotone mapping and let C be a convex, closed and bounded subset of X such that $C \cap D(A) \neq \emptyset$. Moreover, let w^* be any element of X^*. Then there is $u_0 \in C$ such that*

$$\langle v^* + w^*, v - u_0 \rangle \geqslant 0, \quad \forall [v, v^*] \in G(A) \text{ with } v \in C. \tag{4.1}$$

PROOF. For each $[v, v^*] \in G(A)$ with $v \in C$, we put

$$P(v, v^*) := \{u \in C;\ \langle v^* + w^*, v - u \rangle \geqslant 0\}.$$

We note that $P(v, v^*) \neq \emptyset$, since $v \in P(v, v^*)$, and that $P(v, v^*)$ is convex, closed and bounded in X^*. Our claim is to show that

$$\bigcap_{v \in C,\, [v, v^*] \in G(A)} P(v, v^*) \neq \emptyset; \tag{4.2}$$

in fact, $u_0 \in \bigcap_{v \in C, [v, v^*] \in G(A)} P(v, v^*)$ satisfies (4.1). Also, since C is weakly compact in X, in order to get (4.2) it is enough to show that the family $\{P(v, v^*);\ [v, v^*] \in G(A),\ v \in C\}$ has the finite intersection property. To do so, let $[v_i, v_i^*], i = 1, 2, \ldots, n$, be any finite number of elements in $G(A)$ with $v_i \in C$. Now, we define a function $H(x, y)$ on $E \times E$, where $E := \{x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n;\ x_i \geqslant 0,\ i = 1, 2, \ldots, n,\ \sum_{i=1}^n x_i = 1\}$, by

$$H(x, y) = \sum_{i, j=1}^{n} x_i y_j \langle v_j^* + w^*, v_i - v_j \rangle$$

for $x = (x_1, x_2, \ldots, x_n) \in E$, $y = (y_1, y_2, \ldots, y_n) \in E$. Clearly, $H(x, y)$ is continuous on $E \times E$ and affine in x and y. Therefore, by the mini-max lemma, there are $x_0 := (x_{01}, x_{02}, \ldots, x_{0n}) \in S$ and $y_0 := (y_{01}, y_{02}, \ldots, y_{0n}) \in S$ such that

$$H(x_0, y) \leqslant H(x_0, y_0) \leqslant H(x, y_0), \quad \forall x \in E,\ \forall y \in E. \tag{4.3}$$

By an elementary computation we have

$$H(x, x) = \sum_{i > j} x_i x_j \langle v_i^* - v_j^*, v_j - v_i \rangle \leqslant 0, \quad \forall x \in E.$$

Combining this with (4.3), we obtain

$$\begin{aligned} H(x_0, y) &= \sum_{i, j=1}^{n} x_{0i} y_j \langle v_j^* + w^*, v_i - v_j \rangle \\ &= \sum_{j=1}^{n} y_j \left\langle v_j^* + w^*, \sum_{i=1}^{n} x_{0i} v_i - v_j \right\rangle \leqslant 0, \quad \forall y \in E; \end{aligned} \tag{4.4}$$

in particular, taking $y = (y_1, \ldots, y_n)$ with $y_j = 1$ and $y_i = 0$ for $i \neq j$ in (4.4) yields that

$$\left\langle v_j^* + w^*, \sum_{i=1}^{n} x_{0i} v_i - v_j \right\rangle \leqslant 0,$$

so that $u_0 := \sum_{i=1}^n x_{0i} v_i \in P(v_j, v_j^*)$ for $j = 1, 2, \ldots, n$. Thus $u_0 \in \bigcap_{j=1}^n P(v_j, v_j^*)$. Accordingly, (4.2) holds. □

Our proof of Theorem 4.1 needs a fixed-point result (cf. Theorem 3.2) for multivalued mappings.

LEMMA 4.2. *Let $A: X \to X^*$ be a monotone mapping, and $B: D(B) = X \to X^*$ be a singlevalued mapping which is demicontinuous and satisfies for some $v_0 \in D(A)$ that*

$$\lim_{|v|_X \to \infty} \frac{\langle Bv, v - v_0 \rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty. \tag{4.5}$$

Moreover let F be any finite dimensional subspace of X such that $F \cap D(A) \neq \emptyset$. Then, for each $f \in X^*$ there exists $u_F \in F$ such that

$$\langle v^* + Bu_F - f, v - u_F \rangle \geqslant 0, \quad \forall [v, v^*] \in G(A) \text{ with } v \in F. \tag{4.6}$$

PROOF. Now, let B_n be the closed ball about the origin of X with radius $n \in \mathbf{N}$. For any sufficiently large n so that $F \cap D(A) \cap B_n \neq \emptyset$, putting $C_n := B_n \cap F$, we define a mapping $Q_n: C_n \to C_n$ by

$$Q_n w = \left\{ u \in C_n;\ \langle v^* + Bw - f, v - u \rangle \geqslant 0,\ \forall [v, v^*] \in G(A) \text{ with } v \in C_n \right\}$$

for each $w \in C_n$. By Lemma 4.1, $Q_n w \neq \emptyset$ for every $w \in C_n$. Clearly $Q_n w$ is compact and convex in C_n for every $w \in C_n$ and upper semicontinuous in C_n. Therefore, by virtue of Theorem 3.2, Q_n has at least one fixed point $u_n \in C_n$, i.e. $u_n \in Q_n u_n$, which satisfies that

$$\langle v^* + Bu_n - f, v - u_n \rangle \geqslant 0, \quad \forall [v, v^*] \in G(A) \text{ with } v \in C_n. \tag{4.7}$$

Next, we consider the sequence $\{u_n\}$ constructed above. Choosing $v_0 \in D(A) \cap C_n$ with $v_0^* \in Av_0$ and taking them as $v, v^* \in Av$ in (4.7), we see that

$$\frac{\langle v_0^* - f, v_0 - u_n \rangle}{|u_n|_X} \geqslant \frac{\langle Bu_n, u_n - v_0 \rangle}{|u_n|_X}, \quad \forall \text{ large } n.$$

Hence $\{u_n\}$ is bounded in F by (4.5), so that there is a subsequence $\{u_{n_k}\}$ which converges to a point, denoted by u_F, in F as $k \to \infty$. Taking $n = n_k \to \infty$ in (4.7), we have (4.6). □

PROOF OF THEOREM 4.1. First, assume that A is monotone and $R(A + J) = X^*$. Let A' be any monotone mapping from X into X^* such that $G(A) \subset G(A')$. Given $[u, u^*] \in G(A')$, we put $f = u^* + Ju$. Since $R(A + J) \subset R(A' + J)$, we have

$$R(A + J) = R(A' + J) = X^*.$$

Choose $v \in X$ such that $f = v^* + Jv$ with $v^* \in Av$. Then, by the monotonicity of A' and J we see that

$$0 = \langle u^* + Ju - v^* - Jv, u - v \rangle \geqslant \langle Ju - Jv, u - v \rangle \geqslant 0,$$

whence $\langle Ju - Jv, u - v\rangle = 0$. This implies $u = v$ and $v^* = f - Jv = f - Ju = u^* \in Au$. Thus $G(A') \subset G(A)$. Consequently $A = A'$ holds and A is maximal monotone.

Next, assume that $A : X \to X^*$ is maximal monotone. Let f be any element of X^*. Now, take an element $[v_0, v_0^*] \in G(A)$ and choose a finite dimensional subspace F_0 of X with $v_0 \in F_0$. Denote by $\mathcal{F}$ the family of all finite dimensional subspaces of X including F_0. Then it follows from Lemma 4.2 with $B = J$ that for each $F \in \mathcal{F}$ there is $u_F \in F$ such that

$$\langle v^* + Ju_F - f, v - u_F\rangle \geqslant 0, \quad \forall [v, v^*] \in G(A) \text{ with } v \in F. \tag{4.8}$$

Now we denote by $\Lambda(F)$ the set of all pairs $[u_F, Ju_F]$ satisfying (4.8) with $u_F \in F$. In this case $\Lambda(F) \neq \emptyset$ for every $F \in \mathcal{F}$. From (4.8) we infer that

$$\langle v_0^*, v_0\rangle - \langle f, v_0\rangle \geqslant |u_F|_X^2 - |f|_{X^*}|u_F|_X, \quad \forall F \in \mathcal{F},$$

which implies that $\{u_F\}$ is bounded in X. Therefore, there exists a positive number $M > 0$ such that $\Lambda(F) \subset B_M \times B_M^*$ for every $F \in \mathcal{F}$, where B_M and B_M^* are the closed balls about the origin of X and X^* with radius M in X and X^*, respectively. Here we set

$$U(F) = \bigcup_{F' \supset F} \Lambda\big(F'\big), \quad \forall F \in \mathcal{F}.$$

We observe that the family $\{U(F);\ F \in \mathcal{F}\}$ has the finite intersection property, because for any finite number of collections $F_1, F_2, \ldots, F_m \in \mathcal{F}$

$$\bigcap_{i=1}^{m} U(F_i) \supset U\left(\bigcup_{i=1}^{m} F_i\right) \neq \emptyset.$$

Therefore, for the weak-closure $\widetilde{U}(F)$ of $U(F)$ in $X \times X^*$ we have

$$\bigcap_{F \in \mathcal{F}} \widetilde{U}(F) \neq \emptyset,$$

since $B_M \times B_M^*$ is weakly compact in $X \times X^*$ and $\widetilde{U}(F) \subset B_M \times B_M^*$ for every $F \in \mathcal{F}$. Let $[u, \xi]$ be an element $\bigcap_{F \in \mathcal{F}} \widetilde{U}(F)$. Let v be any element in $D(A)$ and choose $F_1 \in \mathcal{F}$ such that $u, v \in F_1$. Then, applying the lemma of choice in Section 1, we find an increasing sequence $\{F_n\}$ with $F_n \supset F_1$ and a sequence $\{[u_n, Ju_n]\}$ such that $[u_n, Ju_n] \in \Lambda(F_n)$ and $u_n \to u$ weakly in X and $Ju_n \to \xi$ weakly in X^*. Then, from (4.8) together with the monotonicity of J it follows that

$$\langle v^* - f, v - u_n\rangle + \langle Ju_n, v - u\rangle \geqslant \langle Ju_n, u_n - u\rangle \geqslant \langle Ju, u_n - u\rangle,$$
$$\forall [v, v^*] \in G(A) \text{ with } v \in F_n.$$

By passing to the limit in n we obtain that

$$\langle v^* + \xi - f, v - u\rangle \geqslant 0, \tag{4.9}$$

and

$$\limsup_{n\to\infty}\langle Ju_n, u_n - u\rangle \leqslant \langle v^* + \xi - f, v - u\rangle. \tag{4.10}$$

It should be noted that (4.9) and (4.10) hold for every $[v, v^*] \in G(A)$. Hence the maximal monotonicity of A implies that $[u, f - \xi] \in G(A)$ and taking $v = u$ in (4.10) yields that

$$\limsup_{n\to+\infty}\langle Ju_n, u_n\rangle \leqslant \langle \xi, u\rangle.$$

Therefore it follows from Propositions 2.2(b), that $\xi \in Ju$. Thus we have $f \in Au + Ju$. This shows $R(A + J) = X^*$. □

COROLLARY 4.1. *Let φ be any proper, l.s.c. convex function on X. Then the subdifferential $\partial\varphi$ is a maximal monotone mapping from X into X^*.*

PROOF. By virtue of Theorem 4.1 it is enough to prove that $R(\partial\varphi + J) = X^*$. To this end, let f be any element of X^*, and consider the function

$$\ell(u) := \varphi(u) + \frac{1}{2}|u|_X^2 - \langle f, u\rangle, \quad \forall u \in X.$$

Clearly, $\ell(\cdot)$ is proper, l.s.c and convex on X as well as $\ell(u) \to \infty$ as $|u|_X \to \infty$, and it has a minimum point $z \in X$, namely z satisfies that

$$\langle f, u - z\rangle \leqslant \varphi(u) + \frac{1}{2}|u|_X^2 - \varphi(z) - \frac{1}{2}|z|_X^2, \quad \forall u \in D(\varphi), \tag{4.11}$$

Here, take $u = z + \varepsilon(v - z)$ in (4.11) for any $v \in D(\varphi)$ and $\varepsilon > 0$ to get

$$\langle f, v - z\rangle \leqslant \varphi(v) - \varphi(z) + \frac{1}{\varepsilon}\left\{\frac{1}{2}\big|z + \varepsilon(v - z)\big|_X^2 - \frac{1}{2}|z|_X^2\right\}. \tag{4.12}$$

Noting that

$$\frac{1}{\varepsilon}\left\{\frac{1}{2}\big|z + \varepsilon(v - z)\big|_X^2 - \frac{1}{2}|z|_X^2\right\} \leqslant \big\langle J\big(z + \varepsilon(v - z)\big), v - z\big\rangle,$$

we derive from (4.12) by letting $\varepsilon \to 0$ that

$$\langle f - Jz, v - z\rangle \leqslant \varphi(v) - \varphi(z), \quad \forall v \in D(\varphi).$$

This shows that $f - Jz \in \partial\varphi(z)$, namely $f \in \partial\varphi(z) + Jz$. Hence $R(\partial\varphi + J) = X^*$. □

COROLLARY 4.2. *Let $A : X \to X^*$ be maximal monotone mapping, and suppose that for some $u_0 \in X$*

$$\inf_{u^* \in Au} \frac{\langle u^*, u - u_0 \rangle}{|u|_X} \longrightarrow +\infty \quad \text{as } |u|_X \to \infty. \tag{4.13}$$

Then $R(A) = X^*$.

PROOF. Let f be any element of X^*. Then, by Theorem 4.1, for each $\lambda \in (0, 1]$ there is $u_\lambda \in X$ such that $f = u_\lambda^* + \lambda J u_\lambda$ with $u_\lambda^* \in A u_\lambda$. Since

$$\frac{\langle f, u_\lambda - u_0 \rangle}{|u_\lambda|_X} = \frac{\langle u_\lambda^*, u_\lambda - u_0 \rangle}{|u_\lambda|_X} + \frac{\lambda \langle J u_\lambda, u_\lambda - u_0 \rangle}{|u_\lambda|_X},$$

it follows from (4.13) that $\{u_\lambda\}$ is bounded in X as $\lambda \to 0$, and hence there exist a sequence $\{\lambda_n\}$ tending to 0 and a point $u \in X$ such that $\{u_n := u_{\lambda_n}\}$ weakly converges to u in X. We note here that $f - \lambda_n J u_n \in A u_n$ and $f - \lambda_n J u_n \to f$ in X. On account of the demiclosedness of A, we have $f \in Au$. Consequently, $R(A) = X^*$. □

In general, condition (4.13) is satisfied, the mapping A is called "*coercive*".

REMARK 4.1. When A is the subdifferential of a proper, l.s.c. and convex function φ on X, we see that $R(\partial\varphi) = X^*$, if $\varphi(u)/|u|_X \to \infty$ as $|u|_X \to \infty$. In fact, from this condition it follows that

$$\inf_{u^* \in Au} \frac{\langle u^*, u - u_0 \rangle}{|u|_X} \geqslant \frac{\varphi(u) - \varphi(u_0)}{|u|_X} \longrightarrow \infty \quad \text{as } |u|_X \to \infty$$

for any fixed $u_0 \in D(\varphi)$.

THEOREM 4.2. *Let A be a monotone mapping from $D(A) = X$ into X^* such that $G(A)$ is demiclosed in $X \times X^*$ and Av is closed and convex in X^* for every $v \in X$. Then A is maximal monotone.*

PROOF. We use Lemma 4.2. Let $[v_0, v_0^*]$ be any element in $G(A)$ and fix it. Choosing a finite dimensional subspace F_0 of X such that $v_0 \in F_0$, we denote by $\mathcal{F}$ the family of all finite dimensional subspaces F, containing F_0, of X. Let f be any element of X^*. On account of Lemma 4.2 with $B = J$, for each $F \in \mathcal{F}$ there exists $u_F \in F$ such that

$$\langle v^* + J u_F - f, v - u_F \rangle \geqslant 0, \quad \forall [v, v^*] \in G(A) \text{ with } v \in F. \tag{4.14}$$

By (4.14) we see that

$$\frac{\langle v_0^* - f, v_0 - u_F \rangle}{|u_F|_X} \geqslant \frac{\langle J u_F, u_F - v_0 \rangle}{|u_F|_X} \geqslant |u_F|_X - |v_0|_X,$$

so that $\{u_F\}$ is bounded in X. Hence there exists a positive number M such that $[u_F, Ju_F] \in B_M \times B_M^*$ for all F. We denote by $\Lambda(F)$ the set of all pairs $[u_F, Ju_F]$ satisfying (4.14), and put

$$U(F) = \bigcup_{F' \supset F} \Lambda\big(F'\big), \quad \forall F \in \mathcal{F}.$$

Then, just as in the proof of Theorem 4.1, we see that

$$B_M \times B_M^* \supset \bigcap_{F \in \mathcal{F}} \widetilde{U}(F) \neq \emptyset,$$

where $\widetilde{U}(F)$ is the weak closure of $U(F)$ in $X \times X^*$. Now, let $[u, \xi]$ be any element in $\bigcap_{F \in \mathcal{F}} \widetilde{U}(F)$. For any $v \in X$, choose $F_1 \in \mathcal{F}$ such that $v, u \in F_1$. Then, applying the lemma of choice in Section 1, we find an increasing sequence $\{F_n\}$ with $F_n \supset F_1$ and a sequence $\{[u_n, Ju_n]\}$ such that $[u_n, Ju_n] \in \Lambda(F_n)$, $u_n \to u$ weakly in X, $Ju_n \to \xi$ weakly in X^*. By taking $v = u$ and passing to the limit as $n \to +\infty$ in (4.14) with $F = F_n$ we have

$$\limsup_{n \to +\infty} \langle Ju_n, u_n - u \rangle \leqslant 0,$$

which implies by Proposition 2.2(b), that

$$Ju = \xi, \quad \lim_{n \to +\infty} \langle Ju_n, u_n \rangle = \langle Ju, u \rangle.$$

Noting this fact and passing to the limit with respect to n in (4.14), we derive

$$\langle v^* + Ju - f, v - u \rangle \geqslant 0, \quad \forall [v, v^*] \in G(A). \tag{4.15}$$

Here, take as v the function $u + tw$ for any $w \in X$ and any $t \in (0, 1)$ in (4.15). Then, by the local boundedness of A (cf. Proposition 2.2(c)) and the demiclosedness of $G(A)$ we can find $\{t_n\} \subset (0, 1)$, tending to 0, and $\{w_n^*\}$ with $w_n^* \in A(u + t_n w)$ such that w_n^* converges weakly in X to a point $u^*(w) \in Au$. Taking $[u + t_n w, w_n^*]$ as $[v, v^*]$ in (4.15) and letting $n \to \infty$, we get $\langle u^*(w) + Ju - f, w \rangle = 0$. Now, put $E := \{u^* + Ju - f;\ u^* \in Au\}$. Then E is a bounded closed and convex subset of X and we have seen above the following statement:

$$\begin{cases} \text{for any } w \in X \text{ there is } z^*(w) \in E \text{ such that} \\ \langle z^*(w), w \rangle = 0. \end{cases} \tag{4.16}$$

This implies by the Hahn–Banach theorem that $0 \in E$, namely $f \in Au + Ju$. Consequently $R(A + J) = X^*$ and A is maximal monotone. □

COROLLARY 4.3. *Let A be a singlevalued, demicontinuous and monotone mapping from $D(A) = X$ into X^*. Then A is maximal monotone.*

Corollary 4.3 is a special case of Theorem 4.2 in which A is singlevalued.

5. Perturbations of monotone type

In this section we study functional equations of the form

$$f \in Au + Bu,$$

where A and B are maximal monotone or pseudo-monotone or of type M. This question is reduced to the investigation of the range $R(A+B)$. We suppose that X is a real reflexive Banach space and X and the dual X^* are strictly convex. Our results of perturbation are stated as follows. The first result is concerned with perturbations for linear maximal monotone mappings.

Before proceeding to the main results in this section we investigate the ranges of mappings of type M without any perturbations.

PROPOSITION 5.1. *Let $A: D(A) = X \to X^*$ be a mapping of type M and suppose that A is bounded and*

$$\inf_{v^* \in Av} \frac{\langle v^*, v\rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty. \tag{5.1}$$

Then $R(A) = X^*$.

PROOF. Let f^* be any element of X^*. By assumption (5.1), there is a positive number M such that

$$\inf_{v^* \in Av} \frac{\langle v^*, v\rangle}{|v|_X} > |f^*|_{X^*}, \quad \forall v \in X \text{ with } |v|_X \geqslant M. \tag{5.2}$$

Let $\mathcal{F}$ be the family of all finite dimensional subspaces of X. For each $F \in \mathcal{F}$ we denote by π_F and π_F^* the natural injection from F into X and the projection from X^* onto F^*, respectively, and consider the mapping $A_F: F \to F^*$ given by $A_F = \pi_F^* A \pi_F$. Then it is easy to see from the property of type M that $A_F v$ is non-empty compact convex in F^* for every $v \in F$ and A_F is upper semicontinuous from F into F^*. We may identify F with an finite dimensional Banach space as well as F^*. In this case, the upper semicontinuity of A_F is checked as follows. If it was not upper semicontinuous at a certain point $v_0 \in F$, then there would be a sequence $\{[v_n, v_n^*]\} \subset G(A_F)$ and a positive number δ such that

$$v_n \to v_0 \quad \text{in } F, \qquad \inf_{v^* \in A_F v_0} |v_n^* - v^*|_{F^*} \geqslant \delta, \quad \forall n. \tag{5.3}$$

Since A is bounded by our assumption, we may assume that there is a sequence $\{\bar{v}_n^*\} \subset X^*$ such that

$$\bar{v}_n^* \in Av_n, \quad v_n^* = \pi_F^* \bar{v}_n^*, \quad \bar{v}_n^* \to \bar{v}^* \quad \text{weakly in } X^*.$$

Clearly,

$$\lim_{n\to\infty} \langle \bar{v}_n^*, v_n\rangle = \langle \bar{v}^*, v_0\rangle,$$

which implies by property (TM2) of type M that $\bar{v}^* \in Av_0$ and $\pi_F^* \bar{v}^* \in A_F v_0$. This contradicts (5.3). Thus A_F is upper semicontinuous from F into F^*.

Now, let B_M be the closed ball about the origin in X with radius M as in (5.2). For each $F \in \mathcal{F}$, by virtue of Theorem 3.2 there exists an element $u_F \in F$ such that

$$\begin{cases} u_F \in B_M \cap F, \quad u_F^* \in A_F u_F; \\ \langle u_F^* - \pi_F^* f^*, u_F - v \rangle \leqslant 0, \quad \forall v \in B_M \cap F. \end{cases} \tag{5.4}$$

Choose $\bar{u}_F^* \in Au_F$ so that $u^* = \pi_F^* \bar{u}_F^*$. Then, it follows from (5.4) that $\langle \bar{u}_F^*, u_F \rangle \leqslant \langle f^*, u_F \rangle$ which shows by (5.2) that $|u_F|_X \leqslant M$. Therefore, by the boundedness of A we see that $\{\bar{u}_F^*\}_{F \in \mathcal{F}}$ is bounded in X^*. Also, by (5.4) again we have that

$$u_F^* = \pi_F^* f^*, \qquad \langle \bar{u}_F^*, v \rangle = \langle f^*, v \rangle, \quad \forall v \in F. \tag{5.5}$$

In order to use the lemma of choice in Section 1, define a mapping $\Lambda_0 : \mathcal{F} \to B_{M_0} \times B_{M_0}^*$, where $M_0 := M + \sup_{v \in B_M,\, v^* \in Av} |v^*|_{X^*}$, by $\Lambda_0(F) = [u_F, \bar{u}_F^*]$ for every $F \in \mathcal{F}$, and put $U_0(F) := \bigcup_{F' \supset F} \Lambda(F')$, $\forall F \in \mathcal{F}$. Then the family $\{U_0(F)\}_{F \in \mathcal{F}}$ has the finite intersection property. Since $B_{M_0} \times B_{M_0}^*$ is weakly compact in $X \times X^*$, we see that

$$\bigcap_{F \in \mathcal{F}} \widetilde{U}_0(F) \neq \emptyset,$$

where $\widetilde{U}_0(F)$ is the weak-closure of $U_0(F)$ in $X \times X^*$. Now, let $\tilde{u}_0$ be in $\bigcap_{F \in \mathcal{F}} \widetilde{U}_0(F)$. Let v be any element in X and take $F_0' \in \mathcal{F}$ such that v and u_0 are in F_0'. By the corollary of the lemma of choice there exists an increasing sequence $\{F_n\} \subset \mathcal{F}$ such that $F_0' \subset F_n$ for all n and $\tilde{u}_n := [u_{F_n}, \bar{u}_{F_n}^*] \to \tilde{u}_0 := [u_0, \bar{u}_0^*]$ weakly in $X \times X^*$ (as $n \to \infty$). Furthermore, on account of (5.5) we observe that

$$\lim_{n \to \infty} \langle \bar{u}_{F_n}^*, u_{F_n} - u_0 \rangle = \lim_{n \to \infty} \langle f^*, u_{F_n} - u_0 \rangle = 0,$$

namely

$$\lim_{n \to \infty} \langle \bar{u}_{F_n}^*, u_{F_n} \rangle = \langle \bar{u}_0^*, u_0 \rangle.$$

Consequently, from the definition of type M it follows that $\bar{u}_0^* \in Au_0$. By (5.5) again we see that $\langle \bar{u}_0^*, v \rangle = \lim_{n \to \infty} \langle \bar{u}_{F_n}^*, v \rangle = \langle f^*, v \rangle$. Since v is an arbitrary element in X, it holds that $\bar{u}_0^* = f^*$ and $f^* \in Au_0$. Thus $R(A) = X^*$. □

COROLLARY 5.1. *Let $A : D(A) = X \to X^*$ be a pseudo-monotone mapping and suppose that A is bounded and*

$$\inf_{v^* \in Av} \frac{\langle v^*, v \rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty.$$

Then $R(A) = X^*$.

Since pseudo-monotone mappings are of type M, the above corollary is a direct consequence of Proposition 5.1.

REMARK 5.1. In Proposition 5.1, condition (5.1) is replaced by the following one: there is an element $v_0 \in X$ such that

$$\inf_{v^* \in Av} \frac{\langle v^*, v - v_0 \rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty. \tag{5.1'}$$

In fact, define a mapping $A' : D(A') = X \to X^*$ by $A'(v) = A(v + v_0)$ for all $v \in X$. Then we see that A' is still of type M. Also, we have by (5.1') that

$$\inf_{v^* \in A'v} \frac{\langle v^*, v \rangle}{|v|_X} = \inf_{v^* \in A(v+v_0)} \frac{\langle v^*, (v + v_0) - v_0 \rangle}{|v + v_0|_X} \cdot \frac{|v + v_0|_X}{|v|_X} \to \infty$$
$$\text{as } |v|_X \to \infty.$$

Thus condition (5.1) holds for the mapping A'. Hence $R(A) = R(A') = X^*$.

THEOREM 5.1. *Let $L : X \to X^*$ be a maximal monotone mapping with linear graph $G(L)$ in $X \times X^*$, and let $B : D(B) = X \to X^*$ be a bounded mapping of type M. Suppose that*

$$\inf_{v^* \in Bv} \frac{\langle v^*, v \rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty. \tag{5.6}$$

Then $R(L + B) = X^*$.

As to perturbations for nonlinear maximal monotone mappings we prove:

THEOREM 5.2. *Let $A : X \to X^*$ be a maximal monotone mapping, and let $B : D(B) = X \to X^*$ be a bounded pseudo-monotone mapping. Suppose that for some $v_0 \in D(A)$*

$$\inf_{v^* \in Bv} \frac{\langle v^*, v - v_0 \rangle}{|v|_X} \to +\infty \quad \text{as } |v|_X \to \infty. \tag{5.7}$$

Then $R(A + B) = X^*$.

COROLLARY 5.2. *Let $A : X \to X^*$ be a maximal monotone mapping and $B : D(B) = X \to X^*$ be a bounded and pseudo-monotone mapping. Suppose that*

(a) *there are a point $v_0 \in D(A)$ and positive constants b_0, b_1 such that*

$$\langle v^*, v - v_0 \rangle \geqslant -b_0 |v|_X - b_1 \quad \forall [v, v^*] \in G(B);$$

(b) $$\inf_{v^* \in Av, v_1^* \in Bv} \frac{\langle v^* + v_1^*, v - v_0 \rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty,\ v \in D(A),$$

where v_0 is as in (a).

Then $R(A + B) = X^*$.

For the proofs of our theorems we prepare the following lemma.

LEMMA 5.1. *Let $A : X \to X^*$ be a maximal monotone mapping. Then we have*:

(i) *We define $A^\circ v = \{v^* \in Av;\ |v^*|_{X^*} = \inf_{v' \in Av} |v'|_{X^*}\}$ for each $v \in D(A)$. Then A° is singlevalued and $D(A^\circ) = D(A)$; A° is called the minimal section of A.*

(ii) *For each $\varepsilon > 0$, the mapping $A_\varepsilon := (A^{-1} + \varepsilon J^{-1})^{-1}$ is a singlevalued, bounded and maximal monotone mapping from X into X^* with $D(A_\varepsilon) = X$. Also, if $v \in D(A)$, then $|A_\varepsilon v|_{X^*} \leqslant |A^\circ v|_{X^*}$, and hence if $[0, 0] \in G(A)$, then $A_\varepsilon 0 = 0$. Moreover, $A_\varepsilon v \to A^\circ v$ weakly in X^* as $\varepsilon \downarrow 0$ for every $v \in D(A)$.*

PROOF. We prove (i). Let v be any element of $D(A)$. Since Av is closed and convex in X^* and X^* is strictly convex by our assumption, there is a unique point v^* in Av such that $|v^*| = \inf_{v' \in Av} |v'|_{X^*}$. Therefore A° is well-defined as a singlevalued mapping from $D(A^\circ) = D(A) \subset X$ into X^*.

Next, we prove (ii). Since $A^{-1} : X^* \to X$ is maximal monotone, it follows from Theorem 4.1 that $R(A^{-1} + \varepsilon J^{-1}) = X$ and $A_\varepsilon : D(A_\varepsilon) = X \to X^*$ is a maximal monotone mapping. Next we show that A_ε is singlevalued. In fact, let u be any point in X and assume $u_i^* \in A_\varepsilon u$, $i = 1, 2$. Then $A^{-1}u_i^* + \varepsilon J^{-1}u_i^* \ni u$, $i = 1, 2$. Therefore, $u = u_i + \varepsilon J^{-1}u_i^*$ for some $u_i \in A^{-1}u_i^*$, $i = 1, 2$. We have

$$\begin{aligned}
0 &= \langle u_1^* - u_2^*, u_1 + \varepsilon J^{-1}u_1^* - u_2 - \varepsilon J^{-1}u_2^* \rangle \\
&= \langle u_1^* - u_2^*, u_1 - u_2 \rangle + \varepsilon \langle u_1^* - u_2^*, J^{-1}u_1^* - J^{-1}u_2^* \rangle \\
&\geqslant \varepsilon \{ |u_1^*|_{X^*}^2 - \langle u_1^*, J^{-1}u_2^* \rangle - \langle u_2^*, J^{-1}u_1^* \rangle + |u_2^*|_{X^*}^2 \} \\
&\geqslant \varepsilon \big(|u_1^*|_{X^*} - |u_2^*|_{X^*} \big)^2 .
\end{aligned}$$

This shows that $|u_1^*|_{X^*} = |u_2^*|_{X^*}$ and $\langle u_2^*, J^{-1}u_1^* \rangle = |u_2^*|_{X^*}^2$, whence $u_1^* = u_2^*$. Thus A_ε is singlevalued.

Next, we show the boundedness of A_ε. Assume that $\{v_n\}$ is a bounded sequence in X. Let $[v_0, v_0^*] \in G(A)$, and put $v_n^* = A_\varepsilon v_n$. Then, by definition, $v_n - \varepsilon J^{-1}v_n^* \in A^{-1}v_n^*$, and the monotonicity of A^{-1} implies that

$$\langle v_n^* - v_0^*, v_n - \varepsilon J^{-1}v_n^* - v_0 \rangle \geqslant 0,$$

whence

$$\varepsilon |v_n^*|_{X^*}^2 \leqslant |v_n^*|_{X^*} \{ \varepsilon |v_0^*|_{X^*} + |v_n - v_0|_X \} + |v_0^*|_{X^*} |v_n - v_0|_X .$$

The last inequality shows that $\{v_n^*\}$ is bounded in X^*. Thus A_ε is bounded.

Next, we show the demicontinuity of A_ε. Let $\{v_n\}$ be any sequence in X such that $v_n \to v$ in X, and $v^* \in X^*$ be any weak cluster point of $\{A_\varepsilon v_n\}$; note that there is at least one weak cluster point on $\{A_\varepsilon v_n\}$, since A_ε is bounded. Now, on account of the demiclosedness of maximal monotone mappings, we see that $v^* = A_\varepsilon v$. Therefore we conclude that $A_\varepsilon v_n \to A_\varepsilon v$. Thus A_ε is demicontinuous.

Finally, assume $v \in D(A)$. Then, $A_\varepsilon v = (A^{-1} + \varepsilon J^{-1})^{-1} v$ implies that $A_\varepsilon v \in A(v - \varepsilon J^{-1}(A_\varepsilon v))$. Therefore, by the monotonicity of A we have for all $v^* \in Av$

$$\big\langle A_\varepsilon v - v^*, -\varepsilon J^{-1}(A_\varepsilon v) \big\rangle \geqslant 0,$$

whence

$$|A_\varepsilon v|_{X^*}^2 \leqslant |v^*|_{X^*} |A_\varepsilon v|_{X^*}, \quad \text{i.e.} \quad |A_\varepsilon v|_{X^*} \leqslant |v^*|_{X^*}.$$

Thus $|A_\varepsilon v|_{X^*} \leqslant |A^\circ v|_{X^*}$. Moreover, by the demiclosedness of A, as $\varepsilon \downarrow 0$, any weak cluster point v^* of $\{A_\varepsilon v\}$ in X belongs to Av and $|v^*|_{X^*} \leqslant |A^\circ v|_{X^*}$. This shows that $v^* = A^\circ v$ and hence $A_\varepsilon v \to A^\circ v$ weakly in X as $\varepsilon \downarrow 0$. □

PROOF OF THEOREM 5.1. We give our proof in three steps.

First step. Denote by $\mathcal{F}$ the family of all finite dimensional subspaces of X. We use the same notation as in the proof of Proposition 5.1; for each F we denote by π_F the injection from F into X and by π_F^* the projection from X^* onto F^*. Let ε be any number in $(0, 1]$ and consider the approximation $L_\varepsilon := (L^{-1} + \varepsilon J^{-1})^{-1}$ of L as constructed in Lemma 5.1. Also, let f^* be any element of X^*. We see from Lemma 5.1 that L_ε is a singlevalued, monotone, bounded and demicontinuous mapping from $D(L_\varepsilon) = X$ into X^* and $L_\varepsilon 0 = 0$. Also, as was seen in the proof of Proposition 5.1, for every $F \in \mathcal{F}$ we observe that $\pi_F^* B \pi_F$ is upper semicontinuous from F into F^* and hence so is $\pi_F^*(L_\varepsilon + B)\pi_F$, and $\pi_F^*(L_\varepsilon v + Bv)$ is a non-empty compact convex subset in F^* for every $v \in F$. By virtue of Proposition 5.1, for each $F \in \mathcal{F}$ there exists $u_{\varepsilon F} \in F$ and $\bar{u}_{\varepsilon F}^* \in B u_{\varepsilon F}$ such that $\pi_F^*(L_\varepsilon u_{\varepsilon F} + \bar{u}_{\varepsilon F}^*) = \pi_F^* f^*$, namely

$$\langle L_\varepsilon u_{\varepsilon F} + \bar{u}_{\varepsilon F}^* - f^*, v \rangle = 0, \quad \forall v \in F. \tag{5.8}$$

By virtue of (5.6) with (5.8) there is a positive constant M such that

$$|u_{\varepsilon F}|_X \leqslant M, \quad \forall \varepsilon \in (0, 1], \ \forall F \in \mathcal{F}, \tag{5.9}$$

and hence there is a positive constant M_1 such that

$$|\bar{u}_{\varepsilon F}^*|_{X^*} \leqslant M_1, \quad \forall \varepsilon \in (0, 1], \ \forall F \in \mathcal{F}. \tag{5.10}$$

Next, since

$$L_\varepsilon u_{\varepsilon F} \in L\big(u_{\varepsilon F} - \varepsilon J^{-1}(L_\varepsilon u_{\varepsilon F})\big), \tag{5.11}$$

it follows from (5.11) with the monotonicity of L that

$$\varepsilon |L_\varepsilon u_{\varepsilon F}|_X^2 = \big\langle L_\varepsilon u_{\varepsilon F}, \varepsilon J^{-1}(L_\varepsilon u_{\varepsilon F}) \big\rangle \leqslant \langle L_\varepsilon u_{\varepsilon F}, u_{\varepsilon F} \rangle = \langle f^* - \bar{u}_{\varepsilon F}^*, u_{\varepsilon F} \rangle,$$

which gives

$$\varepsilon |L_\varepsilon u_{\varepsilon F}|_{X^*}^2 \leqslant M\big(M_1 + |f|_{X^*}\big) =: M_2. \tag{5.12}$$

Fixing $\varepsilon > 0$ we define $\Lambda : \mathcal{F} \to B_M \times B^*_{M_1} \times B^*_{M_\varepsilon} \times B_{M_\varepsilon}$, $M_\varepsilon := \sqrt{M_2/\varepsilon}$, by putting

$$\Lambda(F) = \big\{\big[u_{\varepsilon F}, \bar{u}^*_{\varepsilon F}, L_\varepsilon u_{\varepsilon F}, J^{-1}(L_\varepsilon u_{\varepsilon F})\big];\ (5.8)\text{ holds}\big\}$$

and $U(F) = \bigcup_{F' \supset F} \Lambda(F')$ for every $F \in \mathcal{F}$. Just as in the proof of Proposition 5.1, we see easily that $B_M \times B^*_{M_1} \times B^*_{M_\varepsilon} \times B_{M_\varepsilon} \supset \bigcap_{F \in \mathcal{F}} \widetilde{U}(F) \neq \emptyset$, where $\widetilde{U}(F)$ is the weak-closure of $U(F)$ in $X \times X^* \times X^* \times X$. Let $[u_\varepsilon, g_\varepsilon, \ell_\varepsilon, \rho_\varepsilon]$ be any element in $\bigcap_{F \in \mathcal{F}} \widetilde{U}(F)$ and let v be any element of X. Choosing $F_1 \in \mathcal{F}$ so that $v, u_\varepsilon \in F_1$, by the corollary of the lemma of choice we find an increasing sequence $\{F_n\}$ with $F_n \supset F_1$ and a sequence $\{[u_{\varepsilon F_n}, \bar{u}^*_{\varepsilon F_n}, L_\varepsilon u_{\varepsilon F_n}, J^{-1}(L_\varepsilon u_{\varepsilon F_n})]\}$ such that $[u_{\varepsilon F_n}, \bar{u}^*_{\varepsilon F_n}, L_\varepsilon u_{\varepsilon F_n}, J^{-1}(L_\varepsilon u_{\varepsilon F_n})]$ in $\Lambda(F_n)$ and

$$u_{\varepsilon n} := u_{\varepsilon F_n} \to u_\varepsilon \quad \text{weakly in } X, \qquad \bar{u}^*_{\varepsilon n} := \bar{u}^*_{\varepsilon F_n} \to g_\varepsilon \quad \text{weakly in } X^*,$$
$$L_\varepsilon u_{\varepsilon n} \to \ell_\varepsilon \quad \text{weakly in } X^*, \qquad J^{-1}(L_\varepsilon u_{\varepsilon n}) \to \rho_\varepsilon \quad \text{weakly in } X.$$

By (5.9)–(5.10) and (5.12) we have

$$|u_\varepsilon|_X \leqslant M, \quad |g_\varepsilon|_{X^*} \leqslant M_1, \quad \sqrt{\varepsilon}|\ell_\varepsilon|_{X^*} \leqslant \sqrt{M_2}, \quad \sqrt{\varepsilon}|\rho_\varepsilon|_X \leqslant \sqrt{M_2}. \tag{5.13}$$

Now, by (5.8),

$$\langle L_\varepsilon u_{\varepsilon n} + \bar{u}^*_{\varepsilon n} - f, v\rangle = 0, \quad \forall n = 1, 2, \ldots,$$

whence we get by taking the limit as $n \to \infty$ in this relation that $\langle \ell_\varepsilon + g_\varepsilon - f, v\rangle = 0$. Since v is arbitrary in X, it follows that

$$\ell_\varepsilon + g_\varepsilon = f \quad \text{in } X^*. \tag{5.14}$$

Second step. First we show that $g_\varepsilon \in Bu_\varepsilon$. By (5.8) and the monotonicity of L_ε we see that

$$\langle \bar{u}^*_{\varepsilon n}, u_{\varepsilon n} - u_\varepsilon\rangle = \langle f - L_\varepsilon u_{\varepsilon n}, u_{\varepsilon n} - u_\varepsilon\rangle \leqslant \langle f - L_\varepsilon u_\varepsilon, u_{\varepsilon n} - u_\varepsilon\rangle$$

for all n. Therefore, passing to the limit in n yields that

$$\limsup_{n\to\infty} \langle \bar{u}^*_{\varepsilon n}, u_{\varepsilon n} - u_\varepsilon\rangle \leqslant 0, \quad \text{i.e.} \quad \limsup_{n\to\infty} \langle \bar{u}^*_{\varepsilon n}, u_{\varepsilon n}\rangle \leqslant \langle g_\varepsilon, u_\varepsilon\rangle.$$

By the condition of type M, we have $g_\varepsilon \in Bu_\varepsilon$. Next, we note here that $G(L)$ is weakly closed in $X \times X^*$, since $G(L)$ is linear and closed in $X \times X^*$. Therefore it follows from (5.11) and (5.14) that

$$f - g_\varepsilon = \ell_\varepsilon \in L(u_\varepsilon - \varepsilon\rho_\varepsilon). \tag{5.15}$$

Third step. On account of the estimates (5.13), we can find a sequence $\{\varepsilon_n\}$, tending to 0, and $u \in X$, $g \in X^*$ such that

$$u_n := u_{\varepsilon_n} \to u \quad \text{weakly in } X, \qquad g_n := g_{\varepsilon_n} \to g \quad \text{weakly in } X^*$$

and

$$\varepsilon_n \rho_{\varepsilon_n} \to 0 \quad \text{(strongly) in } X.$$

Use these convergences in (5.15). Then we obtain $f - g \in Lu$, since $G(L)$ is weakly closed in $X \times X^*$. Finally we show $g \in Bu$ as follows. The monotonicity of L implies that

$$\big\langle (f - g_n) - (f - g), u_n - \varepsilon_n \rho_{\varepsilon_n} - u \big\rangle \geqslant 0,$$

so that

$$\limsup_{n\to\infty} \langle g_n, u_n - u \rangle \leqslant \limsup_{n\to\infty} \big\{ \langle g_n, \varepsilon_n \rho_{\varepsilon_n} \rangle + \langle g, u_n - \varepsilon_n \rho_{\varepsilon_n} - u \rangle \big\} = 0,$$

that is, $\limsup_{n\to\infty} \langle g_n, u_n \rangle \leqslant \langle g, u \rangle$. From the property of type M it follows that $g \in Bu$. Thus $f \in Lu + Bu$. □

PROOF OF THEOREM 5.2. Let $\varepsilon \in (0, 1]$ and A_ε be the same approximation of A as in Lemma 5.1. Then, since A_ε is a singlevalued bounded and demicontinuous mapping from X into X^*, it follows from Corollary 2.2 that $A_\varepsilon + B$ is pseudo-monotone from X into X^*. Also, it clearly holds that

$$\inf_{v^* \in Bv} \frac{\langle A_\varepsilon v + v^*, v - v_0 \rangle}{|v|_X} \geqslant \inf_{v^* \in Bv} \frac{\langle v^*, v - v_0 \rangle}{|v|_X} + \frac{\langle A_\varepsilon v_0, v - v_0 \rangle}{|v|_X} \to \infty$$
$$\text{as } |v|_X \to \infty.$$

Therefore, by Corollary 5.1 and Remark 5.1, for any $f \in X^*$ and any $\varepsilon \in (0, 1]$ there is $u_\varepsilon \in X$ and $\bar{u}^*_\varepsilon \in X^*$ such that

$$A_\varepsilon u_\varepsilon + \bar{u}^*_\varepsilon = f, \quad \bar{u}^*_\varepsilon \in Bu_\varepsilon.$$

Moreover,

$$\frac{\langle \bar{u}^*_\varepsilon, u_\varepsilon - v_0 \rangle}{|u_\varepsilon|_X} = \frac{\langle f - A_\varepsilon u_\varepsilon, u_\varepsilon - v_0 \rangle}{|u_\varepsilon|_X} \geqslant \frac{\langle f - A_\varepsilon v_0, u_\varepsilon - v_0 \rangle}{|u_\varepsilon|_X}$$

and the last term of the above relations is bounded with respect to $\varepsilon \in (0, 1]$ on account of Lemma 5.1(ii). This implies by (5.7) that $\{u_\varepsilon\}_{\varepsilon \in (0,1]}$ is bounded in X. Accordingly, taking account of the boundedness of B, we have

$$|u_\varepsilon|_X \leqslant M, \quad |\bar{u}^*_\varepsilon|_{X^*} \leqslant M, \quad \forall \varepsilon \in (0, 1], \tag{5.16}$$

for a certain constant $M > 0$, so that

$$|A_\varepsilon u_\varepsilon|_{X^*} = |f - \bar{u}^*_\varepsilon|_{X^*} \leqslant \big(|f|_{X^*} + M\big) =: M_1. \tag{5.17}$$

By the above observation there exist a sequence $\{\varepsilon_n\}$ tending to 0 (as $n \to \infty$) and $u \in X$, $g, h \in X^*$ with $f = g + h$ such that

$$\begin{aligned} &u_n := u_{\varepsilon_n} \to u \quad \text{weakly in } X, \\ &\bar{u}^*_n := \bar{u}^*_{\varepsilon_n} \to g, \quad \text{weakly in } X^*, \qquad u^*_n := A_{\varepsilon_n} u_{\varepsilon_n} \to h \quad \text{weakly in } X^* \end{aligned}$$

and such that the following limits exist:

$$\chi_1 := \lim_{n\to\infty} \langle \bar{u}^*_n, u_n - u\rangle, \qquad \chi_2 := \lim_{n\to\infty} \langle u^*_n, u_n - u\rangle.$$

Here we note that

$$\chi_1 + \chi_2 = \lim_{n\to\infty} \langle u^*_n + \bar{u}^*_n, u_n - u\rangle = \lim_{n\to\infty} \langle f, u_n - u\rangle = 0, \tag{5.18}$$

whence either $\chi_1 \leqslant 0$ or $\chi_2 \leqslant 0$ holds.

First, consider the case when $\chi_1 \leqslant 0$. In this case, by the pseudo-monotonicity of B we see that $g \in Bu$ and

$$\lim_{n\to\infty} \langle \bar{u}^*_n, u_n\rangle = \langle g, u\rangle. \tag{5.19}$$

It follows from (5.19) together with (5.18) that $\chi_1 = 0$ and $\chi_2 = 0$, which shows that

$$\lim_{n\to\infty} \langle u^*_n, u_n\rangle = \langle h, u\rangle. \tag{5.20}$$

Since A is maximal monotone and $u^*_n \in A(u_n - \varepsilon J^{-1} u^*_n)$, we have

$$\big\langle u^*_n - v^*, u_n - \varepsilon_n J^{-1} u^*_n - v\big\rangle \geqslant 0, \quad \forall [v, v^*] \in G(A). \tag{5.21}$$

Passing to the limit $n \to \infty$ in (5.21) and using (5.20), we see that

$$\langle h - v^*, u - v\rangle \geqslant 0, \quad \forall [v, v^*] \in G(A). \tag{5.22}$$

Consequently, $h \in Au$ by the maximal monotonicity of A and $f \in Au + Bu$. Secondly, consider the case of $\chi_2 \leqslant 0$, namely $\lim_{n\to\infty}\langle u^*_n, u_n\rangle \leqslant \langle h, u\rangle$. Then, we have (5.22) by letting $n \to \infty$ in (5.21). Therefore $h \in Au$ and $\chi_2 = 0$. Hence $\chi_1 = 0$ from which we infer again $g \in Bu$ as well as $f \in Au + Bu$. □

PROOF OF COROLLARY 5.2. For any $\varepsilon \in (0, 1]$ the mapping $B + \varepsilon J : X \to X^*$ with the domain X is bounded and pseudo-monotone (cf. Corollary 2.2). Clearly we see from condition (a) that

$$\inf_{v^* \in Bv} \frac{\langle v^* + \varepsilon J v, v - v_0 \rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty.$$

Therefore, for any $f \in X^*$ it follows from Theorem 5.2 that there are $u_\varepsilon \in D(A)$, $u_\varepsilon^* \in Au_\varepsilon$ and $\bar{u}_\varepsilon^* \in Bu_\varepsilon$ such that

$$u_\varepsilon^* + \bar{u}_\varepsilon^* + \varepsilon J u_\varepsilon = f.$$

We observe by condition (b) that $\{u_\varepsilon\}_{0<\varepsilon\leqslant 1}$ is bounded in X, and hence $\{\bar{u}_\varepsilon^*\}$ and $\{u_\varepsilon^*\}$ are bounded in X^*, so that we find a sequence $\{\varepsilon_n\}$ tending to 0 and $g, h \in X^*$ with $f = g + h$ such that $u_{\varepsilon_n} \to u$ weakly in X, $u_{\varepsilon_n}^* \to h$ weakly in X^*, $\bar{u}_{\varepsilon_n}^* \to g$ weakly in X^* and the limits $\chi_1 := \lim_{n\to\infty} \langle u_{\varepsilon_n}^*, u_{\varepsilon_n} - u \rangle$ and $\chi_2 := \lim_{n\to\infty} \langle \bar{u}_{\varepsilon_n}^*, u_{\varepsilon_n} - u \rangle$ exist. In a way similar to that in the proof of Theorem 5.2 we conclude that $h \in Au$, $g = Bu$ and $f \in Au + Bu$. Thus $R(A + B) = X^*$. □

In the rest of this section we give an existence result for a class of variational inequalities as a corollary to Theorem 5.2.

COROLLARY 5.3. *Let φ be a proper, l.s.c. convex function on X and $B : D(B) = X \to X^*$ be a bounded and pseudo-monotone mapping. Suppose that the same condition* (a) *as in Corollary* 5.2 *is satisfied and that for some $v_0 \in D(\partial\varphi)$*

$$\inf_{v^* \in \partial\varphi(v),\, v_1^* \in Bv} \frac{\langle v^* + v_1^*, v - v_0 \rangle}{|v|_X} \to \infty \quad \text{as } |v|_X \to \infty. \tag{5.23}$$

Then, for any $f \in X^*$ there is at least one $u \in D(\varphi)$ with $u^* \in Bu$ such that

$$\langle u^* - f, u - v \rangle \leqslant \varphi(v) - \varphi(u), \quad \forall v \in D(\varphi). \tag{5.24}$$

PROOF. By virtue of Corollary 5.2 with $A = \partial\varphi$ we have $R(\partial\varphi + B) = X^*$. Therefore, given $f \in X^*$ we find $u \in X$, $\bar{u}^* \in \partial\varphi(u)$ and $u^* \in Bu$ such that $\bar{u}^* + u^* = f$, i.e. $f - u^* \in \partial\varphi(u)$. By the definition of subdifferential the last relation implies that (5.24) holds. □

REMARK 5.2. In the Corollary 5.3, if $D(\varphi)$ is bounded in X, then condition (5.23) is automatically satisfied.

For a non-empty closed convex subset K of X we say that the function $I_K(\cdot)$ given by

$$I_K(v) = \begin{cases} 0 & \text{if } v \in K, \\ \infty & \text{if } v \in X,\ v \notin K \end{cases}$$

is called the indicator function of K. Clearly, I_K is proper, l.s.c. and convex on X, and $\bar{u}^* \in \partial I_K(u)$ if and only if $\langle \bar{u}^*, v-u\rangle \leqslant 0$ for all $v \in K$. Therefore, in the case of $\varphi = I_K$ in Corollary 5.3, (5.24) is written in the form

$$u \in K,\ u^* \in Bu; \quad \langle u^* - f, u - v\rangle \leqslant 0, \quad \forall v \in K.$$

6. Examples of subdifferentials

In this section we give some examples of subdifferentials or pseudo-monotone mappings arising in the theory of nonlinear elliptic partial differential equations; most of them are treated in various real function spaces.

Throughout this section, let Ω be a bounded domain in $\mathbf{R}^N$, $1 \leqslant N < \infty$, with smooth boundary Γ. We use the same notations as indicated in Section 1 for various function spaces.

EXAMPLE 6.1 (Duality mappings). Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(1) The duality mapping J_p from $L^p(\Omega)$ into $L^q(\Omega)$ (= the dual space of $L^p(\Omega)$) is given by the following formula:

$$\langle J_p u, v\rangle = \begin{cases} |u|_p^{2-p} \int_\Omega |u|^{p-2} u v \,\mathrm{d}x & \text{if } u \in L^p(\Omega),\ u \neq 0,\ v \in L^p(\Omega), \\ 0 & \text{if } u = 0,\ v \in L^p(\Omega). \end{cases}$$

Clearly, J_p is continuous from $L^p(\Omega)$ into $L^q(\Omega)$; in particular, if $p = 2$, then J_2 is the identity in $L^2(\Omega)$.

(2) Let $X = W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$. Then, the duality mapping $J_{1,p}$ from X into X^* is given by

$$\langle J_{1,p} u, v\rangle = \begin{cases} |u|_{1,p}^{2-p} \{\sum_{i=1}^N \int_\Omega |u_{x_i}|^{p-2} u_{x_i} v_{x_i} \,\mathrm{d}x + \int_\Omega |u|^{p-2} u v \,\mathrm{d}x\} \\ \quad \text{if } u \in X,\ u \neq 0,\ v \in X, \\ 0 \quad \text{if } u = 0,\ v \in X, \end{cases}$$

The norm $|v|_{1,p}$ is written in the form

$$|v|_{1,p} = \left\{ [v]_{1,p}^p + |v|_p^p \right\}^{1/p},$$

where

$$[v]_{1,p} := \left\{ \sum_{i=1}^N |v_{x_i}|_p^p \right\}^{1/p}. \tag{6.1}$$

By the well-known Poincaré inequality

$$|v|_p \leqslant C_{p,\Omega} [v]_{1,p}, \quad \forall v \in W_0^{1,p}(\Omega),$$

where $C_{p,\Omega}$ is a positive constant depending only on p and the domain Ω, we see that $[\cdot]_{1,p}$ gives a norm on $W_0^{1,p}(\Omega)$ which is equivalent to $|\cdot|_{1,p}$.

When $W_0^{1,p}(\Omega)$ is equipped with the norm $[\cdot]_{1,p}$ given by (6.1), the duality mapping $\tilde{J}_{1,p}$ from $W_0^{1,p}(\Omega)$ into $W^{-1,q}(\Omega)$, equipped with the dual norm to $[\cdot]_{1,p}$, is given by the following formula:

$$\langle \tilde{J}_{1,p}u, v\rangle = \begin{cases} [u]_{1,p}^{2-p}\{\sum_{i=1}^N \int_\Omega |u_{x_i}|^{p-2}u_{x_i}v_{x_i}\,dx\} \\ \quad \text{if } u \in W_0^{1,p}(\Omega),\ u \neq 0,\ v \in W_0^{1,p}(\Omega), \\ 0 \quad \text{if } u = 0,\ v \in W_0^{1,p}(\Omega). \end{cases}$$

EXAMPLE 6.2 (p-Laplacian).

(1) Let $1 < p < +\infty$ and take as X the Sobolev space $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$. We consider a mapping $A_p : X \to X^*$ given by

$$\langle A_p u, v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,dx, \quad \forall u, v \in X. \tag{6.2}$$

It is evident from (6.2) that $D(A_p) = X$ and A_p is singlevalued, bounded, continuous and monotone. Hence, by Theorem 4.2 the mapping A_p is maximal monotone as a mapping from X into X^*. It is easy to see that

$$A_p u = -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) \quad \text{in the distribution sense on } \Omega.$$

The mapping $-A_p$ is called the "p-Laplacian".

Next, consider a continuous convex function $\varphi(\cdot)$ on X defined by

$$\varphi(v) = \frac{1}{p}\int_\Omega |\nabla v|^p\,dx, \quad \forall v \in X. \tag{6.3}$$

Then we have:

PROPOSITION 6.1. *$A_p : D(A_p) = X \to X^*$ is the subdifferential of φ given by* (6.3).

PROOF. We have for any $u, v \in X$

$$\begin{aligned} \langle A_p u, v - u\rangle &= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla(v-u)\,dx \\ &\leqslant \int_\Omega |\nabla u|^{p-1}|\nabla v|\,dx - \int_\Omega |\nabla u|^p\,dx \\ &\leqslant \frac{1}{p}\int_\Omega |\nabla v|^p\,dx - \frac{1}{p}\int_\Omega |\nabla u|^p\,dx = \varphi(v) - \varphi(u). \end{aligned}$$

Hence $A_p u \subset \partial\varphi(u)$ for all $u \in X$, which implies that $A_p = \partial\varphi$, since A_p is maximal monotone as a mapping from X into X^*. □

(2) Now, assume $2 \leqslant p < +\infty$. Then $W_0^{1,p}(\Omega)$ is compactly embedded in $L^2(\Omega)$, and consider the function $\varphi_D : W_0^{1,p}(\Omega) \to [0, \infty]$ given by

$$\varphi_D(v) = \begin{cases} \frac{1}{p}\int_\Omega |\nabla v|^p \, \mathrm{d}x & \text{if } v \in W_0^{1,p}(\Omega), \\ \infty & \text{if } v \in L^2(\Omega),\ v \notin W_0^{1,p}(\Omega). \end{cases} \tag{6.4}$$

This function φ_D is proper, l.s.c and convex on $L^2(\Omega)$. Therefore its subdifferential is defined as a mapping in $L^2(\Omega)$, namely from $L^2(\Omega)$ into itself. We denote it by $\partial\varphi_D$.

PROPOSITION 6.2. *The subdifferential $\partial\varphi_D$ is singlevalued and is characterized by*

$$D(\partial\varphi_D) = \left\{u \in W_0^{1,p}(\Omega);\ -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \in L^2(\Omega)\right\} \tag{6.5}$$

and

$$\partial\varphi_D(u) = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad \forall u \in D(\partial\varphi_D). \tag{6.6}$$

PROOF. Let $[u, u^*]$ be any element of $G(\partial\varphi_D)$. By definition we have

$$\int_\Omega u^*(w-u)\,\mathrm{d}x \leqslant \varphi_D(w) - \varphi_D(u), \quad \forall w \in W_0^{1,p}(\Omega). \tag{6.7}$$

Now, taking $w = u \pm tv$ with $t \in (0, 1]$ and $v \in W_0^{1,p}(\Omega)$ in (6.7) and dividing the both sides by t, we get

$$\pm\int_\Omega u^* v\,\mathrm{d}x \leqslant \frac{1}{t}\left\{\frac{1}{p}\int_\Omega \left|\nabla(u \pm tv)\right|^p \mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x\right\}, \quad \forall v \in W_0^{1,p}(\Omega). \tag{6.8}$$

Pass to the limit as $t \downarrow 0$ in (6.8). Then,

$$\int_\Omega u^* v\,\mathrm{d}x = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\,\mathrm{d}x, \quad \forall v \in W_0^{1,p}(\Omega). \tag{6.9}$$

It is easy to obtain (6.5) and (6.6) from (6.9). □

(3) We now consider the function φ_N given by replacing $W_0^{1,p}(\Omega)$ by $W^{1,p}(\Omega)$ in (6.4), namely

$$\varphi_N(v) = \begin{cases} \frac{1}{p}\int_\Omega |\nabla v|^p \, \mathrm{d}x & \text{if } v \in W^{1,p}(\Omega), \\ \infty & \text{if } v \in L^2(\Omega),\ v \notin W^{1,p}(\Omega). \end{cases} \tag{6.10}$$

In order to describe the subdifferential of φ_N defined by (6.10) we use the following generalized Green's formula (cf. Section 1):

$$\int_\Omega (\operatorname{div} \mathbf{v}) v \, dx + \int_\Omega \mathbf{v} \cdot \nabla v \, dx = \langle \mathbf{B}\mathbf{v}, \hat{v} \rangle_\Gamma \tag{6.11}$$

for all $v \in W^{1,p}(\Omega)$, $\hat{v}$ being the trace of v on Γ, and $\mathbf{v} = (v_1, v_2, \dots, v_N) \in \mathcal{E}^q(\Omega)$, where

$$\frac{1}{p} + \frac{1}{q} = 1, \quad \mathcal{E}^q(\Omega) := \big\{ \mathbf{v} \in \big(L^q(\Omega)\big)^N ; \ \operatorname{div} \mathbf{v} \in L^q(\Omega) \big\},$$

$\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing between $W^{-1/q,q}(\Gamma)$ and $W^{1/q,p}(\Gamma)$, and $\mathbf{B}$ is a continuous linear mapping from $\mathcal{E}^q(\Omega)$ into $W^{-1/q,q}(\Gamma)$ such that

$$\mathbf{B}\mathbf{v} = \mathbf{v} \cdot \mathbf{n} \quad \text{on } \Gamma \text{ for all smooth } \mathbf{v} \text{ on } \overline{\Omega},$$

where $\mathbf{n} = \mathbf{n}(x)$ is the outward unit normal vector to Γ at $x \in \Gamma$.

PROPOSITION 6.3. *The subdifferential of φ_N in $L^2(\Omega)$ is singlevalued and is characterized by*

$$\begin{aligned} D(\partial \varphi_N) = \big\{ & v \in W^{1,p}(\Omega); \ \operatorname{div} |\nabla u|^{p-2} \nabla u) \in L^2(\Omega), \\ & \mathbf{B}\big(|\nabla u|^{p-2} \nabla u\big) = 0 \text{ in } W^{-1/q,q}(\Gamma) \big\} \end{aligned} \tag{6.12}$$

and

$$\partial \varphi_N(u) = -\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big), \quad \forall u \in D(\partial \varphi_N). \tag{6.13}$$

PROOF. Let $[u, u^*]$ be any element of $G(\partial \varphi_N)$. Just as (6.9) in the proof of Proposition 6.2, we have

$$\int_\Omega u^* v \, dx = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx, \quad \forall v \in W^{1,p}(\Omega). \tag{6.14}$$

Here, take any $v \in W_0^{1,p}(\Omega)$ in (6.14) to see that

$$u^* = -\operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) \in L^2(\Omega) \subset L^q(\Omega), \tag{6.15}$$

so that $|\nabla u|^{p-2} \nabla u \in \mathcal{E}^q(\Omega)$. Therefore, by (6.11),

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_\Omega \operatorname{div}\big(|\nabla u|^{p-2} \nabla u\big) v \, dx = \big\langle \mathbf{B}\big(|\nabla u|^{p-2} \nabla u\big), \hat{v} \big\rangle_\Gamma,$$

whence it follows from (6.14) that

$$\int_\Omega u^* v \, dx = -\int_\Omega \operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr) v \, dx + \bigl\langle \mathbf{B}\bigl(|\nabla u|^{p-2}\nabla u\bigr), \hat{v}\bigr\rangle_\Gamma,$$
$$\forall v \in W^{1,p}(\Omega).$$

This implies by (6.15) that

$$\bigl\langle \mathbf{B}\bigl(|\nabla u|^{p-2}\nabla u\bigr), \hat{v}\bigr\rangle_\Gamma = 0, \quad \forall v \in W^{1,p}(\Omega)$$

and hence $\mathbf{B}(|\nabla u|^{p-2}\nabla u) = 0$ in $W^{-1/q,q}(\Gamma)$. Thus we have (6.12) and (6.13). □

EXAMPLE 6.3 (The mapping $-\Delta\beta$). We choose as X the Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$. This is a Hilbert space with the norm $|v|_X$ given by

$$|v|_X := \left\{ \int_\Omega |\nabla v|^2 \, dx + n_0 \int_\Gamma |v|^2 \, d\Gamma \right\}^{1/2}, \quad \forall v \in H^1(\Omega),$$

where $d\Gamma$ is the surface element on the boundary Γ and n_0 is a positive constant; as is well known, this norm $|\cdot|_X$ is equivalent to the usual one $|\cdot|_{1,2}$ in X (cf. [16]). We denote by J_0 the duality mapping from X into the dual space X^* associated with the above norm; it is easy to see that J_0 is linear and

$$\langle J_0 u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx + n_0 \int_\Gamma u v \, d\Gamma, \quad \forall u, v \in X.$$

With the duality mapping J_0, an inner product $(\cdot,\cdot)_*$ in X^* is defined as follows:

$$(u^*, v^*)_* = \bigl\langle u^*, J_0^{-1} v^*\bigr\rangle \quad \bigl(= \bigl\langle v^*, J_0^{-1} u^*\bigr\rangle\bigr), \quad \forall u^*, v^* \in X^*.$$

With this inner product in X^*, we see that X^* is a real Hilbert space and the relations

$$X \subset L^2(\Omega) \subset X^* \quad \text{with densely defined and compact injections} \tag{6.16}$$

hold. In particular, "$L^2(\Omega) \subset X^*$" implies that if $u^* \in L^2(\Omega)$, then the duality $\langle u^*, v\rangle$ between $u^* \in X^*$ and $v \in X$ is nothing but the inner product $\int_\Omega u^* v \, dx$ of u^* and v in $L^2(\Omega)$; namely

$$\langle u^*, v\rangle = \int_\Omega u^* v \, dx.$$

Let β be a maximal monotone mapping in $\mathbf{R}$ and $\hat{\beta}$ be a proper l.s.c. and convex function on $\mathbf{R}$ whose subdifferential $\partial\hat{\beta}$ coincides with β in $\mathbf{R}$. Assume that

$$\hat{\beta}(r) \geqslant c_0 |r|^2 - c_1, \quad \forall r \in \mathbf{R}. \tag{6.17}$$

We see the following facts (C1)–(C4) from the fundamental properties of proper, l.s.c. and convex functions on the real line $\mathbf{R}$; see the monograph [17] for the detailed proofs.

(C1) $D(\hat{\beta})$ is an interval and $\hat{\beta}$ is absolutely continuous on any compact subinterval of the interior of $D(\hat{\beta})$, denoted by $Int\, D(\hat{\beta})$, $\overline{D(\hat{\beta})} = \overline{D(\beta)}$ and $\hat{\beta}$ has the right-derivative $\hat{\beta}'_+(r)$ and the left-derivative $\hat{\beta}'_-(r)$ at every point $r \in Int\, D(\hat{\beta})$.

(C2) If the left end-point r_0 (resp. right end-point r^0) of $\overline{D(\beta)}$ belongs to $D(\beta)$, then the right-derivative $\hat{\beta}'_+(r_0)$ (resp. left-derivative $\hat{\beta}'_-(r^0)$) exists.

(C3) The right-derivative and left derivative of $\hat{\beta}$ are non-decreasing on $D(\beta)$, satisfying

$$\hat{\beta}'_+(r_1) \leqslant \hat{\beta}'_-(r_2) \leqslant \hat{\beta}'_+(r_2) \leqslant \hat{\beta}'_-(r_3), \quad \text{if } r_1, r_2, r_3 \in D(\beta),\ r_1 < r_2 < r_3.$$

(C4) $r' \in \beta(r)$ if and only if

$$\begin{cases} r' \in [\hat{\beta}'_-(r), \hat{\beta}'_+(r)] & \text{if } r \in Int\, D(\beta), \\ r' \in (-\infty, \hat{\beta}'_+(r)] & \text{if } r = \text{the left endpoint of } D(\beta), \\ r' \in [\hat{\beta}'_-(r), +\infty) & \text{if } r = \text{the right endpoint of } D(\beta). \end{cases} \tag{6.18}$$

Now, consider the function φ on X^* given by the following formula:

$$\varphi(v) = \begin{cases} \int_\Omega \hat{\beta}(v)\,\mathrm{d}x & \text{if } v \in L^2(\Omega),\ \hat{\beta}(v) \in L^1(\Omega), \\ \infty & \text{otherwise.} \end{cases} \tag{6.19}$$

For simplicity, we denote by the same notation φ as (6.19) for the restriction onto $L^2(\Omega)$ and X. Then, taking account of (6.16), we see from assumption (6.17) that φ is proper, lower semicontinuous and convex on any spaces X, $L^2(\Omega)$ and X^*. Therefore, three subdifferentials of φ are defined as mappings from X^* into $X(= X^{**})$, in $L^2(\Omega)$ and in X^*. We denote them respectively by $\partial_1\varphi$, $\partial_2\varphi$ and $\partial_*\varphi$. We recall below their definitions:

(i) $g \in \partial_1\varphi(u)$ if and only if $\hat{\beta}(u) \in L^1(\Omega)$ (hence $u \in L^2(\Omega)$), $g \in X$ and

$$\langle v - u, g\rangle \leqslant \varphi(v) - \varphi(u), \quad \forall v \in D(\varphi),$$

where in order to show the duality between X^* and $X = X^{**}$ we used the same notation $\langle\cdot,\cdot\rangle$ as that between X and X^*.

(ii) $g \in \partial_2\varphi(u)$ if and only if $\hat{\beta}(u) \in L^1(\Omega)$ (hence $u \in L^2(\Omega)$), $g \in L^2(\Omega)$ and

$$\int_\Omega g(v - u)\,\mathrm{d}x \leqslant \varphi(v) - \varphi(u), \quad \forall v \in D(\varphi),$$

(iii) $g^* \in \partial_*\varphi(u)$ if and only if $\hat{\beta}(u) \in L^1(\Omega)$ (hence $u \in L^2(\Omega)$), $g^* \in X^*$ and

$$(g^*, v - u)_* \leqslant \varphi(v) - \varphi(u), \quad \forall v \in D(\varphi).$$

We are going to give characterizations of these subdifferentials $\partial_1\varphi$, $\partial_2\varphi$, $\partial_*\varphi$.

PROPOSITION 6.4.

(a) *$g \in \partial_2\varphi(u)$ if and only if $u, g \in L^2(\Omega)$ and $g \in \beta(u)$ a.e. on Ω.*

(b) *$g \in \partial_1\varphi(u)$ if and only if $u \in L^2(\Omega)$, $g \in X$ and $g \in \beta(u)$ a.e. on Ω.*

(c) *$g^* \in \partial_*\varphi(u)$ if and only if $u \in L^2(\Omega)$ and there exists $g \in X$ such that $g \in \beta(u)$ a.e. on Ω and $g^* = J_0 g$. Hence $\partial_*\varphi = J_0(\partial_1\varphi)$.*

PROOF. When the effective domain of $\hat{\beta}$ is a single point, the assertions of the proposition are trivial, we consider the case the interior of $D(\hat{\beta})$ is an open interval (r_0, r^0), $-\infty \leqslant r_0 < r^0 \leqslant +\infty$, in the rest of our proof.

First assume that $g \in \partial_2\varphi(u)$. Then, by definition,

$$\int_\Omega g(v-u)\,\mathrm{d}x \leqslant \int_\Omega \{\hat{\beta}(v) - \hat{\beta}(u)\}\,\mathrm{d}x,$$
$$\forall v \in L^2(\Omega) \text{ with } \hat{\beta}(v) \in L^1(\Omega). \tag{6.20}$$

Next, take as v in (6.20) any convex combination $u + t(w-u)$ with $w \in L^2(\Omega)$, $\hat{\beta}(w) \in L^1(\Omega)$ and $t \in (0,1)$ and divide its resultant by t to get

$$\int_\Omega g(w-u)\,\mathrm{d}x \leqslant \frac{1}{t}\int_\Omega \{\hat{\beta}(u + t(w-u)) - \hat{\beta}(u)\}\,\mathrm{d}x. \tag{6.21}$$

Choose any (sufficiently small) positive number δ and take a function w_δ of the following form as the function w in (6.21):

$$w_\delta(x) = \begin{cases} u(x) \pm \delta & \text{if } r_0 + \delta \leqslant u(x) \leqslant r^0 - \delta, \\ u(x) & \text{otherwise.} \end{cases}$$

Then, letting $t \downarrow 0$ yields that

$$g(x) \in [\hat{\beta}'_-(u(x)), \hat{\beta}'_+(u(x))] \quad \text{for a.e. } x \text{ with } r_0 + \delta \leqslant u(x) \leqslant r^0 - \delta,$$

so that

$$g(x) \in [\hat{\beta}'_-(u(x)), \hat{\beta}'_+(u(x))] \quad \text{for a.e. } x \text{ with } r_0 < u_(x) < r^0.$$

Similarly, if we take as w in (6.21)

$$w_{0\delta}(x) = \begin{cases} u(x) + \delta & \text{if } u(x) = r_0 > -\infty, \\ u(x) & \text{otherwise,} \end{cases}$$

and

$$w^0_\delta(x) = \begin{cases} u(x) - \delta & \text{if } u(x) = r^0 < +\infty, \\ u(x) & \text{otherwise,} \end{cases}$$

then we obtain from (6.18) that $g(x) \in (-\infty, \hat{\beta}'_+(u(x))]$ for a.e. x with $u(x) = r_0 > -\infty$ as well as $g(x) \in [\hat{\beta}'_-(u(x)), +\infty)$ for a.e. x with $u(x) = r^0 < +\infty$. Consequently $g(x) \in \beta(u(x))$ for a.e. $x \in \Omega$. Conversely, assume that $u, g \in L^2(\Omega)$ and $g(x) \in \beta(u(x))$ for a.e. $x \in \Omega$. Then $g(x)(v(x) - u(x)) \leqslant \hat{\beta}(v(x)) - \hat{\beta}(u(x))$ a.e. for $x \in \Omega$. By integrating this in x over Ω, we have (6.20) and hence $g \in \partial_2\varphi(u)$. Thus (a) holds and simultaneously (b) holds.

Finally we show (c). Assume that $g^* \in \partial_*\varphi(u)$ and put $g = J_0^{-1}g^* \in X$. Then, by definition, $\hat{\beta}(u) \in L^1(\Omega)$ (hence $u \in L^2(\Omega)$ by (6.17)) and

$$(v - u, g^*)_* = \langle v - u, g\rangle = \int_\Omega g(v - u)\,\mathrm{d}x \leqslant \int_\Omega \{\hat{\beta}(v) - \hat{\beta}(u)\}\,\mathrm{d}x,$$
$$\forall v \in L^2(\Omega) \text{ with } \hat{\beta}(v) \in L^1(\Omega).$$

Therefore, $g \in \partial_2\varphi(u)$ as well as $g \in \partial_1\varphi(u)$, and it follows from (a) that $g \in \beta(u)$ a.e. on Ω, and $g^* = J_0 g$. The converse is also easily obtained. □

Let g^* be any element in X^* of the form

$$\langle g^*, v\rangle = \int_\Omega f v\,\mathrm{d}x + \int_\Gamma h v\,\mathrm{d}\Gamma, \quad \forall v \in X$$

for $f \in L^2(\Omega)$ and $h \in L^2(\Gamma)$. Since X^* is a Hilbert space with inner product $(\cdot,\cdot)_*$, the duality mapping from X^* into X^* is the identity I and $R(\partial_*\varphi + I) = X^*$ by Theorem 4.1. Therefore there exists $u \in X^*$ such that $g^* \in \partial_*\varphi(u) + u$. In this case, $u \in L^2(\Omega)$ is a solution of the following boundary-value problem in a weak sense:

$$-\Delta\tilde{u} + u = f, \quad \tilde{u} \in \beta(u), \text{ in } \Omega, \tag{6.22}$$

with boundary condition

$$\frac{\partial\tilde{u}}{\partial n} + n_0\tilde{u} = h \quad \text{on } \Gamma. \tag{6.23}$$

In fact, according to Proposition 6.4, (c), there is $\tilde{u} \in X$ such that $g^* = J_0\tilde{u} + u$ and $\tilde{u} \in \beta(u)$ a.e. on Ω, so that $\langle J_0\tilde{u}, v\rangle = \langle g^* - u, v\rangle$ for all $v \in X$, namely

$$\int_\Omega \nabla\tilde{u}\cdot\nabla v\,\mathrm{d}x + n_0\int_\Gamma \tilde{u}v\,\mathrm{d}\Gamma = \int_\Omega (f - u)v\,\mathrm{d}x + \int_\Gamma h\hat{v}\,\mathrm{d}\Gamma,$$
$$\forall v \in X, \tag{6.24}$$

where $\hat{v}$ is the trace of v on Γ. By taking $v \in \mathcal{D}(\Omega)$ in (6.24), we derive that $-\Delta\tilde{u} + u = f$ in the distribution sense on Ω. This shows that $\operatorname{div}\nabla\tilde{u} = \Delta\tilde{u} = u - f \in L^2(\Omega)$, namely (6.22) holds, and by the generalized Green's formula

$$\int_\Omega \nabla\tilde{u}\cdot\nabla v\,\mathrm{d}x + \int_\Omega \Delta\tilde{u}v\,\mathrm{d}x = \left\langle\frac{\partial\tilde{u}}{\partial n}, \hat{v}\right\rangle_\Gamma, \tag{6.25}$$

where $\partial\tilde{u}/\partial n$ is defined in $W^{-1/2,2}(\Gamma)$ as the mapping $\mathbf{B}(\nabla\tilde{u})$ in the case of $p=2$. Substituting (6.25) in (6.24) and using (6.22), we get

$$\left\langle \frac{\partial\tilde{u}}{\partial n}, \hat{v} \right\rangle_{\Gamma} + n_0 \int_{\Gamma} \tilde{u}\hat{v}\,\mathrm{d}\Gamma = \int_{\Gamma} h\hat{v}\,\mathrm{d}\Gamma, \quad \forall \hat{v} \in W^{1/2,2}(\Gamma),$$

so that $\partial\tilde{u}/\partial n = h - n_0\tilde{u} \in L^2(\Gamma)$. Thus (6.23) is obtained.

EXAMPLE 6.4 (Convex functions on the boundary Γ). Let us consider $X := W^{1,p}(\Omega)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and the trace space $\widehat{X} := W^{1/q,p}(\Gamma)$. We denote the trace on Γ of any function $v \in X$ by $\hat{v}$. Let φ be a proper l.s.c. and convex function on X such that

$$\varphi(v) \geqslant C_0|v|_{1,p}^p - C_1, \quad \forall v \in X, \tag{6.26}$$

where C_0 and C_1 are positive constants. Now, define a function $\hat{\varphi}$ on $\widehat{X}$, associated with φ, by the following formula:

$$\hat{\varphi}(h) = \begin{cases} \inf_{v\in X,\, \hat{v}=h} \varphi(v) & \text{if } h \in \widehat{X}, \\ \infty & \text{otherwise.} \end{cases} \tag{6.27}$$

PROPOSITION 6.5. *Given a proper l.s.c., convex function φ on X satisfying* (6.26), *let $\hat{\varphi}$ be a function on $\widehat{X}$ given by* (6.27). *Then we have*:

(a) *$\hat{\varphi}$ is proper, l.s.c. and convex on $\widehat{X}$ and satisfies that*

$$\hat{\varphi}(h) \geqslant C_2|h|_{1/q,p}^p - C_3, \quad \forall h \in \widehat{X}. \tag{6.28}$$

(b) $R(\partial\hat{\varphi}) = \widehat{X}$.

PROOF. Since the trace mapping $v \to \hat{v}$, which assigns to each $v \in X$ its trace $\hat{v}$, is linear and continuous from X onto $\widehat{X}$, there is a positive constant C_4 such that $|\hat{v}|_{1/q,p} \leqslant C_4|v|_{1,p}$ for all $v \in X$. By the definition (6.27) of $\hat{\varphi}$ and (6.26) we see that $\hat{\varphi}$ is not identically ∞ on $\widehat{X}$, $D(\hat{\varphi}) \subset \widehat{X}$ and

$$\hat{\varphi}(h) = \inf_{v\in X,\, \hat{v}=h} \varphi(v) \geqslant C_0 \inf_{v\in X,\, \hat{v}=h} |v|_{1,p}^p - C_1 \geqslant \frac{C_0}{C_4}|h|_{1/q,p}^p - C_1$$

for any $h \in \widehat{X}$. Hence $\hat{\varphi}$ satisfies (6.28) with $C_2 = C_0/C_4$ and $C_3 = C_1$.

Next we show the convexity of $\hat{\varphi}$. Let $h_1, h_2 \in X$ such that $\hat{\varphi}(h_1) < \infty$, $\hat{\varphi}(h_2) < \infty$ and $0 \leqslant r \leqslant 1$. Then, for any $\varepsilon > 0$ there are functions $u_1, u_2 \in D(\varphi)$ such that

$$h_i = \hat{u}_i, \quad \hat{\varphi}(h_i) \geqslant \varphi(u_i) - \varepsilon, \quad i = 1, 2.$$

Therefore

$$r\hat{\varphi}(h_1) + (1-r)\hat{\varphi}(h_1) \geqslant r\varphi(u_1) + (1-r)\varphi(u_2) - \varepsilon$$

$$\geqslant \varphi\big(ru_1 + (1-r)u_2\big) - \varepsilon$$
$$\geqslant \hat{\varphi}\big(rh_1 + (1-r)h_2\big) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $\hat{\varphi}(rh_1 + (1-r)h_2) \leqslant r\hat{\varphi}(h_1) + (1-r)\hat{\varphi}(h_1)$. Thus $\hat{\varphi}$ is convex on $\widehat{X}$. Next, we show the lower semicontinuity of $\hat{\varphi}$ on $\widehat{X}$. Let $\{h_n\} \subset D(\hat{\varphi})$ be a sequence such that

$$h_n \to h \quad \text{weakly in } \widehat{X}, \qquad \liminf_{n\to\infty} \hat{\varphi}(h_n) < \infty.$$

Then, given $\varepsilon > 0$, we can find a subsequence $\{h_{n_k}\}$ of $\{h_n\}$, a sequence $\{v_k\} \subset D(\varphi)$ and $v \in X$ such that

$$\hat{v}_k = h_{n_k}, \quad \hat{\varphi}(h_{n_k}) \geqslant \varphi(v_k) - \varepsilon, \quad v_k \to v \quad \text{weakly in } X,$$
$$\liminf_{n\to\infty} \hat{\varphi}(h_n) = \lim_{k\to\infty} \hat{\varphi}(h_{n_k}).$$

Since $\hat{v} = h$ and $\liminf_{k\to\infty} \varphi(v_k) \geqslant \varphi(v)$ by the lower semicontinuity of φ on X, we get

$$\liminf_{n\to\infty} \hat{\varphi}(h_n) \geqslant \varphi(v) - \varepsilon \geqslant \hat{\varphi}(h) - \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, this shows that $\hat{\varphi}$ is lower semicontinuous on $\widehat{X}$. Thus we have (a).

The assertion (b) follows immediately from Corollary 4.2. □

For instance, consider the case where

$$\varphi(v) = \frac{1}{p}\sum_{i=1}^{N} \int_\Omega \hat{a}_i(v_{x_i})\,\mathrm{d}x + \int_\Omega a_0(v)\,\mathrm{d}x, \quad \forall v \in X,$$

where $\hat{a}_i(\cdot)$, $i = 0, 1, 2, \ldots, N$, are convex functions on $\mathbf{R}$ such that

$$C_5|r|^p \leqslant \hat{a}_i(r) \leqslant C_6|r|^p, \quad \forall r \in \mathbf{R},\ i = 0, 1, 2, \ldots, N, \tag{6.29}$$

where C_5 and C_6 are positive constants. Clearly φ is proper, l.s.c. and convex on X. Associated with φ, a proper, l.s.c. and convex function $\hat{\varphi}$ on $\widehat{X}$ is defined by (6.27) and $D(\partial\hat{\varphi}) = D(\hat{\varphi}) = \widehat{X}$. In this case, $h^* \in \partial\hat{\varphi}(h)$ if and only if there exists $u \in X$ such that

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(u_{x_i}) + a_0(u) = 0 \quad \text{in the distribution sense in } \Omega, \tag{6.30}$$

$$\hat{u} \text{ (the trace of } u) = h \quad \text{a.e. on } \Gamma \tag{6.31}$$

and

$$\mathbf{Ba}(u) = h^* \quad \text{in } \widehat{X}^*, \tag{6.32}$$

where $a_i(r) := \mathrm{d}\hat{a}_i(r)/\mathrm{d}r$, $i = 1, 2, \ldots$, $\mathbf{a}(u) := (a_1(u_{x_1}), a_2(u_{x_2}), \ldots, a_N(u_{x_N})) \in \mathcal{E}^q(\Omega)$ and $\mathbf{B}$ is the linear continuous mapping from $\mathcal{E}^q(\Omega)$ into $\widehat{X}^*$ whose definition is given in (4) of Section 1. In fact, assume that $h^* \in \partial\hat{\varphi}(h)$. On account of (6.29), the minimization problem

$$\min_{v \in X,\, \hat{v}=h} \varphi(v)$$

has a solution $u \in X$ which is characterized by (6.30) and (6.31). By the definition (6.27) we note that $\hat{\varphi}(h) = \varphi(u)$. In this case, we see that $\operatorname{div}\mathbf{a}(u) = a_0(u) \in L^q(\Omega)$. Hence $\mathbf{a}(u) \in \mathcal{E}^q(\Omega)$, and by the generalized Green's formula

$$\int_\Omega \big(\operatorname{div}\mathbf{a}(u)\big) w \,\mathrm{d}x + \int_\Omega \mathbf{a}(u) \cdot \nabla w \,\mathrm{d}x = \big\langle \mathbf{B}\mathbf{a}(u), \widehat{w} \big\rangle_\Gamma, \quad \forall w \in X. \tag{6.33}$$

Now, for any $h_1 \in \widehat{X}$, take a function $u_1 \in X$ with $\hat{u}_1 = h_1$. Then

$$\langle h^*, h_1 - h \rangle_\Gamma \leqslant \hat{\varphi}(h_1) - \hat{\varphi}(h) \leqslant \varphi(u_1) - \varphi(u). \tag{6.34}$$

Substituting as h_1 any convex combination $h + r(h_1 - h)$, $r \in (0, 1)$, in (6.34), we get

$$\begin{aligned} \langle h^*, h_1 - h \rangle_\Gamma \leqslant \sum_{i=1}^N \int_\Omega \frac{1}{r} \big\{ & \hat{a}_i\big(u_{x_i} + r(u_{1x_i} - u_{x_i})\big) - \hat{a}_i(u_{x_i}) \\ & + \hat{a}_0\big(u + r(u_1 - u)\big) - \hat{a}_0(u) \big\} \,\mathrm{d}x. \end{aligned}$$

Letting $r \downarrow 0$ in this inequality and using the generalized Green's formula (6.33) with (6.30) and (6.31), we obtain that

$$\begin{aligned} \langle h^*, h_1 - h \rangle_\Gamma &\leqslant \sum_{i=1}^N \int_\Omega \big\{ a_i(u_{x_i})(u_{1x_i} - u_{x_i}) + a_0(u)(u_1 - u) \big\} \,\mathrm{d}x \\ &= \big\langle \mathbf{B}\mathbf{a}(u), h_1 - h \big\rangle_\Gamma, \end{aligned}$$

which holds for any $h_1 \in \widehat{X}$. Therefore $\langle h^*, h_1 - h \rangle_\Gamma = \langle \mathbf{B}\mathbf{a}(u), h_1 - h \rangle_\Gamma$ for all $h_1 \in \widehat{X}$, namely $h^* = \mathbf{B}\mathbf{a}(u)$. Thus (6.32) holds. It is similar to see $h^* \in \partial\hat{\varphi}(h)$ from (6.30), (6.31) and (6.32).

As was observe above, $\partial\hat{\varphi}(\cdot)$ is a generalized version of the co-normal derivative $\mathbf{B}\mathbf{a}(u)$, formally $\mathbf{a}(u) \cdot \mathbf{n}$ on Γ.

EXAMPLE 6.5 (The mapping $-\Delta + \beta$). Let $\hat{\beta}$ be a proper l.s.c. convex function on $\mathbf{R}$ satisfying the same condition (6.17) as in Example 6.3; denote by β the subdifferential of $\hat{\beta}$ in the one-dimensional space $\mathbf{R}$ into itself.

We consider the function ψ on $L^2(\Omega)$ given by

$$\psi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \int_\Omega \hat{\beta}(v) \,\mathrm{d}x & \text{for } v \in H^1(\Omega),\ \hat{\beta}(v) \in L^1(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{6.35}$$

Clearly ψ is proper l.s.c. and convex on $L^2(\Omega)$, and hence the subdifferential $\partial\psi$ is defined as a mapping from $L^2(\Omega)$ into itself.

Now we decompose ψ as the sum of the following two functions ψ_0 and ψ_1:

$$\psi_0(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v|^2 \, dx & \text{if } v \in H^1(\Omega), \\ \infty & \text{otherwise} \end{cases}$$

and

$$\psi_1(v) = \begin{cases} \int_\Omega \hat{\beta}(v) \, dx & \text{if } v \in L^2(\Omega),\ \hat{\beta}(v) \in L^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Both of ψ_0 and ψ_1 are proper, l.s.c. and convex on $L^2(\Omega)$, and their subdifferentials in $L^2(\Omega)$ are denoted by $\partial\psi_0$ and $\partial\psi_1$.

By Proposition 6.4(a), we see that

$$\partial\psi_1(v) = \big\{g \in L^2(\Omega);\ g \in \beta(v) \text{ a.e. on } \Omega\big\}.$$

Also, by Proposition 6.2, $\partial\psi_0$ is singlevalued in $L^2(\Omega)$ and $\partial\psi_0(v) = -\Delta v \in L^2(\Omega)$ with $\partial v/\partial n$ $(= \mathbf{B}(\nabla v)) = 0$ in $H^{-1/2}(\Gamma)$, which implies by the regularity result (cf. [14]) for linear elliptic partial differential equations that $v \in H^2(\Omega)$ and $\partial v/\partial n \in H^{1/2}(\Gamma)$. Accordingly we have

$$\partial\psi_0(v) = -\Delta v \quad \text{with } \frac{\partial v}{\partial n} = 0 \text{ a.e. on } \Gamma,$$

and $D(\partial\psi_0) = \{v \in H^2(\Omega);\ \partial v/\partial n = 0 \text{ in } H^{1/2}(\Gamma)\}$.

We are going to prove the following proposition.

PROPOSITION 6.6. $\partial\psi = \partial\psi_0 + \partial\psi_1$, *namely*

$$D(\partial\psi) := \Big\{v \in H^2(\Omega);\ \exists g \in L^2(\Omega) \textit{ with } g \in \beta(v) \textit{ a.e. on } \Omega, \ \frac{\partial v}{\partial n} = 0 \textit{ a.e. on } \Gamma\Big\} \tag{6.36}$$

and

$$\partial\psi(v) = \big\{-\Delta v + g;\ g \in L^2(\Omega),\ g \in \beta(v) \textit{ a.e. on } \Omega\big\}, \quad \forall v \in D(\partial\psi). \tag{6.37}$$

PROOF. By assumption (6.17), we have $R(\beta) = \mathbf{R}$ (cf. Corollaries 4.1 and 4.2). By replacing β by $\beta(\cdot + r_0) - r_0^*$ with $[r_0, r_0^*] \in G(\beta)$, if necessary, we may assume without

loss of generality that $[0, 0] \in G(\beta)$ as well as $\hat{\beta}(0) = 0$. Now, for each $\varepsilon \in (0, 1]$, consider the approximation $\beta_\varepsilon := (\beta^{-1} + \varepsilon I)^{-1}$ of β, I being the identity in $\mathbf{R}$. Then by Lemma 5.1 we know that

(a) For all $r \in D(\beta)$, $|\beta_\varepsilon(r)| \leqslant |\beta^\circ(r)|$ and $\beta_\varepsilon(r) \to \beta^\circ(r)$ as $\varepsilon \downarrow 0$.

(b) β_ε is Lipschitz continuous on $\mathbf{R}$ with a Lipschitz constant $1/\varepsilon$.

In fact, for any $r_1, r_2 \in \mathbf{R}$, since $\beta_\varepsilon(r_i) \in \beta(r_i - \varepsilon\beta_\varepsilon(r_i))$, $i = 1, 2$, we have

$$\big(\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2)\big)\big(r_1 - \varepsilon\beta_\varepsilon(r_1) - r_2 + \varepsilon\beta_\varepsilon(r_2)\big) \geqslant 0$$

which gives

$$\varepsilon\big|\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2)\big|^2 \leqslant \big|\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2)\big||r_1 - r_2|,$$

hence $\varepsilon|\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2)| \leqslant |r_1 - r_2|$.

(c) $\beta_\varepsilon(r)r \geqslant (c_0/2)r^2 - c_1$, $\forall r \in \mathbf{R}$, $\forall \varepsilon \in (0, \varepsilon_0]$, where c_0 and c_1 are the same positive constants as in (6.17) and ε_0 is a small positive number. In fact, since $\beta_\varepsilon(r) \in \beta(r - \varepsilon\beta_\varepsilon(r))$ and $\beta_\varepsilon(0) = 0$, we have $\beta_\varepsilon(r)(r - \varepsilon\beta_\varepsilon(r)) \geqslant 0$ and $\beta_\varepsilon(r)r \geqslant 0$ from which we derive

$$\begin{aligned}
\beta_\varepsilon(r)r &\geqslant \beta_\varepsilon(r)r - \varepsilon\beta_\varepsilon(r)^2 \\
&= \beta_\varepsilon(r)\big(r - \varepsilon\beta_\varepsilon(r)\big) \\
&\geqslant \hat{\beta}\big(r - \varepsilon\beta_\varepsilon(r)\big) \\
&\geqslant c_0\big|r - \varepsilon\beta_\varepsilon(r)\big|^2 - c_1 \\
&= c_0 r^2 - 2c_0\varepsilon\beta_\varepsilon(r)r + c_0\varepsilon^2\beta_\varepsilon(r)^2 - c_1.
\end{aligned}$$

Hence

$$(1 + 2c_0\varepsilon)\beta_\varepsilon(r)r \geqslant c_0 r^2 - c_1$$

and we may take as ε_0 any positive number smaller than $\min\{1, 1/(2c_0)\}$.

Now, for each $\varepsilon \in (0, \varepsilon_0]$ we define a function $\psi_{1\varepsilon}$ on $L^2(\Omega)$ by putting

$$\psi_{1\varepsilon}(w) := \int_\Omega \hat{\beta}_\varepsilon(w)\,\mathrm{d}x, \quad \forall w \in L^2(\Omega),$$

where $\hat{\beta}_\varepsilon(\cdot)$ is the primitive of $\beta_\varepsilon(\cdot)$ with $\hat{\beta}_\varepsilon(0) = 0$. It is easy to check that $\psi_{1\varepsilon}$ is finite and continuous on $L^2(\Omega)$ and $\partial\psi_{1\varepsilon}(w) = \beta_\varepsilon(w)$ for every $w \in L^2(\Omega)$. Therefore, by virtue of Theorem 5.2, it follows that $R(\partial\psi_0 + \partial\psi_{1\varepsilon}) = L^2(\Omega)$. This shows that for any function f in $L^2(\Omega)$ there is a function u_ε such that

$$\partial\psi_0(u_\varepsilon) + \partial\psi_{1\varepsilon}(u_\varepsilon) = f \quad \text{in } L^2(\Omega), \tag{6.38}$$

namely $u_\varepsilon \in H^2(\Omega)$ and

$$-\Delta u_\varepsilon + \beta_\varepsilon(u_\varepsilon) = f \quad \text{a.e. on } \Omega, \tag{6.39}$$

$$\frac{\partial u_\varepsilon}{\partial n} = 0 \quad \text{a.e. on } \Gamma. \tag{6.40}$$

Here, multiply (6.39) by $\beta_\varepsilon(u_\varepsilon)$ and use the boundary condition (6.40) to get

$$\int_\Omega \beta_\varepsilon'(u_\varepsilon)|\nabla u_\varepsilon|^2 \,\mathrm{d}x + \int_\Omega \left|\beta_\varepsilon(u_\varepsilon)\right|^2 \mathrm{d}x = \int_\Omega f\beta_\varepsilon(u_\varepsilon)\,\mathrm{d}x.$$

From this with the non-negativeness of $\beta_\varepsilon'(\cdot)$ (= the derivative of $\beta_\varepsilon(\cdot)$) we observe that $|\beta_\varepsilon(u_\varepsilon)|_2 \leqslant |f|_2$, i.e. $\{\beta_\varepsilon(u_\varepsilon)\}_{0<\varepsilon\leqslant\varepsilon_0}$ is bounded in $L^2(\Omega)$ and $\{u_\varepsilon\}$ is bounded in $H^2(\Omega)$ by (c) and (6.39). Hence we can extract a sequence $\{\varepsilon_n\}$, tending to 0, and find functions $u \in H^2(\Omega)$ and $\tilde{u} \in L^2(\Omega)$ such that

$$u_n := u_{\varepsilon_n} \to u \quad \text{weakly in } H^2, \qquad \beta_{\varepsilon_n}(u_n) \to \tilde{u} \quad \text{weakly in } L^2(\Omega).$$

Now, passing to the limit as $n \to +\infty$ in (6.39) and (6.40) with $\varepsilon = \varepsilon_n$, we obtain that

$$-\Delta u + \tilde{u} = f \quad \text{a.e. on } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{a.e. on } \Gamma.$$

Moreover, by the usual compact embedding results (cf. [1]) note that $u_n \to u$ in $C(\overline{\Omega})$ and that $\varepsilon_n\beta_{\varepsilon_n}(u_n) \to 0$ in $L^2(\Omega)$. Also, since $\beta_{\varepsilon_n}(u_n) \in \partial\psi_1(u_n - \varepsilon_n\beta_{\varepsilon_n}(u_n))$ we see from the demiclosedness of $\partial\psi_1$ in $L^2(\Omega) \times L^2(\Omega)$ that $\tilde{u} \in \partial\psi_1(u)$, i.e., $\tilde{u} \in \beta(u)$ a.e. on Ω. Thus $f \in \partial\psi_0(u) + \partial\psi_1(u)$ and hence $R(\partial\psi_0 + \partial\psi_1) = L^2(\Omega)$.

Replacing $\psi_1(v)$ by $\psi_1(v) + \frac{1}{2}|v|_2^2$ and repeating the same argument as above, we obtain that $R(\partial\psi_0 + \partial\psi_1 + I) = L^2(\Omega)$, where I is here the identity in $L^2(\Omega)$. This implies that $\partial\psi_0 + \partial\psi_1$ is maximal monotone. Since $\partial\psi \supset \partial\psi_0 + \partial\psi_1$ is trivial, we conclude that $\partial\psi = \partial\psi_0 + \partial\psi_1$ and (6.36) and (6.37) hold. □

7. Convergence of maximal monotone mappings

In this section we introduce a notion of convergence of monotone mappings which is very useful in the theory of elliptic partial differential equations, and moreover we study its characterization by means of their smooth approximations.

Throughout this section, let X be a reflexive Banach space. Assume that X and its dual X^* are strictly convex and that X and X^* satisfy the following property $(R) := \{(R_X), (R_{X^*})\}$:

- (R_X) For a sequence $\{v_n\}$ in X and a point $v \in X$, if $v_n \to v$ weakly in X and $|v_n|_X \to |v|_X$ (as $n \to \infty$), then $v_n \to v$ (strongly) in X.
- (R_{X^*}) For a sequence $\{v_n^*\}$ in X^* and a point $v^* \in X$, if $v_n^* \to v^*$ weakly in X^* and $|v_n^*|_{X^*} \to |v^*|_{X^*}$, then $v_n^* \to v^*$ (strongly) in X.

We notice that any uniformly convex Banach space X and its dual X^* satisfy property (R). If property (R) is satisfied, the duality mappings $J : X \to X^*$ and $J^{-1} : X^* \to X$ are continuous in the strong topology.

We begin our study with the concept of regular approximation for monotone mappings.

Now, let A be a maximal monotone mapping from X into X^*. Then, for any $u \in X$ and any positive number $\lambda > 0$, there exists a unique element $u_\lambda \in X$ such that

$$J(u_\lambda - u) + \lambda A u_\lambda \ni 0 \quad \text{or equivalently} \quad \frac{1}{\lambda} J(u - u_\lambda) \in A u_\lambda. \tag{7.1}$$

In fact, defining a mapping $A_u : X \to X^*$ by $A_u v = A(v + u)$ for every $v \in D(A_u) = -u + D(A)$, we see that A_u is also maximal monotone and hence $R(\lambda A_u + J) = X^*$ (cf. Theorem 4.1), namely there exists an element $\tilde{u}_\lambda$ with $\tilde{u}_\lambda + u \in D(A)$ such that $0 \in \lambda A_u(\tilde{u}_\lambda) + J\tilde{u}_\lambda$ or equivalently $0 \in \lambda A(u_\lambda) + J(u_\lambda - u)$ with $u_\lambda = \tilde{u}_\lambda + u$. Moreover, such an element u_λ is uniquely determined thanks to the strict monotonicity of J. On the basis of this observation we give the following definition.

DEFINITION 7.1. Let A be a maximal monotone mapping from X into X^* and λ be any positive number. Then, for each $u \in X$, the (unique) element u_λ satisfying (7.1) is denoted by $R_\lambda^A u$. In this way, a mapping $R_\lambda^A : D(R_\lambda^A) = X \to X$ is defined. We call R_λ^A the resolvent, with index λ, of A. Also, we define $A_\lambda : D(A_\lambda) = X \to X^*$ by

$$A_\lambda v = \frac{1}{\lambda} J\big(v - R_\lambda^A v\big), \quad \forall v \in X. \tag{7.2}$$

We call A_λ the Yosida-approximation, with index λ, of A.

Next we investigate some properties of R_λ^A and A_λ.

THEOREM 7.1. *Let A be a maximal monotone mapping from X into X^*. Then we have*:

(a) *For each $\lambda > 0$, R_λ^A is bounded and continuous in X, and $A_\lambda : D(A_\lambda) = X \to X^*$ is a singlevalued, bounded, continuous and monotone mapping; hence A_λ is maximal monotone from X into X^*.*

(b) *$|A_\lambda v|_{X^*} \leqslant |A^0 v|_{X^*} := \inf_{v^* \in Av} |v^*|_{X^*}$ for each $v \in D(A)$ and $\lambda > 0$. Moreover, $A_\lambda v \to A^0 v$ in X^* as $\lambda \downarrow 0$ for each $v \in D(A)$.*

(c) *$R_\lambda^A v \to v$ in X as $\lambda \downarrow 0$ for each $v \in \overline{D(A)}$. More generally, it holds that $R_\lambda^A v \to \Pr_{\overline{D(A)}}(v)$ in X as $\lambda \downarrow 0$ for each $v \in X$, where $\Pr_{\overline{D(A)}}(\cdot)$ is the projection from X onto $\overline{D(A)}$.*

(d) *For each $[u, u^*] \in G(A)$, there exists a family $\{v_\lambda\}_{0<\lambda\leqslant 1}$ such that*

$$v_\lambda \to u \quad \text{in } X, \quad A_\lambda v_\lambda \to u^* \quad \text{in } X^* \quad \text{as } \lambda \downarrow 0.$$

PROOF. Let $\lambda \in (0, 1]$. For simplicity we write simply R_λ for R_λ^A. For any $u, v \in X$ we observe from (7.1) with the monotonicity of A that

$$\eta_{\lambda 1}(u, v) := \left\langle \frac{1}{\lambda} J(u - R_\lambda u) - \frac{1}{\lambda} J(v - R_\lambda v),\, R_\lambda u - R_\lambda v \right\rangle \geqslant 0, \tag{7.3}$$

and from the definition of duality mapping J that

$$\eta_{\lambda 2}(u, v) := \left\langle \frac{1}{\lambda} J(u - R_\lambda u) - \frac{1}{\lambda} J(v - R_\lambda v), (u - R_\lambda u) - (v - R_\lambda v) \right\rangle \geqslant 0. \tag{7.4}$$

Adding (7.3) and (7.4) yields

$$\eta_{\lambda 3}(u, v) := \left\langle \frac{1}{\lambda} J(u - R_\lambda u) - \frac{1}{\lambda} J(v - R_\lambda v), u - v \right\rangle \geqslant 0. \tag{7.5}$$

We now prove (a). For a fixed $[u, u^*] \in G(A)$ we have by the monotonicity of A

$$\left\langle \frac{1}{\lambda} J(v - R_\lambda v) - u^*, R_\lambda v - u \right\rangle \geqslant 0,$$

namely

$$|v - R_\lambda v|_X^2 \leqslant \lambda \langle u^*, R_\lambda v - u \rangle + \big\langle J(v - R_\lambda v), v - u \big\rangle. \tag{7.6}$$

Therefore

$$|v - R_\lambda v|_X^2 \leqslant 2|u - v|_X^2 + 2\lambda^2 |u^*|_{X^*}^2 + 2\lambda |u^*|_{X^*} |u - v|_X, \quad \forall v \in X. \tag{7.7}$$

This shows that R_λ is a bounded mapping in X as well as A_λ is a bounded mapping from X into X^*. By definition, A_λ is singlevalued. Also, the monotonicity of A_λ immediately follows from (7.5). To see the continuity of and R_λ and A_λ we use again (7.3)–(7.5) again. Let $\{u_n\}$ be a sequence in X such that $u_n \to u$ in X. By the boundedness of R_λ we may assume that $R_\lambda u_n \to \tilde{u}$ weakly in X for a certain $\tilde{u} \in X$ and $u_n^* := \frac{1}{\lambda} J(u_n - R_\lambda u_n) \to u^*$ weakly in X^*. Therefore,

$$\begin{aligned} \langle u_n^*, R_\lambda u_n - \tilde{u} \rangle &= \frac{1}{\lambda} \big\langle J(u_n - R_\lambda u_n), R_\lambda u_n - \tilde{u} \big\rangle \\ &= -\frac{1}{\lambda} |u_n - R_\lambda u_n|_X^2 + \frac{1}{\lambda} \big\langle J(u_n - R_\lambda u_n), u_n - \tilde{u} \big\rangle \\ &\leqslant \frac{1}{2\lambda} |u_n - \tilde{u}|_X^2 - \frac{1}{2\lambda} |u_n - R_\lambda u_n|_X^2, \end{aligned}$$

which implies that

$$\limsup_{n \to \infty} \langle u_n^*, R_\lambda u_n - \tilde{u} \rangle \leqslant \frac{1}{2\lambda} \limsup_{n \to \infty} \big\{ |u_n - \tilde{u}|_X^2 - |u_n - R_\lambda u_n|_X^2 \big\} \leqslant 0,$$

because $u_n - \tilde{u} \to u - \tilde{u}$ in X and $u_n - R_\lambda u_n \to u - \tilde{u}$ weakly in X. Since $u_n^* \in A R_\lambda u_n$, it follows from the above inequalities that $u^* \in A\tilde{u}$ and $\lim_{n \to \infty} |u_n - R_\lambda u_n|_X = |u - \tilde{u}|_X$.

Hence, by condition (R), we have $u_n - R_\lambda u_n \to u - \tilde{u}$ and $R_\lambda u_n \to \tilde{u}$ in X. This shows that $u^* = \frac{1}{\lambda} J(u - \tilde{u}) \in A\tilde{u}$, namely $\tilde{u} = R_\lambda u$. Thus R_λ as well as A_λ is continuous in X.

Now we prove (b). Let v be any point in $D(A)$. For any $v^* \in Av$ we have by the monotonicity of A that

$$\left\langle v^* - \frac{1}{\lambda} J(v - R_\lambda v), v - R_\lambda v \right\rangle \geqslant 0,$$

since $v^* \in Av$ and $A_\lambda v = \frac{1}{\lambda} J(v - R_\lambda v) \in A(R_\lambda v)$. From this inequality we get

$$|A_\lambda v|_{X^*} = \frac{1}{\lambda} \big| J(v - R_\lambda v) \big|_{X^*} \leqslant |v^*|_{X^*}; \tag{7.8}$$

and $R_\lambda v \to v$ in X. By the arbitrariness of $v^* \in Av$ in (7.8), we have

$$|A_\lambda v|_{X^*} \leqslant \inf_{v^* \in Av} |v^*|_{X^*} =: \big| A^0 v \big|_{X^*}. \tag{7.9}$$

Next, let $\{\lambda_n\}$ be any sequence in $(0, 1]$ tending to 0 (as $n \to \infty$). We may assume by (7.9) that $A_{\lambda_n} v (= \frac{1}{\lambda_n} J(v - R_{\lambda_n} v)) \to a^*$ weakly in X^* for a certain $a^* \in X^*$. Since

$$\langle A_{\lambda_n} v - u^*, R_{\lambda_n} v - u \rangle \geqslant 0, \quad \forall [u, u^*] \in G(A),$$

it follows by letting $n \to \infty$ in the above relation that

$$\langle a^* - u^*, v - u \rangle \geqslant 0, \quad \forall [u, u^*] \in G(A).$$

This implies together with the maximal monotonicity of A that $a^* \in Av$ and $|a^*|_{X^*} \geqslant |A^0 v|_{X^*}$. Hence

$$\big| A^0 v \big|_{X^*} \leqslant |a^*|_{X^*} \leqslant \liminf_{n \to \infty} |A_{\lambda_n} v|_{X^*} \leqslant \limsup_{n \to \infty} |A_{\lambda_n} v|_{X^*} \leqslant \big| A^0 v \big|_{X^*}.$$

Consequently we obtain that $a^* = A^0 v$ and

$$A_{\lambda_n} \to A^0 v \quad \text{weakly in } X^*, \qquad \lim_{n \to \infty} |A_{\lambda_n} v|_{X^*} = \big| A^0 v \big|_{X^*},$$

so that $A_{\lambda_n} v \to A^0 v$ in X^*. This shows that $A_\lambda v \to A^0 v$ in X^* as $\lambda \downarrow 0$.

We proceed the proof of (c). Let v be any point in X. Going back to (7.6) with an arbitrary $u \in D(A)$ and taking a sequence $\lambda_n \downarrow 0$ such that $R_{\lambda_n} v \to \tilde{v}$ weakly in X and $J(R_{\lambda_n} v - v) \to b^*$ weakly in X^*, we observe that

$$\limsup_{n \to \infty} |R_{\lambda_n} v - v|_X^2 \leqslant \langle b^*, u - v \rangle, \quad \forall u \in D(A). \tag{7.10}$$

Here note that the inequality in (7.10) holds true for every $u \in \mathrm{co}(\overline{D(A)})$, which implies that $R_{\lambda_n} v \to v$ in X for every $v \in \mathrm{co}(\overline{D(A)})$. Since $R_{\lambda_n} v \in D(A)$, it follows

that $\mathrm{co}(\overline{D(A)}) \subset \overline{D(A)}$. Thus $\overline{D(A)}$ is convex in X. Moreover, noting that $|b^*|_{X^*} \leqslant \liminf_{n\to\infty} |R_{\lambda_n} v - v|_X$, we get from (7.10) that

$$\limsup_{n\to\infty} |R_{\lambda_n} v - v|_X \leqslant |u - v|_X, \quad \forall v \in X,\ \forall u \in \overline{D(A)}.$$

This implies by $\tilde{v} \in \overline{D(A)}$ that

$$|\tilde{v} - v|_X = \lim_{n\to\infty} |R_{\lambda_n} v - v|_X = \left|\mathrm{Pr}_{\overline{D(A)}}(v) - v\right|_X. \tag{7.11}$$

Consequently, $\tilde{v} = \mathrm{Pr}_{\overline{D(A)}}(v)$ and $R_{\lambda_n} v - v \to \mathrm{Pr}_{\overline{D(A)}}(v) - v$ in X, namely $R_{\lambda_n} v \to \mathrm{Pr}_{\overline{D(A)}}(v)$ in X. Clearly, the above observation is valid for the whole $\lambda \downarrow 0$.

Finally we prove (d). Let $[u, u^*]$ be any element of $G(A)$. Now, for any $\lambda > 0$ we put

$$u_\lambda = u + \lambda J^{-1}(u^*), \quad \text{or equivalently} \quad u^* = \frac{1}{\lambda} J(u_\lambda - u). \tag{7.12}$$

Since $\frac{1}{\lambda} J(u_\lambda - u) + Au \ni 0$, it follows by the definition R_λ that $u = R_\lambda u_\lambda$ and

$$A_\lambda u_\lambda = \frac{1}{\lambda} J(u_\lambda - u) = u^*.$$

It is clear that $u_\lambda \to u$ in X as $\lambda \downarrow 0$. Thus the family $\{u_\lambda\}$ given by (7.12) is a required one. □

REMARK 7.1. In the proof of (c) of Theorem 7.1 we obtained that for any maximal monotone mapping $A : X \to X^*$ the closure $\overline{D(A)}$ of the domain $D(A)$ is convex in X.

Next, we give a notion of convergence of maximal monotone mappings.

DEFINITION 7.2. We are given a sequence of maximal monotone mappings $\{A_n\}$ from X into X^*. It is said that $\{A_n\}$ converges to a maximal monotone mapping $A : X \to X^*$ in the graph sense (as $n \to \infty$), if the following statement holds: for each $[u, u^*] \in G(A)$ there exits a sequence $\{[u_n, u_n^*]\}$ in $X \times X^*$ such that $[u_n, u_n^*] \in G(A_n)$ for all $n = 1, 2, \ldots$, and $u_n \to u$ in X and $u_n^* \to u^*$ in X^*. This convergence is shortly denoted by

$$A_n \to A \quad \text{in } X \times X^* \text{ in the graph sense.}$$

According to Theorem 7.1(d), the Yosida approximation A_λ of a maximal monotone mapping $A : X \to X^*$ converges to A in the graph sense as any sequence $\lambda_n > 0$ tends to 0.

LEMMA 7.1. *Let $A_n : X \to X^*$, $n = 1, 2, \ldots$, and $A : X \to X^*$ be maximal monotone mappings. Assume that $A_n \to A$ in $X \times X^*$ in the graph sense. If $[u_n, u_n^*] \in G(A_n)$, $n = 1, 2, \ldots$, and if*

$$\begin{aligned} &u_n \to u \quad \textit{weakly in } X, \quad u_n^* \to u^* \quad \textit{weakly in } X^*, \\ &\limsup_{n\to\infty} \langle u_n^*, u_n \rangle \leqslant \langle u^*, u \rangle, \end{aligned} \tag{7.13}$$

then $[u, u^*] \in G(A)$ *and*

$$\lim_{n\to\infty} \langle u_n^*, u_n \rangle = \langle u^*, u \rangle. \tag{7.14}$$

PROOF. Let $[v^*, v]$ be any element in $G(A)$. By assumption, there is a sequence $\{[v_n, v_n^*]\}$ with $[v_n, v_n^*] \in G(A_n)$ such that $v_n \to v$ in X and $v_n^* \to v^*$ in X^*. Since $\langle v_n^* - u_n^*, v_n - u_n \rangle \geqslant 0$ by the monotonicity of A_n, we infer by letting $n \to \infty$ that $\langle v^* - u^*, v - u \rangle \geqslant 0$ for all $[v, v^*] \in G(A)$. The maximal monotonicity of A implies that $u^* \in Au$. Now, use our assumption to find a sequence $[\tilde{u}_n, \tilde{u}_n^*] \in G(A_n)$, $n = 1, 2, \ldots$, such that $\tilde{u}_n \to u$ in X and $\tilde{u}_n^* \to u^*$ in X^*. By the monotonicity of A_n we have $\langle v_n^* - \tilde{u}_n^*, v_n - \tilde{u}_n \rangle \geqslant 0$. Letting $n \to \infty$ in it, we get $\liminf_{n\to\infty} \langle v_n^*, v_n \rangle \geqslant \langle u^*, u \rangle$. This together with (7.13) implies (7.14). □

We now state a theorem which gives some equivalent properties to graph convergence.

THEOREM 7.2. *Let* $A_n : X \to X^*$, $n = 1, 2, \ldots$, *and* $A : X \to X^*$ *be maximal monotone mappings. Then the following statements* (i)–(iii) *are equivalent to each other*:

(i) $A_n \to A$ *in* $X \times X^*$ *as* $n \to \infty$ *in the graph sense.*

(ii) $R_\lambda^{A_n} u \to R_\lambda^A u$ *in* X *as* $n \to \infty$ *for any* $u \in X$ *and any* $\lambda > 0$.

(iii) $A_{n\lambda} u \to A_\lambda u$ *in* X *as* $n \to \infty$ *for any* $u \in X$ *and any* $\lambda > 0$, *where* $A_{n\lambda}$ *is the Yosida approximation of* A_n.

For our proof of Theorem 7.2 we use the following lemma.

LEMMA 7.2. *Suppose that the same assumptions as in Theorem* 7.2 *are satisfied. Let* λ *be any positive number. Then we have*:

(1) *For a given bounded set* E *in* X *there is a positive number* $C(\lambda, E)$, *depending on* λ *and* E, *such that* $|R_\lambda^{A_n} u|_X \leqslant C(\lambda, E)$ *for all* $u \in E$ *and* $n = 1, 2, \ldots$.

(2) *If* $R_\lambda^{A_n} u \to a$ *weakly in* X *and* $A_{n\lambda} u \to a^*$ *weakly in* X^* *as* $n \to \infty$, *then* $a^* \in Aa$ *and*

$$\lim_{n\to\infty} \left\langle A_{n\lambda} u, R_\lambda^{A_n} u \right\rangle = \langle a^*, a \rangle. \tag{7.15}$$

PROOF. For simplicity we write R_λ^n for $R_\lambda^{A_n}$. Let $[u_0, u_0^*]$ be an element of $G(A)$ and choose $[u_{0n}, u_{0n}^*] \in G(A_n)$ so that $u_{0n} \to u_0$ in X and $u_{0n}^* \to u^*$ in X^*. Then, just as (7.7), we have

$$\left| v - R_\lambda^n v \right|_X^2 \leqslant 2|u_{0n} - v|_X^2 + 2\lambda^2 |u_{0n}^*|_{X^*}^2 + 2\lambda |u_{0n}^*|_{X^*} |u_{0n} - v|_X, \quad \forall v \in X,$$

which shows that R_λ^n, $n = 1, 2, \ldots$, are bounded on each bounded subset of X uniformly in n. Thus (1) holds.

Next we show (2). For simplicity we write R_λ^n for $R_\lambda^{A_n}$. For any two positive integers n, m we see by the monotonicity of A that $\langle A_{n\lambda} u - A_{m\lambda} u, (u - R_\lambda^n u) - (u - R_\lambda^m u) \rangle \geqslant 0$, and hence

$$\left\langle A_{n\lambda} u - A_{m\lambda} u, R_\lambda^n u - R_\lambda^m u \right\rangle \leqslant 0, \tag{7.16}$$

since $A_{k\lambda}u = \frac{1}{\lambda}J(u - R^k_\lambda u)$ for $k = n, m$. Write (7.16) in the following form:

$$\langle A_{n\lambda}u, R^n_\lambda u\rangle + \langle A_{m\lambda}u, R^m_\lambda u\rangle \leqslant \langle A_{n\lambda}u, R^m_\lambda u\rangle + \langle A_{m\lambda}u, R^n_\lambda u\rangle.$$

First fix m and pass to the limit $n \to \infty$, and then the limit $m \to \infty$ to get

$$\limsup_{n\to\infty}\langle A_{n\lambda}u, R^n_\lambda u\rangle \leqslant \langle a^*, a\rangle. \tag{7.17}$$

Now, let $[v, v^*]$ be any element of $G(A)$, and by using our assumption, take a sequence $[v_n, v^*_n] \in G(A_n)$ such that $v_n \to v$ in X and $v^*_n \to v^*$ in X^* (as $n \to \infty$). Then we have by the monotonicity of A_n that

$$\langle A_{n\lambda}u - v^*_n, R^n_\lambda u - v_n\rangle \geqslant 0. \tag{7.18}$$

Noting (7.17) and passing to the limit $n \to \infty$ in (7.18), we obtain that

$$\langle a^* - v^*, a - v\rangle \geqslant 0.$$

Since this inequality is valid for all $[v, v^*] \in G(A)$, the maximal monotonicity implies that $[a, a^*] \in G(A)$, i.e. $a^* \in Aa$. Moreover, in the above argument, if we take $[a, a^*]$ as $[v, v^*] \in G(A)$, we derive from (7.18) again that $\liminf_{n\to\infty}\langle A_{n\lambda}u, R^n_\lambda u\rangle \geqslant \langle a^*, a\rangle$. Combining this with (7.17), we have (7.15). □

PROOF OF THEOREM 7.2. By the formula $A_{n\lambda} = \frac{1}{\lambda}J(I - R^n_\lambda)$, clearly (ii) and (iii) are equivalent. Therefore it is enough to show the equivalence between (i) and the set {(ii), (iii)}.

First assume (i) holds. Let u be any point in X. By (1) of Lemma 7.2 $\{R^n_\lambda u\}_{n=1}^\infty$ is bounded in X as well as $\{A_{n\lambda}u\}_{n=1}^\infty$ in X^*. Therefore we can extract a subsequence $\{n_k\}$ from $\{n\}$ so that

$$R^{n_k}_\lambda u \to a \quad \text{weakly in } X, \quad A_{n_k\lambda}u \to a^* \quad \text{weakly in } X^* \quad \text{as } k \to \infty.$$

Here, apply (2) of Lemma 7.2 to these subsequences $\{R^{n_k}_\lambda u\}$ and $\{A_{n_k\lambda}u\}$ to see that

$$a^* \in Aa, \quad \lim_{k\to\infty}\langle A_{n_k\lambda}u, R^{n_k}_\lambda u\rangle = \langle a^*, a\rangle.$$

Hence it holds that

$$\lim_{k\to\infty}\left\langle A_{n_k\lambda}u, \frac{1}{\lambda}\big(u - R^{n_k}_\lambda u\big)\right\rangle = \left\langle a^*, \frac{1}{\lambda}(u - a)\right\rangle,$$

whence

$$\lim_{k\to\infty}\left|\frac{1}{\lambda}J\big(u - R^{n_k}_\lambda u\big)\right|^2_{X^*} = \lim_{k\to\infty}\left|\frac{1}{\lambda}\big(u - R^{n_k}_\lambda u\big)\right|^2_X \leqslant |a^*|_{X^*}\left|\frac{1}{\lambda}(u - a)\right|_X. \tag{7.19}$$

Since $\frac{1}{\lambda}(u - R_\lambda^{n_k} u) \to \frac{1}{\lambda}(u - a)$ weakly in X, it follows that

$$\liminf_{k\to\infty} \left| \frac{1}{\lambda}\big(u - R_\lambda^{n_k} u\big) \right|_X \geqslant \left| \frac{1}{\lambda}(u - a) \right|_X .$$

On account of this fact we derive from (7.19) that

$$\lim_{k\to\infty} |A_{n_k\lambda} u|_{X^*} = \lim_{k\to\infty} \left| \frac{1}{\lambda} J\big(u - R_\lambda^{n_k} u\big) \right|_{X^*} \leqslant |a^*|_{X^*}.$$

Since $A_{n_k\lambda} u \to a^*$ weakly in X^*, condition (R) implies that $A_{n_k\lambda} u = \frac{1}{\lambda} J(u - R_\lambda^{n_k} u) \to a^*$ in X^*, and at the same time $R_\lambda^{n_k} u \to a$ in X as $k \to \infty$. Moreover, $\frac{1}{\lambda} J(u - a) = a^* \in Aa$, and hence it follows that

$$a = R_\lambda^A u, \qquad a^* = A_\lambda u.$$

Noting that $[a, a^*]$ is independent of the choice of subsequence $\{n_k\}$, we conclude that $R_\lambda^n u \to R_\lambda^A u$ in X and $A_{n\lambda} u \to A_\lambda u$ as $n \to \infty$ for each $u \in X$. Thus (ii) and (iii) hold.

Conversely, assume that (ii) and (iii) hold. Let $[u, u^*]$ be any element of $G(A)$, and put $v = u + \lambda J^{-1} u^*$, i.e. $u^* = \frac{1}{\lambda} J(v - u)$. Since $u^* \in Au$, the last relation implies that

$$u = R_\lambda^A v, \qquad u^* = A_\lambda v.$$

By assumption (ii), we have $R_\lambda^n v \to R_\lambda^A v$ in X as $n \to \infty$. Now, take

$$u_n := R_\lambda^n v, \qquad u_n^* := A_{n\lambda} v.$$

Then, $[u_n, u_n^*] \in G(A_n)$, $u_n \to u$ in X and $u_n^* \to u^*$ in X^*. Thus (i) is obtained. □

Finally we discuss the continuous dependence of the set of solutions u of $f \in Au + Bu$ upon the maximal monotone mapping A, where B is a fixed pseudo-monotone mapping.

THEOREM 7.3. *Let $A_n : X \to X^*$, $n = 1, 2, \ldots$, be maximal monotone mappings and suppose that A_n converges to a maximal monotone mapping A in $X \times X^*$ in the graph sense as $n \to \infty$. Also, let $B : D(B) = X \to X^*$ be a bounded pseudo-monotone mapping. Furthermore suppose that there are a continuous real function $\rho(\cdot) : [0, \infty) \to (-\infty, \infty)$, satisfying $\rho(r) \to \infty$ as $r \to \infty$, and a bounded set $\{a_n\}$ in X with $a_n \in D(A_n)$, $n = 1, 2, \ldots$, and positive constants b_0, b_1 such that*

$$\langle v^*, v - a_n \rangle \geqslant -b_0 |v|_X - b_1, \quad \forall [v, v^*] \in G(B),\ n = 0, 1, 2, \ldots, \tag{7.20}$$

$$\inf_{u^* \in A_n u,\ v^* \in Bu} \langle u^* + v^*, u - a_n \rangle \geqslant \rho\big(|u|_X\big)|u|_X, \quad \forall n = 1, 2, \ldots. \tag{7.21}$$

Then we have:

(i) *For each sequence* $\{g_n^*\}$ *in* X^* *such that* $g_n^* \to g^*$ *in* X^* *as* $n \to \infty$, *denote the set* $\{u \in X;\ g_n^* \in A_n u + Bu\}$ *by* $S_n(g_n^*)$. *Then* $\{S_n(g_n^*)\}$ *is a family of non-empty and uniformly bounded sets in* X.

(ii) $S(g^*) = \{u \in X;\ g^* \in Au + Bu\}$ *is non-empty. Moreover, if* $u_n \in S_n(g_n^*)$, $n = 1, 2, \ldots$, *and* $u_n \to u$ *weakly in* X, *then* $u \in S(g^*)$.

PROOF. We first prove (i). It follows from Corollary 5.2 that $S_n(g_n^*)$, $n = 1, 2, \ldots$, is non-empty. Also, assuming that $u_n \in S_n(g_n^*)$ and $g_n^* = u_n^* + v_n^*$, $u_n^* \in A_n u_n$ and $v_n^* \in Bu_n$, we observe from (7.20) and (7.21) that

$$\begin{aligned} \rho\big(|u_n|_X\big)|u_n|_X &\leqslant \langle u_n^* + v_n^*, u_n - a_n\rangle = \langle g_n^*, u_n - a_n\rangle \\ &\leqslant |g_n^*|_{X^*}\big(|u_n|_X + |a_n|_X\big) \end{aligned} \tag{7.22}$$

for all n. Now, put

$$R_0 := \sup_{n\geqslant 0}\big\{|g_n^*|_{X^*}\big(1 + |a_n|_X\big)\big\} + \sup_{0\leqslant r\leqslant 1}\big|\rho(r)\big|.$$

We see from (7.22) that

$$\rho\big(|u_n|_X\big) \leqslant R_0, \quad \forall n = 1, 2, \ldots.$$

Therefore, denoting $\sup\{r \in [0, \infty);\ \rho(r) \leqslant R_0\}$ by $R_* \geqslant 0$, we have $|u_n|_X \leqslant R_*$ for all n, namely $S_n(g_n^*) \subset B_{R_*}$ for all n, where B_{R_*} is the closed ball about the origin of X with radius R_*.

Next, we show (ii). Assume again that $u_n \in S_n(g_n^*)$ and $g_n^* = u_n^* + v_n^*$ with $u_n^* \in A_n u_n$ and $v_n^* \in Bu_n$. Further assume that $u_n \to u$ weakly in X. Now, extracting any subsequence $\{n_k\}$ from $\{n\}$ so that $\chi_1 := \lim_{k\to\infty}\langle u_{n_k}^*, u_{n_k} - u\rangle$ and $\chi_2 := \lim_{k\to\infty}\langle v_{n_k}^*, u_{n_k} - u\rangle$ exist, we have

$$\chi_1 + \chi_2 = \lim_{k\to\infty}\langle u_{n_k}^* + v_{n_k}^*, u_{n_k} - u\rangle = \lim_{k\to\infty}\langle g_{n_k}^*, u_{n_k} - u\rangle = 0.$$

Therefore $\chi_1 \leqslant 0$ or $\chi_2 \leqslant 0$. In the case of $\chi_1 \leqslant 0$, it follows from Lemma 7.1 that $[u, u^*] \in G(A)$ and $\chi_1 = 0$, i.e. $\lim_{k\to\infty}\langle u_{n_k}^*, u_{n_k}\rangle = \langle u^*, u\rangle$. Hence $\chi_2 = 0$ and it follows from the pseudo-monotonicity of B that $v^* \in Bu$ and $g^* = u^* + v^* \in Au + Bu$. In the case of $\chi_2 = 0$ we have similarly $g^* = u^* + v^* \in Au + Bu$. Thus $u \in S(g^*)$ and $S(g^*)$ is non-empty. □

REMARK 7.2. Throughout this section condition (R) for Banach space X and its dual X^* is always supposed. But, as is easily checked, the conclusion of Theorem 7.3 is valid without condition (R).

REMARK 7.3. We do not assume explicitly the coerciveness (7.21) for A and B. If B has the following property:

$$\begin{cases} \text{for any } [v, v^*] \in G(B) \text{ and any sequence } \{v_n\} \text{ in } X \text{ with } v_n \to v \text{ in } X, \\ \text{there exists a sequence } [v_n, v_n^*] \in G(B) \text{ such that } v_n^* \to v^* \text{ in } X^*, \end{cases} \tag{7.23}$$

then A and B satisfies that

$$\inf_{u^*\in Au,\, v^*\in Bu} \langle u^*+v^*, u-a_0\rangle \geqslant \rho\big(|u|_X\big)|u|_X, \quad \forall n=1,2,\ldots, \tag{7.24}$$

where a_0 is a weak limit point of $\{a_n\}$ in X. This is checked as follows. We may assume by take a subsequence of $\{a_n\}$ if necessary that $a_n \to a_0$ weakly in X for a certain $a_0 \in X$. Now, let $[u, \bar{u}^*]$ and $[u, \bar{v}^*]$ be elements of $G(A)$ and $G(B)$, respectively, such that

$$\langle \bar{u}^*+\bar{v}^*, u-a_0\rangle = \inf_{u^*\in Au,\, v^*\in Bu} \langle u^*+v^*, u-a_0\rangle.$$

Then, by our assumption of Theorem 7.3 and (7.23), there sequences $[u_n, u_n^*] \in G(A_n)$ and $[u_n, v_n^*] \in G(B)$, $n=1,2,\ldots$, such that $u_n \to u$ in X and $u_n^* \to \bar{u}^*$, $v_n^* \to \bar{v}^*$ in X^*. Since

$$\langle u_n^*+v_n^*, u_n-a_n\rangle \geqslant \rho\big(|u|_X\big)|u|_X,$$

it follows by letting $n\to\infty$ that

$$\langle \bar{u}^*+\bar{v}^*, u-a_0\rangle \geqslant \rho\big(|u|_X\big)|u|_X.$$

Hence (7.24) holds. In case (7.24) is satisfied, we see directly from Corollary 5.2 that $S(g^*)$ is non-empty. For some other investigations on the convergence of monotone mappings we refer to H. Attouch [2] and V. Barbu [4].

8. Convergence of convex functions

In this section we study convergence of convex functions in connection with that of subdifferentials and resolvents. Throughout this section let X be a real reflexive Banach space and assume that X and X^* are strictly convex and satisfy condition (R) as in the previous section.

Let φ be a proper, l.s.c. and convex function on X. Then, for each real number $\lambda>0$, the resolvent $R_\lambda^{\partial\varphi}$ is defined for the subdifferential $\partial\varphi$ as before by

$$\frac{1}{\lambda} J\big(R_\lambda^{\partial\varphi}u-u\big) + \partial\varphi\big(R_\lambda^{\partial\varphi}u\big) \ni 0 \quad \text{in } X^*. \tag{8.1}$$

Clearly, (8.1) is equivalent to the following relation

$$\frac{1}{2\lambda}\big|R_\lambda^{\partial\varphi}u-u\big|_X^2 + \varphi\big(R_\lambda^{\partial\varphi}u\big) = \min_{v\in X}\left\{\frac{1}{2\lambda}|u-v|_X^2+\varphi(v)\right\}. \tag{8.2}$$

Now we define a function φ_λ on X by putting

$$\varphi_\lambda(u) := \frac{1}{2\lambda}\big|R_\lambda^{\partial\varphi}u-u\big|_X^2 + \varphi\big(R_\lambda^{\partial\varphi}u\big), \quad \forall u\in X,\ \forall\lambda>0. \tag{8.3}$$

We see that φ_λ is everywhere finite on X, since

$$\varphi(v) + C_\varphi\big(|v|_X + 1\big) \geqslant 0, \quad \forall v \in X, \tag{8.4}$$

holds for a certain positive constant C_φ. Also, φ_λ is convex on X and it is verified as follows. For any $u_1, u_2 \in X$ and any $r \in (0, 1)$ it holds that

$$\begin{aligned}
&\varphi_\lambda\big(ru_1 + (1-r)u_2\big) \\
&\quad\leqslant \frac{1}{2\lambda}\big|rR_\lambda^{\partial\varphi}u_1 + (1-r)R_\lambda^{\partial\varphi}u_2 - ru_1 - (1-r)u_2\big|_X^2 \\
&\qquad + \varphi\big(rR_\lambda^{\partial\varphi}u_1 + (1-r)R_\lambda^{\partial\varphi}u_2\big) \\
&\quad\leqslant \frac{r}{2\lambda}\big|R_\lambda^{\partial\varphi}u_1 - u_1\big|_X^2 + r\varphi\big(R_\lambda^{\partial\varphi}u_1\big) + \frac{(1-r)}{2\lambda}\big|R_\lambda^{\partial\varphi}u_2 - u_2\big|_X^2 \\
&\qquad + (1-r)\varphi\big(R_\lambda^{\partial\varphi}u_2\big) \\
&\quad= r\varphi_\lambda(u_1) + (1-r)\varphi_\lambda(u_2).
\end{aligned}$$

Moreover, φ_λ is bounded on X, i.e. bounded on each bounded subsets of X, due to the definition (8.3). Therefore $D(\varphi_\lambda) = X$ and φ_λ is locally Lipschitz continuous on X by the general theory on convex functions.

LEMMA 8.1. *Let φ be a proper, l.s.c. and convex function on X. Then we have*:

(a) *$\varphi(R_\lambda^{\partial\varphi}(\cdot))$ is bounded and continuous on X for every $\lambda > 0$.*

(b) *$\varphi_\lambda(u) \to \varphi(u)$ and $\varphi(R_\lambda^{\partial\varphi}u) \to \varphi(u)$ as $\lambda \downarrow 0$ for all $u \in X$.*

(c) *For any $\lambda > 0$, the subdifferential $\partial\varphi_\lambda$ of φ_λ coincides with the Yosida approximation of $\partial\varphi$, namely we see that*

$$\partial\varphi_\lambda(u) = (\partial\varphi)_\lambda(u) = \frac{1}{\lambda}J\big(u - R_\lambda^{\partial\varphi}u\big), \quad \forall u \in X.$$

(d) *For any $\lambda > 0$ and any $u, v \in X$, the function $\varphi_\lambda(u + tv)$ is continuously differentiable in $t \in \mathbf{R}$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda(u + tv) = \big\langle \partial\varphi_\lambda(u + tv), v\big\rangle, \quad -\infty < t < \infty.$$

PROOF. For simplicity we write R_λ for $R_\lambda^{\partial\varphi}$. We show (a). By (a) of Theorem 7.1, R_λ is bounded and continuous in X. From the relation $\varphi(R_\lambda u) = \varphi_\lambda(u) - \frac{1}{2\lambda}|u - R_\lambda u|_X^2$ for all $u \in X$ it follows that $\varphi(R_\lambda(\cdot))$ is bounded and continuous on X, too. Next we show (b). It follows from the definition of φ and the lower semicontinuity of φ that

$$\varphi(u) \geqslant \limsup_{\lambda\downarrow 0}\varphi_\lambda(u) \geqslant \liminf_{\lambda\downarrow 0}\varphi_\lambda(u) \geqslant \liminf_{\lambda\downarrow 0}\varphi(R_\lambda u) \geqslant \varphi(u),$$

so that

$$\lim_{\lambda\downarrow 0}\varphi_\lambda(u)=\varphi(u),\quad \forall u\in X.$$

We proceed to show (c). We observe for any $u, v\in X$ that

$$\begin{aligned}
&\left\langle\frac{1}{\lambda}J(u-R_\lambda u),u-v\right\rangle\\
&\quad=\left\langle\frac{1}{\lambda}J(u-R_\lambda u),u-R_\lambda u\right\rangle+\left\langle\frac{1}{\lambda}J(u-R_\lambda u),R_\lambda u-R_\lambda v\right\rangle\\
&\qquad+\left\langle\frac{1}{\lambda}J(u-R_\lambda u),R_\lambda v-v\right\rangle\\
&\quad\geqslant\frac{1}{\lambda}|u-R_\lambda u|_X^2+\varphi(R_\lambda u)-\varphi(R_\lambda v)-\frac{1}{2\lambda}|u-R_\lambda u|_X^2-\frac{1}{2\lambda}|v-R_\lambda v|_X^2\\
&\quad=\frac{1}{2\lambda}|u-R_\lambda u|_X^2+\varphi(R_\lambda u)-\frac{1}{2\lambda}|v-R_\lambda v|_X^2-\varphi(R_\lambda v)\\
&\quad=\varphi_\lambda(u)-\varphi_\lambda(v),
\end{aligned}$$

namely,

$$\left\langle\frac{1}{\lambda}J(u-R_\lambda u),u-v\right\rangle\geqslant\varphi_\lambda(u)-\varphi_\lambda(v),\quad\forall u,v\in X.$$

This implies that $(\partial\varphi)_\lambda(u):=\frac{1}{\lambda}J(u-R_\lambda)\in\partial\varphi_\lambda(u)$ for all $u\in X$. Therefore, $(\partial\varphi)_\lambda=\partial\varphi_\lambda$, because $(\partial\varphi)_\lambda$ is maximal monotone from X into X^*. Finally we show (d). For any $u, v\in X$ and $s, t\in\mathbf{R}$ with $s<t$ we have

$$\begin{aligned}
\big\langle\partial\varphi_\lambda(u+tv),v\big\rangle&=\frac{1}{(t-s)}\big\langle\partial\varphi_\lambda(u+tv),(u+tv)-(u+sv)\big\rangle\\
&\geqslant\frac{1}{(t-s)}\big\{\varphi_\lambda(u+tv)-\varphi_\lambda(u+sv)\big\}.
\end{aligned}$$

Letting $s\uparrow t$ in the above relations, we get

$$\big\langle\partial\varphi_\lambda(u+tv),v\big\rangle\geqslant\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda(u+tv);$$

the derivative of the right-hand side exists a.e. on $\mathbf{R}$, because φ_λ is locally Lipschitz continuous on X. By considering the case of $s>t$ we get the opposite inequality, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_\lambda(u+tv)=\big\langle\partial\varphi_\lambda(u+tv),v\big\rangle$$

for a.e. $t \in \mathbf{R}$. Since the right-hand side is continuous in t, we conclude that $\varphi_\lambda(u + tv)$ of C^1-class. □

COROLLARY 8.1. *Let φ be a proper l.s.c. and convex function on X. Then $D(\partial\varphi)$ is a dense subset of $D(\varphi)$, namely $\overline{D(\partial\varphi)} = \overline{D(\varphi)}$.*

PROOF. Let u be any point in $D(\varphi)$. Then, by (b) of Lemma 8.1, $\varphi R_\lambda u \to \varphi(u)$ as $\lambda \downarrow 0$, which implies that $u \in \overline{D(\partial\varphi)}$. Therefore we obtain the lemma. □

As was seen in Lemma 8.1, φ_λ is a sort of smooth approximation of φ. We call φ_λ the Moreau–Yosida approximation with index λ of φ.

DEFINITION 8.1. Let $\{\varphi_n\}$ be a sequence of proper, l.s.c. convex functions on X. Then $\{\varphi_n\}$ converges to a proper, l.s.c. convex function φ on X in the sense of *Mosco* (cf. [15]), if the following two conditions (M1) and (M2) are satisfied:

(M1) Let $\{n_k\}$ be any subsequence of $\{n\}$. If $\{v_k\}$ is a sequence in X and $v \in X$ such that $v_k \to v$ weakly in X as $k \to \infty$, then

$$\liminf_{k\to\infty} \varphi_{n_k}(v_k) \geqslant \varphi(v).$$

(M2) For each $v \in D(\varphi)$, there is a sequence $\{v_n\}$ in X such that

$$v_n \to v \quad \text{in } X, \quad \varphi_n(v_n) \to \varphi(v) \quad \text{as } n \to \infty.$$

Now we give a result on very useful characterizations, which is due to H. Attouch [2], for the Mosco convergence of convex functions.

THEOREM 8.1. *Let $\varphi_n, n = 1, 2, \ldots,$ and φ be proper l.s.c. convex functions on X. Then the following statements* (i), (ii) *and* (iii) *are equivalent to each other*:

(i) *$\varphi_n \to \varphi$ on X as $n \to \infty$ in the sense of Mosco.*

(ii) *$R_\lambda^{\partial\varphi_n} u \to R_\lambda^{\partial\varphi} u$ as $n \to \infty$ for each $\lambda > 0$ and each $u \in X$. Moreover, the following normalization condition (N) is satisfied*:

(N) *there are a sequence $[a_n, a_n^*] \in G(\partial\varphi_n)$, $n = 1, 2, \ldots,$ and $[a, a^*] \in G(\partial\varphi)$ such that $a_n \to a$ in X, $a_n^* \to a^*$ in X^* and $\varphi_n(a_n) \to \varphi(a)$ as $n \to \infty$.*

(iii) *$\varphi_{n\lambda}(u) \to \varphi_\lambda(u)$ as $n \to \infty$ for each $\lambda > 0$ and each $u \in X$, where $\varphi_{n\lambda}$ is the Moreau–Yosida approximation with index λ of φ_n.*

COROLLARY 8.2. *$\varphi_n, n = 1, 2, \ldots,$ and φ be proper, l.s.c. convex functions on X. Then each of* (i), (ii) *and* (iii) *of Theorem* 8.1 *is equivalent to the following statement* (iv), *too*:

(iv) *$\partial\varphi_n \to \partial\varphi$ in $X \times X^*$ in the graph sense as $n \to \infty$, and the normalization condition (N) is satisfied.*

For our proof of Theorem 8.1 we prepare the following lemma.

LEMMA 8.2. *Let $\varphi_n, n = 1, 2, \ldots,$ and φ be proper l.s.c. convex functions on X, and suppose that statement* (i) *or* (ii) *of Theorem* 8.1 *is satisfied. Then there is a positive constant C_0 such that*

$$\varphi_n(v) + C_0\big(|v|_X + 1\big) \geqslant 0, \quad \forall v \in X,\ \forall n = 1, 2, \ldots. \tag{8.5}$$

PROOF. First we consider the case when (i) of Theorem 8.1 is satisfied. Assume that the assertion of the lemma is false, namely for each positive integer k there are a point $v_k \in X$ and positive integer $n(k)$ such that

$$\varphi_{n(k)}(v_k) + k\big(|v_k|_X + 1\big) < 0. \tag{8.6}$$

Since $\{n(k)\}_{k=1}^{\infty}$ is a unbounded set of positive integers, we may assume without loss of generality that $n(k) \uparrow \infty$ as $k \to \infty$. We have the following two cases:

(1) $\{v_k\}$ is bounded in X;

(2) $\{v_k\}$ is unbounded in X.

Consider the case (1). Then there is a subsequence of $\{v_k\}$ which weakly converges to some v in X. For simplicity we denote it by $\{v_k\}$ again. By property (M1) of the Mosco convergence (cf. Definition 8.1), we have

$$\liminf_{k\to\infty} \varphi_{n(k)}(v_k) \geqslant \varphi(v) > -\infty. \tag{8.7}$$

On the other hand, it follows from (8.6) that $\liminf_{k\to\infty} \varphi_{n(k)}(v_k) = -\infty$. This contradicts (8.7).

Next consider the case (2). We may assume by extracting a suitable subsequence from $\{v_k\}$ that $|v_k|_X \uparrow \infty$ (as $k \to \infty$). Also, let $u_0 \in D(\varphi)$ and choose a sequence $\{u_k\}$ such that $u_k \in D(\varphi_{n(k)})$, $k = 1, 2, \ldots$, $u_k \to u_0$ in X and $\varphi_{n(k)}(u_k) \to \varphi(u_0)$. We now put

$$t_k := \frac{1}{\sqrt{k}\,|v_k - u_k|_X}, \quad w_k := t_k v_k + (1 - t_k)u_k \quad \text{for large } k.$$

We note that $t_k \to 0$ because of $|v_k - u_k|_X \to \infty$ and

$$|w_k - u_k|_X = t_k|v_k - u_k|_X = \frac{1}{\sqrt{k}} \to 0, \quad \text{and hence } w_k \to u_0 \text{ in } X.$$

Moreover we observe that

$$\begin{aligned}
\varphi_{n(k)}(w_k) &\leqslant t_k\varphi_{n(k)}(v_k) + (1 - t_k)\varphi_{n(k)}(u_k) \\
&\leqslant \frac{-k\,(|v_k|_X + 1)}{\sqrt{k}\,|v_k - u_k|_X} + (1 - t_k)\varphi_{n(k)}(u_k) \\
&\leqslant \frac{-\sqrt{k}\,(|v_k|_X + 1)}{|v_k - u_k|_X} + \varphi(u_0) + 1
\end{aligned}$$

for large k. Hence $\lim_{k\to\infty}\varphi_{n(k)}(w_k) = -\infty$. On the other hand, by property (M1), we have $\liminf_{k\to\infty}\varphi_{n(k)}(w_k) \geqslant \varphi(u_0) > -\infty$, which is a contradiction.

Next consider the case when (ii) of Theorem 8.1 is satisfied. We simply write R_λ^n and R_λ for the resolvents of $\partial\varphi_n$ and $\partial\varphi$, respectively, for each $\lambda > 0$. Fix a positive number λ. Let $[a_n, a_n^*]$, $n = 1, 2, \ldots$, and $[a, a^*]$ be as in condition (N) of (ii) of Theorem 8.1, and put

$$u_n := a_n + \lambda J^{-1}(a_n^*), \qquad u_0 := a + \lambda J^{-1}(a^*).$$

Then we observe that

$$a_n = R_\lambda^n u_n, \qquad \frac{1}{\lambda}J(u_n - a_n) = a_n^* \in \partial\varphi_n(a_n),$$

$$a = R_\lambda u_0, \qquad \frac{1}{\lambda}J(u_0 - a) = a^* \in \partial\varphi(a).$$

By our assumption,

$$u_n \to u_0 \quad \text{in } X, \quad a_n^* = \frac{1}{\lambda}J(u_n - a_n) \to a^* = \frac{1}{\lambda}J(u_0 - a) \quad \text{in } X^*.$$

From the subdifferential inequality it follows that

$$\varphi_n(v) \geqslant \varphi_n(a_n) + \langle a_n^*, v - a_n\rangle, \quad \forall v \in X,\ \forall n = 1, 2, \ldots. \tag{8.8}$$

Hence (8.5) holds for

$$C_0 = \sup_{n\geqslant 1}\left\{\frac{1}{\lambda}|u_n - a_n|_X + \left|\langle a_n^*, a_n\rangle\right| + \left|\varphi_n(a_n)\right|\right\},$$

which is finite. □

PROOF OF (i) → (ii) OF THEOREM 8.1. By assumption, take some $a \in D(\varphi)$ and a sequence $\{a_n\}$ in X such that $a_n \in D(\varphi_n)$, $a_n \to a$ in X and $\varphi_n(a_n) \to \varphi(a)$ (as $n \to \infty$). Now, let λ be any positive number and u be any point in X. First, we show that $\{R_\lambda^n u\}$ is bounded in X, where we write simply R_λ^n for the resolvent $R_\lambda^{\partial\varphi_n}$. Also, we put $u_n := R_\lambda^n u$. Our claim is to show that $u_n \to R_\lambda u$ (as $n \to \infty$), where R_λ denotes the resolvent $R_\lambda^{\partial\varphi}$. Since $\frac{1}{\lambda}J(u - u_n) \in \partial\varphi_n(u_n)$, it follows that

$$\begin{aligned}\varphi_n(a_n) &\geqslant \varphi_n(u_n) + \left\langle \frac{1}{\lambda}J(u - u_n), a_n - u_n\right\rangle \\ &= \varphi_n(u_n) + \left\langle \frac{1}{\lambda}J(u - u_n), a_n - u\right\rangle + \frac{1}{\lambda}|u - u_n|_X^2,\end{aligned}$$

from which we derive with the help of (8.5) of Lemma 8.2 that

$$\varphi_n(a_n) + C_0\big(|u_n|_X + 1\big) + \frac{1}{2\lambda}|a_n - u|_X^2 \geqslant \frac{1}{2\lambda}|u - u_n|_X^2 \tag{8.9}$$

and

$$\varphi_n(u_n) \leqslant \varphi_n(a_n) + \frac{1}{2\lambda}|a_n - u|_X^2; \quad \text{hence } \limsup_{n\to\infty} \varphi_n(u_n) < \infty. \tag{8.10}$$

Now (8.9) shows that $\{u_n\}$ is bounded in X and (8.10) shows that $\{\varphi_n(u_n)\}$ is bounded for each $\lambda > 0$.

Next we prove that $u_n \to u_0 := R_\lambda u$ in X. This is shown as follows. By the boundedness of $\{u_n\}$ in X, we can extract a subsequence $\{u_{n_k}\}$ from $\{u_n\}$ such that $u_{n_k} \to \tilde{u}_0$ weakly in X as $k \to \infty$ for a certain $\tilde{u}_0 \in X$. Here we note by (8.10) that $\tilde{u}_0 \in D(\varphi)$, because

$$\varphi(\tilde{u}_0) \leqslant \liminf_{k\to\infty} \varphi_{n_k}(u_{n_k}) < \infty.$$

Let v be any element of $D(\varphi)$ and take a sequence $\{v_k\}$ such that $v_k \to v$ in X and $\varphi_{n_k}(v_k) \to \varphi(v)$ as $k \to \infty$. We observe from the definition of $\varphi_{n\lambda}$ that

$$\varphi_{n_k}(u_{n_k}) + \frac{1}{2\lambda}|u - u_{n_k}|_X^2 \leqslant \varphi_{n_k}(v_k) + \frac{1}{2\lambda}|u - v_k|_X^2,$$

whence

$$\begin{aligned}
\varphi(\tilde{u}_0) + \frac{1}{2\lambda}|u - \tilde{u}_0|_X^2 &\leqslant \liminf_{k\to\infty} \varphi_{n_k}(u_{n_k}) + \liminf_{k\to\infty} \frac{1}{2\lambda}|u - u_{n_k}|_X^2 \\
&\leqslant \liminf_{k\to\infty} \left\{ \varphi_{n_k}(u_{n_k}) + \frac{1}{2\lambda}|u - u_{n_k}|_X^2 \right\} \\
&\leqslant \limsup_{k\to\infty} \left\{ \varphi_{n_k}(u_{n_k}) + \frac{1}{2\lambda}|u - u_{n_k}|_X^2 \right\} \\
&\leqslant \limsup_{k\to\infty} \left\{ \varphi_{n_k}(v_k) + \frac{1}{2\lambda}|u - v_k|_X^2 \right\} \\
&= \lim_{k\to\infty} \varphi_{n_k}(v_k) + \lim_{k\to\infty} \frac{1}{2\lambda}|u - v_k|_X^2 \\
&= \varphi(v) + \frac{1}{2\lambda}|u - v|_X^2
\end{aligned}$$

for every $v \in X$. This implies that $\tilde{u}_0 = R_\lambda u$. In particular, if we take $v = u_0 := R_\lambda u$ and repeat the same computation as above, then we obtain

$$\liminf_{k\to\infty} \varphi_{n_k}(u_{n_k}) = \varphi(u_0), \qquad \lim_{k\to\infty} \frac{1}{2\lambda}|u - u_{n_k}|_X^2 = \frac{1}{2\lambda}|u - u_0|_X^2. \tag{8.11}$$

Noting that $u_{n_k} \to u_0$ weakly in X as $k \to \infty$, by condition (R) we infer from the last relation of (8.11) that $u_{n_k} \to u_0$ (strongly) in X. Moreover, since the limits $\varphi(u_0)$ and $\frac{1}{2\lambda}|u - u_0|_X^2$ are independent of the choice of subsequence of $\{u_n\}$, we conclude that the

whole sequence $\{u_n\}$ converges to u_0 in X, and the first relation of (8.11) for the whole sequence $\{u_n\}$ gives the normalization condition (N). □

PROOF OF (ii) → (iii) OF THEOREM 8.1. We use the same simple notations R_λ^n and R_λ as in the proof of (i) → (ii). Let λ be any positive number and fix it. Let $[a_n, a_n^*]$, $n = 1, 2, \ldots$, and $[a, a^*]$ be as in the normalization condition (N), and put $u_n := a_n + \lambda J^{-1}(a_n^*)$ as well as $u_0 := a + \lambda J^{-1}(a^*)$; note that $a_n = R_\lambda^n u_n$, $a = R_\lambda u_0$ and $u_n \to u_0$ in X (as $n \to \infty$). Then we first claim that

$$\varphi_{n\lambda}(u_0) \to \varphi_\lambda(u_0). \tag{8.12}$$

To this end we observe that

$$\varphi_{n\lambda}(u_n) = \frac{1}{2\lambda}|u_n - a_n|_X^2 + \varphi_n(a_n) \to \frac{1}{2\lambda}|u_0 - a|_X^2 + \varphi(a) = \varphi_\lambda(u_0). \tag{8.13}$$

Moreover we notice that for each $\lambda > 0$ and each bounded set E in X there is a positive constant $C(\lambda, E)$ such that

$$\big|\partial\varphi_{n\lambda}(v)\big|_{X^*} \leqslant C(\lambda, E), \quad \big|R_\lambda^n(v)\big|_X \leqslant C(\lambda, E), \quad \forall v \in E. \tag{8.14}$$

In fact, by $\frac{1}{\lambda}J(v - R_\lambda^n v) = \partial\varphi_{n\lambda}(v)$ and (8.5) of Lemma 8.1 we have

$$\begin{aligned}
\varphi_{n\lambda}(u_n) &\geqslant \varphi_{n\lambda}(v) + \left\langle \frac{1}{\lambda} J\big(v - R_\lambda^n v\big), u_n - v \right\rangle \\
&= \varphi_n\big(R_\lambda^n v\big) + \frac{1}{2\lambda}\big|v - R_\lambda^n v\big|_X^2 + \left\langle \frac{1}{\lambda} J\big(v - R_\lambda^n v\big), u_n - v \right\rangle \\
&\geqslant -C_0\big(\big|R_\lambda^n v\big|_X + 1\big) + \frac{1}{2\lambda}\big|v - R_\lambda^n v\big|_X^2 + \left\langle \frac{1}{\lambda} J\big(v - R_\lambda^n v\big), u_n - v \right\rangle.
\end{aligned}$$

From the above relations we easily derive that

$$\frac{1}{4\lambda}\big|v - R_\lambda^n v\big|_X^2 \leqslant \frac{2}{\lambda}|u_n - v|_X^2 + 2\lambda C_0^2 + C_0\big(|v|_X + 1\big) + \varphi_n(u_n).$$

Given a bounded set E in X, if we take

$$C_1(\lambda, E) := 4 \sup_{n \geqslant 1,\, v \in E} \left\{ \frac{2}{\lambda^2}|u_n - v|_X^2 + 2C_0^2 + \frac{C_0}{\lambda}\big(|v|_X + 1\big) + \frac{1}{\lambda}\varphi_n(u_n) \right\},$$

then we get

$$\left|\frac{1}{\lambda}\big(v - R_\lambda^n v\big)\right|_X = \big|\partial\varphi_{n\lambda}(v)\big|_{X^*} \leqslant \sqrt{C_1(\lambda, E)}, \quad \forall v \in E,$$

and

$$\left|R_\lambda^n v\right|_X \leqslant \sup_{v\in E}|v|_X + \lambda\sqrt{C_1(\lambda, E)} =: C_2(\lambda, E), \quad \forall v \in E.$$

Therefore (8.14) holds for $C(\lambda, E) := \sqrt{C_1(\lambda, E)} + C_2(\lambda, E)$. Also, we use the integral formula (cf. (d) of Lemma 8.1)

$$\varphi_{n\lambda}(w) = \varphi_{n\lambda}(v) + \int_0^1 \left\langle \partial\varphi_{n\lambda}\big(v + t(w - v)\big), w - v\right\rangle \mathrm{d}t, \quad \forall v, w \in X. \tag{8.15}$$

Now, by (8.15) with $w = u_0$ and $v = u_n$ we get

$$\left|\varphi_{n\lambda}(u_0) - \varphi_{n\lambda}(u_n)\right| \leqslant C(\lambda, E)|u_n - u_0|_X \to 0 \quad \text{as } n \to \infty,$$

where E is a closed, convex and bounded subset of X including all of $u_n, n = 1, 2, \ldots,$ and u_0. Combining this with (8.13), we obtain (8.12). Finally, by (8.15) again it follows for any $u \in X$ that

$$\begin{aligned}\lim_{n\to\infty}\varphi_{n\lambda}(u) &= \lim_{n\to\infty}\varphi_{n\lambda}(u_0) + \lim_{n\to\infty}\int_0^1 \left\langle \partial\varphi_{n\lambda}\big(u_0 + t(u - u_0)\big), u - u_0\right\rangle \mathrm{d}t \\ &= \varphi_\lambda(u_0) + \int_0^1 \left\langle \partial\varphi_\lambda\big(u_0 + t(u - u_0)\big), u - u_0\right\rangle \mathrm{d}t \\ &= \varphi_\lambda(u).\end{aligned}$$

We applied the Lebesgue's dominated convergence theorem to see the second equality in the above relations; in fact, we have by (8.14)

$$\left|\partial\varphi_{n\lambda}\big(u_0 + t(u - u_0)\big)\right|_{X^*} \leqslant C(\lambda, E), \quad \forall t \in [0, 1],\ \forall n = 1, 2, \ldots,$$

where $E := \{u_0 + t(u - u_0);\ 0 \leqslant t \leqslant 1\}$, and by our assumption (ii)

$$\partial\varphi_{n\lambda}\big(u_0 + t(u - u_0)\big) \to \partial\varphi_\lambda\big(u_0 + t(u - u_0)\big) \quad \text{in } X^*,\ \forall t \in [0, 1]. \qquad \square$$

The implication (iii) $\to$ (i) of Theorem 8.1 is the most crucial step. We prepare some lemmas.

LEMMA 8.3. *Let $\{\eta_n\}$ be a sequence of convex functions of C^1-class on $(0, \infty)$ which converges pointwise on $(0, \infty)$ to a convex function η of C^1-class on $(0, \infty)$ as $n \to \infty$. Then, η_n' converges pointwise to η' on $(0, \infty)$, where η_n' and η' are the first derivatives of η_n and η, respectively.*

PROOF. We note from the convexity of η_n and η (cf. [17, Chapter 1]) that for any $0 < x_1 < x_2 < x < y < y_1 < y_2 < \infty$

$$\frac{\eta_n(x_2) - \eta_n(x_1)}{x_2 - x_1} \leqslant \eta_n'(x) \leqslant \eta_n'(y) \leqslant \frac{\eta_n(y_2) - \eta_n(y_1)}{y_2 - y_1}$$

as well as

$$\frac{\eta(x_2)-\eta(x_1)}{x_2-x_1} \leqslant \eta'(x) \leqslant \eta'(y) \leqslant \frac{\eta(y_2)-\eta(y_1)}{y_2-y_1}.$$

Since

$$\frac{\eta_n(x_2)-\eta_n(x_1)}{x_2-x_1} \to \frac{\eta(x_2)-\eta(x_1)}{x_2-x_1} \quad \text{and} \quad \frac{\eta_n(y_2)-\eta_n(y_1)}{y_2-y_1} \to \frac{\eta(y_2)-\eta(y_1)}{y_2-y_1},$$

we see from the above inequalities that $\{\eta_n'\}$ a uniformly bounded family of increasing functions on each compact interval of $(0,\infty)$. Hence, with the help of the Ascoli–Arzela's theorem we can show that η_n converges uniformly to η on each compact interval of $(0,\infty)$. Here, according to Helly's theorem of choice, it is possible to extract a subsequence $\{\eta_{n_k}'\}$ from $\{\eta_n'\}$ which converges pointwise to an increasing function on $(0,\infty)$. In this case it is easy to see that the limit function is η'. Consequently we conclude that the whole sequence $\{\eta_n'\}$ converges pointwise to η' on $(0,\infty)$. □

LEMMA 8.4. *Let φ be a proper, l.s.c. convex function on X. Then we have*:

(a) *For each fixed $v \in X$, the function $\lambda \to R_\lambda^{\partial\varphi} v$ is continuous from $(0,\infty)$ into X and $\lambda \to \varphi(R_\lambda^{\partial\varphi} v)$ is continuous on $(0,\infty)$.*

(b) *For each fixed $v \in X$, the function $\lambda \to -\lambda\varphi_\lambda(v)$ is convex and of C^1-class on $(0,\infty)$, and*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\{\lambda\varphi_\lambda(v)\} = J\big(R_\lambda^{\partial\varphi} v\big), \quad \forall \lambda > 0.$$

PROOF. We prove (a). Write simply R_λ for $R_\lambda^{\partial\varphi}$. Let $v \in X$ and $\{\lambda_n\}$ be a sequence of positive numbers such that $\lambda_n \to \lambda$ for a positive number λ. Then, recalling (7.7) in the proof of Theorem 7.1, we see that $\{R_{\lambda_n} v\}_{n=1}^\infty$ is bounded in X. Therefore, there is a subsequence $\{\lambda_{n_k}\}$ such that $v_k := R_{\lambda_{n_k}} v \to v_0$ weakly in X. By definition, v_k satisfies that

$$\begin{aligned}\liminf_{k\to\infty} \varphi_{\lambda_{n_k}}(v) &= \liminf_{k\to\infty}\left\{\varphi(v_k) + \frac{1}{2\lambda_{n_k}}|v-v_k|_X^2\right\} \\ &\geqslant \varphi(v_0) + \frac{1}{2\lambda}|v-v_0|_X^2 \geqslant \varphi_\lambda(v).\end{aligned}$$

Combining these inequalities with the following one:

$$\begin{aligned}\limsup_{k\to\infty} \varphi_{\lambda_{n_k}}(v) &\leqslant \limsup_{k\to\infty}\left\{\varphi(R_\lambda v) + \frac{1}{2\lambda_{n_k}}|v-R_\lambda v|_X^2\right\} \\ &= \limsup_{k\to\infty}\left\{\varphi_\lambda(v) + \frac{1}{2}\left(\frac{1}{\lambda_{n_k}} - \frac{1}{\lambda}\right)|v-R_\lambda v|_X^2\right\} = \varphi_\lambda(v),\end{aligned}$$

we conclude that $\varphi_\lambda(v) = \varphi(v_0) + \frac{1}{2\lambda}|v - v_0|_X^2$, namely $v_0 = R_\lambda v$. This shows that the whole sequence $\{R_{\lambda_n} v\}$ weakly converges to $R_\lambda v$ in X as $n \to \infty$. Moreover, as a by-product of the above argument we obtain that

$$\lim_{n\to\infty} \varphi(R_{\lambda_n} v) = \varphi(R_\lambda v), \qquad \lim_{n\to\infty} \frac{1}{2\lambda_n}|v - R_{\lambda_n} v|_X^2 = \frac{1}{2\lambda}|v - R_\lambda v|_X^2.$$

Accordingly it turns out that $R_{\lambda_n} v \to R_\lambda v$ in X as $n \to \infty$. Thus (a) has been seen.

Next we give a proof of (b). Let $v \in X$. For $\lambda_i > 0$, $i = 1, 2$, and $r \in (0, 1)$, one observes that

$$\begin{aligned}
&r\lambda_1\varphi_{\lambda_1}(v) + (1-r)\lambda_2\varphi_{\lambda_2}(v) \\
&\quad \leqslant r\left\{\lambda_1\varphi(R_{(r\lambda_1+(1-r)\lambda_2)}v) + \frac{1}{2}|v - R_{(r\lambda_1+(1-r)\lambda_2)}v|_X^2\right\} \\
&\qquad + (1-r)\left\{\lambda_2\varphi(R_{(r\lambda_1+(1-r)\lambda_2)}v) + \frac{1}{2}|v - R_{(r\lambda_1+(1-r)\lambda_2)}v|_X^2\right\} \\
&\quad = \big(r\lambda_1 + (1-r)\lambda_2\big)\varphi_{(r\lambda_1+(1-r)\lambda_2)}(v).
\end{aligned}$$

This shows that $\lambda \to -\lambda\varphi_\lambda(v)$ is finite and convex on $(0, \infty)$, and hence it is locally Lipschitz continuous on $(0, \infty)$.

Let $v \in X$, $\lambda_i > 0$, $i = 1, 2$, again. Then, we note that

$$\begin{aligned}
\lambda_1\varphi_{\lambda_1}(v) - \lambda_2\varphi_{\lambda_2}(v) &\leqslant \left\{\lambda_1\varphi_{\lambda_1}(R_{\lambda_2}v) + \frac{1}{2}|v - R_{\lambda_2}v|_X^2\right\} \\
&\qquad - \left\{\lambda_2\varphi_{\lambda_2}(R_{\lambda_2}v) + \frac{1}{2}|v - R_{\lambda_2}v|_X^2\right\} \\
&= (\lambda_1 - \lambda_2)\varphi(R_{\lambda_2}v),
\end{aligned}$$

and similarly

$$\lambda_1\varphi_{\lambda_1}(v) - \lambda_2\varphi_{\lambda_2}(v) \geqslant (\lambda_1 - \lambda_2)\varphi(R_{\lambda_1}v).$$

From the above inequalities we derive that

$$(\lambda_1 - \lambda_2)\varphi(R_{\lambda_1}v) \leqslant \lambda_1\varphi_{\lambda_1}(v) - \lambda_2\varphi_{\lambda_2}(v) \leqslant (\lambda_1 - \lambda_2)\varphi(R_{\lambda_2}v). \tag{8.16}$$

Since $\varphi(R_\lambda v)$ is continuous in $\lambda \in (0, \infty)$ on account of (a) of this lemma, it follows by dividing (8.16) by $\lambda_1 - \lambda_2$ and by letting $\lambda_2 \to \lambda_1$ that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\big\{\lambda\varphi_\lambda(v)\big\}_{|\lambda=\lambda_1} = \varphi(R_{\lambda_1}v).$$

This is nothing but what we want to require. □

PROOF OF (III) → (I) OF THEOREM 8.1. Assuming that (iii) of Theorem 8.1, we verify conditions (M1) and (M2) in the definition of Mosco convergence for $\{\varphi_n\}$. We use still simple notations R_λ^n and R_λ for $R_\lambda^{\partial\varphi_n}$ and $R_\lambda^{\partial\varphi}$, respectively. Prior to these verifications we mention the following important remark which results from our assumption (iii) with the help of Lemmas 8.3 and 8.4. That is:

$$\begin{cases} \partial\varphi_{n\lambda}(u) \to \partial\varphi_\lambda(u) \quad \text{in } X^* \text{ as } n \to \infty,\ \forall u \in X,\ \forall \lambda > 0, \\ \text{or equivalently} \\ R_\lambda^n u \to R_\lambda u \quad \text{in } X \text{ as } n \to \infty,\ \forall u \in X,\ \forall \lambda > 0. \end{cases} \tag{8.17}$$

In fact, apply Lemma 8.3 for $\eta_n(\lambda) = -\lambda\varphi_{n\lambda}(u)$ and $\eta(\lambda) = -\lambda\varphi_\lambda(u)$. Then, from our assumption (iii) and Lemma 8.4 it follows that $\eta_n(\lambda) \to \eta(\lambda)$ for each $\lambda > 0$, and hence turns out that

$$\eta_n'(\lambda) = \varphi_n\big(R_\lambda^n u\big) \to \eta'(\lambda) = \varphi(R_\lambda u) \quad \text{pointwise in } \lambda \in (0, \infty). \tag{8.18}$$

Now, since

$$\varphi_{n\lambda}(u) = \varphi_n\big(R_\lambda^n u\big) + \frac{1}{2\lambda}\big|u - R_\lambda^n u\big|_X^2 \to \varphi_\lambda(u) = \varphi(R_\lambda u) + \frac{1}{2\lambda}|u - R_\lambda u|_X^2, \tag{8.19}$$

our assumption (iii) and (8.18) imply that

$$\lim_{n\to\infty} \frac{1}{2\lambda}\big|u - R_\lambda^n u\big|_X^2 = \frac{1}{2\lambda}|u - R_\lambda u|_X^2;$$

hence

$$\lim_{n\to\infty} \frac{\lambda}{2}\big|\partial\varphi_{n\lambda}(u)\big|_{X^*}^2 = \frac{\lambda}{2}\big|\partial\varphi_\lambda(u)\big|_{X^*}^2$$

and $\{\partial\varphi_{n\lambda}(u)\}$ is bounded in X. Therefore, according to condition (R), in order to obtain (8.17) it is enough to check that $\partial\varphi_{n\lambda}(u)$ weakly converges to $\partial\varphi_\lambda(u)$ in X^* as $n \to \infty$. This is done as follows. By the boundedness of $\{\partial\varphi_{n\lambda}(u)\}$, there is a weak limit point $\tilde{u}$ of this sequence. Take a subsequence $\{\partial\varphi_{n_k\lambda}(u)\}$ which weakly converges to $\tilde{u}$ in X as $k \to \infty$. Then, by the subdifferential inequality for $\partial\varphi_{n_k\lambda}$ we have

$$\varphi_{n_k\lambda}(v) \geqslant \varphi_{n_k\lambda}(u) + \big\langle\partial\varphi_{n_k\lambda}(u), v - u\big\rangle, \quad \forall v \in X.$$

Letting $k \to \infty$ in this inequality, we see by our assumption (iii) that

$$\varphi_\lambda(v) \geqslant \varphi_\lambda(u) + \langle\tilde{u}, v - u\rangle, \quad \forall v \in X,$$

which implies that $\tilde{u} = \partial\varphi_\lambda(u)$. Therefore we can conclude that the whole sequence $\{\partial\varphi_{n\lambda}(u)\}$ weakly converges to $\partial\varphi_\lambda(u)$ in X^* as $n \to \infty$. Thus (8.17) has been proved.

We now derive (i) from (iii). Fix $u \in D(\varphi)$. By our assumption (iii) and (b) of Lemma 8.1 we have that

$$\varphi_{n\lambda}(u) \to \varphi_\lambda(u) \quad \text{as } n \to \infty,\ \forall \lambda > 0,$$

and

$$\varphi_\lambda(u) \to \varphi(u) \quad \text{as } \lambda \downarrow 0.$$

Hence, by the usual diagonal argument we can find a sequence $\{\lambda_n\}$ of positive numbers tending to 0 (as $n \to \infty$) such that $\varphi_{n\lambda_n}(u) \to \varphi(u)$. In this case, by (8.17) we have

$$\begin{aligned} \infty > \varphi(u) &= \lim_{n\to\infty} \varphi_{n\lambda_n}(u) = \lim_{n\to\infty} \left\{ \varphi_n(R^n_{\lambda_n} u) + \frac{\lambda_n}{2} |\partial \varphi_{n\lambda_n} u|^2_X \right\} \\ &\geqslant \limsup_{n\to\infty} \varphi_n\big(R^n_{\lambda_n} u\big). \end{aligned} \tag{8.20}$$

Next, let $\{n_k\}$ be any subsequence in $\{n\}$ and $\{v_k\}$ be any sequence in X such that $v_k \to v$ weakly in X as $k \to \infty$ for some $v \in X$. Then, by the subdifferential inequality for $\partial\varphi_{n_k\lambda}$ we get

$$\varphi_{n_k}(v_k) \geqslant \varphi_{n_k\lambda}(v_k) \geqslant \varphi_{n_k\lambda}(v) + \big\langle \partial\varphi_{n_k\lambda}(v), v_k - v \big\rangle.$$

Now, pass to the limit $k \to \infty$ to obtain

$$\liminf_{k\to\infty} \varphi_{n_k}(v_k) \geqslant \varphi_\lambda(v), \quad \forall \lambda > 0;$$

we used here the strong convergence property (8.17). Furthermore, letting $\lambda \downarrow 0$ yields

$$\liminf_{k\to\infty} \varphi_{n_k}(v_k) \geqslant \varphi(v). \tag{8.21}$$

Thus conditions (M1) and (M2) for the Mosco convergence of $\{\varphi_n\}$ have been checked by (8.20), (8.21) and (8.17). □

The assertion of Corollary 8.2 is an immediate consequence of Theorems 7.2 and 8.1.

9. Variational inequalities

In this section we discuss variational inequalities for a concrete class of pseudo-monotone mappings and monotone mappings in Sobolev spaces. We use the same notations for function spaces $W^{1,p}(\Omega)$ and its trace space $W^{1/q,p}(\Gamma)$, $1/p + 1/q = 1$, $W^{1,p}_0(\Omega)$ and their dual spaces as in Section 1, supposing as well that Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary Γ.

Let us consider one of typical examples of pseudo-monotone mappings from $X := W_0^{1,p}(\Omega)$ or $W^{1,p}(\Omega)$, $1 < p < +\infty$, into X^*.

Let $a_0(x,\zeta)$ and $a_i(x,\zeta,\xi)$, $i = 1, 2, \ldots, N$, be functions on $\Omega \times \mathbf{R}$ and $\Omega \times \mathbf{R} \times \mathbf{R}^N$, respectively, satisfying that

(a1) for all $(\zeta,\xi) \in \mathbf{R} \times \mathbf{R}^N$ the functions $x \to a_0(x,\zeta)$ and $x \to a_i(x,\zeta,\xi)$ are measurable on Ω;

(a2) for a.e. $x \in \Omega$ the functions $\zeta \to a_0(x,\zeta)$ and $(\zeta,\xi) \to a_i(x,\zeta,\xi)$ are continuous on $\mathbf{R}$ and $\mathbf{R} \times \mathbf{R}^N$, respectively;

(a3) there are positive constants c_0 and c_1 such that

$$c_0\big(|\zeta|^{p-1} - 1\big) \leqslant a_0(x,\zeta) \leqslant c_1\big(|\zeta|^{p-1} + 1\big), \tag{9.1}$$

$$c_0\big(|\xi|^{p-1} - 1\big) \leqslant a_i(x,\zeta,\xi) \leqslant c_1\big(|\xi|^{p-1} + 1\big), \quad i = 1, 2, \ldots, N,$$
$$\text{a.e. } x \in \Omega,\ \forall \zeta \in \mathbf{R},\ \forall \xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbf{R}^N; \tag{9.2}$$

(a4) the following monotonicity property is satisfied:

$$\sum_{i=1}^{N} \big(a_i(x,\zeta,\xi) - a_i(x,\zeta,\bar{\xi})\big)(\xi_i - \bar{\xi}_i) \geqslant 0, \quad \text{a.e. } x \in \Omega,\ \forall \zeta \in \mathbf{R},$$
$$\forall \xi = (\xi_1, \xi_2, \ldots, \xi_N),\ \forall \bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_N) \in \mathbf{R}^N. \tag{9.3}$$

We define a singlevalued mapping $\tilde{A}: D(\tilde{A}) = X \times X \to X^*$ by putting

$$\big\langle \tilde{A}(v,u), w \big\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, v, \nabla u) w_{x_i}\, \mathrm{d}x + \int_\Omega a_0(x,v) w\, \mathrm{d}x,$$
$$\forall u, v, w \in X; \tag{9.4}$$

note here that functions $a_i(x, v, \nabla u)$ and $a_0(x, v)$ are measurable on Ω by conditions (a1) and (a2), and the integrals in the left side of (9.4) are integrable by condition (a3). It is clear by (9.1) and (9.2) that $\tilde{A}$ is bounded and continuous from $X \times X$ into X^*.

PROPOSITION 9.1. *The mapping $\tilde{A}$ given by* (9.4) *is semi-monotone from $X \times X$ into X^*. Hence $A: D(A) = X \to X^*$ given by $Au = \tilde{A}(u,u)$ for all $u \in X$ is pseudo-monotone. Moreover, for any $v_0 \in X$,*

$$\frac{\langle Av, v - v_0 \rangle}{|v|_X} \to +\infty \quad \text{as } |v|_X \to +\infty. \tag{9.5}$$

PROOF. We check two properties (SM1) and (SM2) in Definition 2.3. Clearly $\tilde{A}$ is continuous from $X \times X$ into X^*. For any fixed $v \in X$, the mapping $\tilde{A}(v,\cdot): D(\tilde{A}(v,\cdot)) = X \to X^*$ is monotone by (9.3) of (a4). Therefore it follows from Corollary 4.3 that $\tilde{A}(v,\cdot)$ is maximal monotone from X into X^*. Thus (SM1) is satisfied.

Next we check (SM2). Let $\{v_n\}$ be a sequence in X such that $v_n \to v$ weakly in X and let u be any function in X. Since the embedding from X into $L^p(\Omega)$ is compact, we have $v_n \to v$ in $L^p(\Omega)$, so that

$$a_i(\cdot, v_n, \nabla u) \to a_i(\cdot, v, \nabla u), \quad i = 1, 2, \ldots, N,$$
$$a_0(\cdot, v_n) \to a_0(\cdot, v) \quad \text{in } L^q(\Omega),$$

which implies that $\tilde{A}(v_n, u) \to \tilde{A}(v, u)$ in X^*. Thus (SM2) holds, and (9.5) immediately follows from (9.1) and (9.2). □

Now we consider some variational inequalities for the mapping A constructed from the semi-monotone mapping $\tilde{A}$ given by (9.4).

EXAMPLE 9.1 (Interior obstacle problem). Let $X := W_0^{1,p}(\Omega)$, $1 < p < +\infty$, and

$$K_i(v_c) = \{v \in X;\ v \geqslant v_c \text{ a.e. on } \Omega\},$$

where v_c is a function given in X. Clearly $K_i(v_c)$ is non-empty, closed and convex in X. Therefore, by virtue of Theorem 5.2, for any $f \in L^q(\Omega)$, $1/p + 1/q = 1$, there is a function $u \in X$ such that

$$u \in K_i(v_c); \tag{9.6-1}$$
$$\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u)(u_{x_i} - v_{x_i})\,\mathrm{d}x + \int_\Omega a_0(x, u)(u - v)\,\mathrm{d}x$$
$$\leqslant \int_\Omega f(u - v)\,\mathrm{d}x, \quad \forall v \in K_i(v_c). \tag{9.6-2}$$

The variational inequality (9.6) is equivalent to the following system:

$$u \in K_i(v_c), \tag{9.7-1}$$
$$\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u)\eta_{x_i}\,\mathrm{d}x + \int_\Omega a_0(x, u)\eta\,\mathrm{d}x \geqslant \int_\Omega f\eta\,\mathrm{d}x,$$
$$\forall \eta \in X \text{ with } \eta \geqslant 0 \text{ a.e. on } \Omega, \tag{9.7-2}$$
$$\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u)(u_{x_i} - v_{cx_i})\,\mathrm{d}x + \int_\Omega a_0(x, u)(u - v_c)\,\mathrm{d}x$$
$$= \int_\Omega f(u - v_c)\,\mathrm{d}x. \tag{9.7-3}$$

In fact, taking as v in (9.6-2) any function of the form $u + \eta$ with $\eta \in X$ and $\eta \geqslant 0$ a.e. on Ω, we get (9.7-2). Also, taking as v in (9.6-2) functions v_c and $2u - v_c$, we have (9.7-3).

Conversely we derive from (9.7) that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)(u_{x_i}-v_{x_i})\,\mathrm{d}x+\int_{\Omega} a_0(x,u)(u-v)\,\mathrm{d}x-\int_{\Omega} f(u-v)\,\mathrm{d}x\\
&\quad=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)(u_{x_i}-v_{cx_i})\,\mathrm{d}x+\int_{\Omega} a_0(x,u)(u-v_c)\,\mathrm{d}x\\
&\qquad-\int_{\Omega} f(u-v_c)\,\mathrm{d}x+\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)(v_{cx_i}-v_{x_i})\,\mathrm{d}x\\
&\qquad+\int_{\Omega} a_0(x,u)(v_c-v)\,\mathrm{d}x-\int_{\Omega} f(v_c-v)\,\mathrm{d}x\leqslant 0,
\end{aligned}$$

for every $v\in K_i(v_c)$. Thus (9.6) holds. Furthermore (9.7) is formally equivalent to the following system:

$$\begin{cases}
u\geqslant v_c \quad \text{on } \Omega, \quad u=0 \text{ on } \Gamma,\\
-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u,\nabla u)+a_0(x,u)\geqslant f \quad \text{on } \Omega,\\
(-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u,\nabla u)+a_0(x,u)-f)(u-v_c)=0 \quad \text{on } \Omega.
\end{cases}\tag{9.8}$$

EXAMPLE 9.2 (Gradient obstacle problem). Let $X:=W_0^{1,p}(\Omega)$, $1<p<\infty$, and

$$K_g(v_g)=\{v\in X;\ |\nabla v|\leqslant v_g \text{ a.e. on } \Omega\},$$

where v_g is a given non-negative function in $L^p(\Omega)$. Then $K_g(v_g)$ is non-empty, closed and convex in X. By Theorem 5.2, for any $f\in L^q(\Omega)$ there is a function $u\in X$ such that

$$u\in K_g(v_g),\tag{9.9-1}$$

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)(u_{x_i}-v_{x_i})\,\mathrm{d}x+\int_{\Omega} a_0(x,u)(u-v)\,\mathrm{d}x\\
&\quad\leqslant\int_{\Omega} f(u-v)\,\mathrm{d}x, \quad \forall v\in K_g(v_g).
\end{aligned}\tag{9.9-2}$$

EXAMPLE 9.3 (Boundary obstacle problem). Let $X=W^{1,p}(\Omega)$, $1<p<+\infty$, and $\hat{v}_c\in W^{1/q,p}(\Gamma)$, $1/p+1/q=1$. Then we put

$$K_b(\hat{v}_c)=\{v\in X;\ \hat{v}\geqslant\hat{v}_c \text{ a.e. on } \Gamma\},\tag{9.10}$$

where $\hat{v}$ is the trace of $v \in X$ on Γ. Also, given $f \in L^q(\Omega)$ and $h \in L^q(\Gamma)$, we define an element $g \in X^*$ by

$$\langle g, w\rangle = \int_\Omega f w \,\mathrm{d}x + \int_\Gamma h\widehat{w} \,\mathrm{d}x, \quad \forall w \in X. \tag{9.11}$$

Then, by Theorem 5.2 again there exists a function $u \in X$ such that

$$u \in K_b(\hat{v}_c), \tag{9.12-1}$$

$$\sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)(u_{x_i} - v_{x_i}) \,\mathrm{d}x + \int_\Omega a_0(x, u)(u - v) \,\mathrm{d}x$$

$$\leqslant \int_\Omega f(u - v) \,\mathrm{d}x + \int_\Gamma h(\hat{u} - \hat{v}) \,\mathrm{d}\Gamma, \quad \forall v \in K_b(\hat{v}_c). \tag{9.12-2}$$

Take as v in (9.12-2) any function $u + \eta$ with $\eta \in X$ and $\eta \geqslant 0$ a.e. on Γ to obtain

$$\sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)\eta_{x_i} \,\mathrm{d}x + \int_\Omega a_0(x, u)\eta \,\mathrm{d}x \geqslant \int_\Omega f\eta \,\mathrm{d}x + \int_\Gamma h\hat{\eta} \,\mathrm{d}\Gamma,$$

$$\forall \eta \in X \text{ with } \hat{\eta} \geqslant 0 \text{ a.e. on } \Gamma. \tag{9.13}$$

In particular, if we take any $\eta \in W_0^{1,p}(\Omega)$, then (9.13) implies

$$-\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u) = f \quad \text{a.e. on } \Omega, \tag{9.14}$$

so that

$$\mathbf{a}(u) := \big(a_1(\cdot, u, \nabla u), a_2(\cdot, u, \nabla u), \ldots, a_N(\cdot, u, \nabla u)\big) \in \mathcal{E}^q(\Omega).$$

Therefore the generalized Green's formula holds (cf. (5) of Section 1):

$$\int_\Omega \big(\operatorname{div} \mathbf{a}(u)\big)v \,\mathrm{d}x + \int_\Omega \mathbf{a}(u) \cdot \nabla u \,\mathrm{d}x = \big\langle \mathbf{Ba}(u), \hat{v}\big\rangle_\Gamma, \quad \forall v \in X. \tag{9.15}$$

Using (9.15), we derive from (9.13) and (9.14) that

$$\big\langle \mathbf{Ba}(u), \hat{\eta}\big\rangle_\Gamma \geqslant \int_\Gamma h\hat{\eta} \,\mathrm{d}\Gamma, \quad \forall \hat{\eta} \in X \text{ with } \eta \geqslant 0 \text{ a.e. on } \Gamma. \tag{9.16}$$

Next, choose a function $v_c \in X$ such that $v_c = \hat{v}_c$ a.e. on Γ, and take v_c and $2u - v_c$ as v in (9.12-2). Then we immediately see that

$$\sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)(u_{x_i} - v_{cx_i}) \,\mathrm{d}x + \int_\Omega a_0(x, u)(u - v_c) \,\mathrm{d}x$$

$$= \int_\Omega f(u - v_c)\,\mathrm{d}x + \int_\Gamma h(\hat{u} - \hat{v}_c)\,\mathrm{d}x. \tag{9.17}$$

With the help of (9.15) we infer from (9.14) and (9.17) that

$$\langle \mathbf{Ba}(u), \hat{u} - \hat{v}_c \rangle_\Gamma = \int_\Gamma h(\hat{u} - \hat{v}_c)\,\mathrm{d}\Gamma. \tag{9.18}$$

Now, it is easy to see that (9.12) is equivalent to {(9.14), (9.16), (9.18)} together with $u \in K_b(\hat{v}_c)$. Therefore (9.12) is formally equivalent to the following system:

$$\begin{cases} u \geqslant \hat{v}_c \quad \text{on } \Gamma, \\ -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + a_0(x, u) = f \quad \text{a.e. on } \Omega, \\ \mathbf{Ba}(u) \geqslant h \quad \text{on } \Gamma, \\ (\mathbf{Ba}(u) - h)(u - \hat{v}_c) = 0 \quad \text{on } \Gamma. \end{cases}$$

EXAMPLE 9.4 (Problems with mass constraint). We consider a simple case of variational inequalities in $L^2(\Omega)$ with constraint

$$\int_\Omega w\,\mathrm{d}x = m_0,$$

where m_0 is a given real number. We put

$$K_{m_0} = \left\{ w \in L^2(\Omega);\ \int_\Omega w\,\mathrm{d}x = m_0 \right\},$$

and denote by π_0 the projection from $L^2(\Omega)$ onto $K_0 = \{w \in L^2(\Omega);\ \int_\Omega w\,\mathrm{d}x = 0\}$; actually we have

$$\pi_0(w)(\cdot) = w(\cdot) - \frac{1}{|\Omega|} \int_\Omega w\,\mathrm{d}x \quad \text{a.e. on } \Omega,\ \forall w \in L^2(\Omega),$$

where $|\Omega|$ stands for the volume of Ω. The variational inequality, which we treat here, is of the form:

$$\begin{cases} u \in H^1(\Omega) \cap K_{m_0}, \\ \int_\Omega \nabla u \cdot \nabla(u - w)\,\mathrm{d}x \leqslant \int_\Omega f(u - w)\,\mathrm{d}x, \quad \forall w \in H^1(\Omega) \cap K_{m_0}, \end{cases} \tag{9.19}$$

where f is given in $L^2(\Omega)$. This is equivalent to

$$\begin{cases} u \in H^1(\Omega) \cap K_{m_0}, \\ \int_\Omega \nabla u \cdot \nabla w\,\mathrm{d}x = \int_\Omega f w\,\mathrm{d}x, \quad \forall w \in H^1(\Omega) \cap K_0. \end{cases} \tag{9.20}$$

PROPOSITION 9.2. *Problem* (9.19) *has one and only one solution* u. *Moreover* u *is a unique solution of the following system*: $u \in H^2(\Omega)$ $(= W^{2,2}(\Omega))$ *and*

$$-\Delta u = \pi_0(f) \quad \text{a.e. on } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{a.e. on } \Gamma, \qquad \int_\Omega u \, dx = m_0. \tag{9.21}$$

PROOF. We define a function $\psi(\cdot)$ on $L^2(\Omega)$ by

$$\psi(w) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx + \frac{1}{2}(\int_\Omega w \, dx)^2 & \text{for } w \in H^1(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

This is proper, l.s.c. and convex on $L^2(\Omega)$ and on $H^1(\Omega)$. As is easily checked, the subdifferential $\partial\psi$ from $H^1(\Omega)$ into its dual $H^1(\Omega)^*$ is characterized as follows: $g^* \in \partial\psi(v)$ if and only if $v \in H^1(\Omega)$ and

$$\langle g^*, w\rangle = \int_\Omega \nabla v \cdot \nabla w \, dx + \left(\int_\Omega v \, dx\right)\left(\int_\Omega w \, dx\right), \quad \forall w \in H^1(\Omega);$$

hence, $\partial\psi$ is singlevalued. Also, we have

$$\frac{\psi(w)}{|w|_{1,2}} \to \infty \quad \text{as } |w|_{1,2} \to \infty,$$

because $\sqrt{\psi(w)}$ is equivalent to the norm of $H^1(\Omega)$. Therefore, by Corollary 4.2, $R(\partial\psi) = H^1(\Omega)^*$, so that for the element $g^* \in H^1(\Omega)$ given by

$$\langle g^*, w\rangle = \int_\Omega \pi_0(f) w \, dx + m_0 \int_\Omega w \, dx, \quad \forall w \in H^1(\Omega),$$

there exists $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla w \, dx + \left(\int_\Omega u \, dx\right)\left(\int_\Omega w \, dx\right) = \int_\Omega \pi_0(f) w \, dx + m_0 \int_\Omega w \, dx,$$
$$\forall w \in H^1(\Omega). \tag{9.22}$$

Now, taking $w \equiv 1$ in (9.22), we get $\int_\Omega u \, dx = m_0$, since $\int_\Omega \pi_0(f) \, dx = 0$. Accordingly, it follows from (9.22) that

$$\int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega \pi_0(f) w \, dx, \quad \forall w \in H^1(\Omega), \tag{9.23}$$

which is nothing but (9.20) and hence (9.19). The uniqueness of the solution u is easily seen by the monotonicity argument. Furthermore, (9.23) implies that $-\Delta u = \pi_0(f)$ a.e. on Ω, whence $\partial u/\partial n = 0$ a.e. on Γ. Consequently (9.21) is obtained. We see from the usual regularity result for linear elliptic equations (cf. [14]) that $u \in H^2(\Omega)$. □

EXAMPLE 9.5 (Problems with mass constraint and boundary obstacles). In the last example of variational inequalities we consider a problem having the mass constraint and boundary obstacle in the space $H^1(\Omega)$. Given functions $f \in L^2(\Omega)$ and $\hat{v}_c \in H^{\frac{1}{2}}(\Gamma)$, our variational inequality is of the form

$$u \in K_0(\hat{v}_c); \quad \int_\Omega \nabla u \cdot \nabla(u - w)\,\mathrm{d}x \leqslant \int_\Omega f(u - w)\,\mathrm{d}x, \quad \forall w \in K_0(\hat{v}_c), \tag{9.24}$$

where

$$K_0(\hat{v}_c) := \left\{ w \in H^1(\Omega);\ \int_\Omega w\,\mathrm{d}x = 0,\ \widehat{w} \geqslant \hat{v}_c \text{ a.e. on } \Gamma \right\}.$$

This variational inequality is equivalent to the relation "$f \in \partial\varphi(u)$", where φ is a proper, l.s.c. and convex function on $L^2(\Omega)$ defined by

$$\varphi(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x & \text{if } u \in K_0(\hat{v}_c), \\ \infty & \text{otherwise.} \end{cases} \tag{9.25}$$

PROPOSITION 9.3. *A function u is a solution of* (9.24), *or equivalently $f \in \partial\varphi(u)$, where φ is a proper, l.s.c. convex function on $L^2(\Omega)$ given by* (9.25), *if and only if the following three conditions* (a), (b) *and* (c) *are satisfied*:

(a) $-\Delta u + \ell_u = f$ *in* $L^2(\Omega)$, *where ℓ_u is a constant depending on u.*
(b) $\int_\Omega u\,\mathrm{d}x = 0$.
(c) $\partial u/\partial n \geqslant 0$ (*in* $H^{1/2}(\Gamma)$), $\hat{u} \geqslant \hat{v}_c$ *a.e. on* Γ, $\langle \partial u/\partial n, \hat{u} - \hat{v}_c \rangle_\Gamma = 0$.

PROOF. We consider an approximation φ_n, $n = 1, 2, \ldots$, of φ by penalty method

$$\varphi_n(v) = \begin{cases} \frac{1}{2}\int_\Omega |\nabla v|^2\,\mathrm{d}x + \frac{n}{2}\left(\int_\Omega v\,\mathrm{d}x\right)^2 & \text{if } u \in K(\hat{v}_c), \\ \infty & \text{otherwise,} \end{cases} \tag{9.26}$$

where $K(\hat{v}_c) := \{v \in H^1(\Omega);\ \hat{v} \geqslant \hat{v}_c\}$. Then, since

$$\left\{ \frac{1}{2}\int_\Omega |\nabla w|^2\,\mathrm{d}x + \frac{n}{2}\left(\int_\Omega w\,\mathrm{d}x\right)^2 \right\}^{1/2}$$

is an equivalent norm on $H^1(\Omega)$, $\partial\varphi_n$ satisfies all the conditions of Corollary 4.2, and hence there exists a unique $u_n \in H^1(\Omega)$ such that $f \in \partial\varphi_n(u_n)$; the function u_n is characterized by $u_n \in K(\hat{v}_c)$ and

$$\begin{aligned} &\int_\Omega \nabla u_n \cdot \nabla(u_n - w)\,\mathrm{d}x + n\left(\int_\Omega u_n\,\mathrm{d}x\right)\left(\int_\Omega (u_n - w)\,\mathrm{d}x\right) \\ &\quad \leqslant \int_\Omega f(u_n - w)\,\mathrm{d}x, \quad \forall w \in K(\hat{v}_c). \end{aligned} \tag{9.27}$$

Taking as w of (9.27) a function $u_n \pm \eta$ with any smooth function η having compact support in Ω, we see that

$$\int_\Omega \nabla u_n \cdot \nabla \eta + n\left(\int_\Omega u_n \,\mathrm{d}x\right)\left(\int_\Omega \eta \,\mathrm{d}x\right) = \int_\Omega f\eta \,\mathrm{d}x,$$

that is,

$$-\Delta u_n + n\int_\Omega u_n \,\mathrm{d}x = f \quad \text{in the distribution sense in } \Omega; \tag{9.28}$$

note here that $\Delta u_n \in L^2(\Omega)$ and hence $\nabla u_n \in \mathcal{E}^2(\Omega)$ and $\partial u_n/\partial n$ is defined as $\mathbf{B}(\nabla u_n) \in H^{-1/2}(\Gamma)$. Furthermore, just as in Example 9.3, we see that u_n satisfies the following inequalities:

$$\frac{\partial u_n}{\partial n} \geqslant 0, \quad \hat{u}_n \geqslant \hat{v}_c \quad \text{a.e. on } \Gamma, \qquad \left\langle \frac{\partial u_n}{\partial n}, \hat{u}_n - \hat{v}_c \right\rangle_\Gamma = 0. \tag{9.29}$$

We now give some uniform estimates of $\{u_n\}$. To this end, choose a function $w_0 \in K(\hat{v}_b)$ with $\int_\Omega w_0 \,\mathrm{d}x = 0$ and substitute it as w in (9.27) to obtain

$$\begin{aligned}&\frac{1}{2}\int_\Omega |\nabla u_n|^2 \,\mathrm{d}x + n\left(\int_\Omega u_n \,\mathrm{d}x\right)^2 - \int_\Omega f u_n \,\mathrm{d}x \\ &\quad\leqslant \frac{1}{2}\int_\Omega |\nabla w_0|^2 \,\mathrm{d}x - \int_\Omega f w_0 \,\mathrm{d}x.\end{aligned}$$

This shows that $\{u_n\}$ is bounded in $H^1(\Omega)$. Next, choose a smooth function η_1 with compact support in Ω such that $\int_\Omega \eta_1 \,\mathrm{d}x = 1$ and take $u_n \pm \eta_1$ as w in (9.27). Then we get

$$n\left|\int_\Omega u_n \,\mathrm{d}x\right| \leqslant \left|\int_\Omega \nabla u_n \cdot \nabla \eta_1 \,\mathrm{d}x\right| + \left|\int_\Omega f\eta_1 \,\mathrm{d}x\right|,$$

whence $\{n\int_\Omega u_n \,\mathrm{d}x\}$ is bounded. Now we extract a subsequence $\{n_k\}$ of $\{n\}$ such that $\{u_{n_k}\}$ weakly converges in $H^1(\Omega)$ and $\{n_k \int_\Omega u_{n_k} \,\mathrm{d}x\}$ converges, too, as $k \to \infty$. In this case, denoting by u the weak limit of $\{u_{n_k}\}$ in $H^1(\Omega)$ and by ℓ_u the limit of $n_k \int_\Omega u_{n_k} \,\mathrm{d}x$, we see from (9.28) and (9.29) that assertions (a), (b) and (c) hold. □

10. Quasi-variational inequalities

Let X be a real Banach space and X^* be its dual space, and assume that X and X^* are strictly convex.

Given a mapping $A : X \to X^*$, a non-empty closed convex subset K of X and an element $g^* \in X^*$, we consider the problem to find an element u such that

$$u \in K,\ u^* \in Au; \quad \langle u^* - g^*, u - v\rangle \leqslant 0, \quad \forall v \in K. \tag{10.1}$$

As was seen in Corollary 5.3, if A is bounded, coercive and pseudo-monotone, then this problem has at least one solution u. Moreover, if A is strictly monotone, then the solution of (10.1) is unique.

Our objective of this section is to discuss the so-called quasi-variational inequality, as a generalization of (10.1), for a nonlinear mapping A from X into X^*, which is of the following form:

$$u \in K(u),\ u^* \in Au; \quad \langle u^* - g^*, u - v\rangle \leqslant 0, \quad \forall v \in K(u). \tag{10.2}$$

This is a variational inequality with the state constraint $K = K(u)$ depending upon the unknown u. As to problem (10.2) we prove the following abstract existence result.

THEOREM 10.1. *Let $\tilde{A}$ be a bounded semi-monotone mapping from $X \times X$ into X^*; put $Au = \tilde{A}(u,u)$ for every $u \in X$. Let K_0 be a bounded, closed and convex set in X. Suppose that to each $v \in K_0$ a non-empty, bounded, closed and convex subset $K(v)$ of K_0 is assigned, and the mapping $v \to K(v)$ satisfies the following continuity properties* (K1) *and* (K2):

(K1) *If $v_n \in K_0$, $v_n \to v$ weakly in X (as $n \to +\infty$), then for each $w \in K(v)$ there is a sequence $\{w_n\}$ in X such that $w_n \in K(v_n)$ and $w_n \to w$ (strongly) in X.*

(K2) *If $v_n \to v$ weakly in X, $w_n \in K(v_n)$ and $w_n \to w$ weakly in X, then $w \in K(v)$.*

Then, for any $g^* \in X^*$, the quasi-variational inequality (10.2) *has at least one solution u.*

PROOF. We prove the above theorem in the following two cases (A) and (B).

(A) The case when for each v the mapping $\tilde{A}(v,\cdot)$ is strictly monotone from

$$D\big(\tilde{A}(v,\cdot)\big) = X$$

into X^*.

(B) The general case as in the statement of Theorem 10.1.

In the case of (A). We define a mapping S from K_0 into itself as follows. Let v be any element in K_0 and consider the variational inequality with state constraint $K(v)$, namely

$$u \in K(v),\ u^*(v) \in \tilde{A}(v,u); \quad \big\langle u^*(v) - g^*, u - w\big\rangle \leqslant 0, \quad \forall w \in K(v). \tag{10.3}$$

We note that for each v the mapping $\tilde{A}(v,\cdot): D(\tilde{A}(v,\cdot)) = X \to X^*$ is a maximal monotone mapping. For any $v_0 \in K(v)$, $w \in X$, $w^* \in \tilde{A}(v,w)$ and $v_0^* \in \tilde{A}(v,v_0)$ we have

$$\langle w^*, w - v_0\rangle \geqslant \langle v_0^*, w - v_0\rangle \geqslant -|v_0^*|_{X^*}|w|_X - |v_0^*|_{X^*}|v_0|_X.$$

Also, the condition

$$\inf_{w^* \in \partial I_{K(v)}(w),\ w_1^* \in \tilde{A}(v,w)} \frac{\langle w^* + w_1^*, w - v_0\rangle}{|w|_X} \to +\infty \quad \text{as } |w|_X \to +\infty,\ w \in K(v)$$

is automatically satisfied, because $K(v)$ is bounded. Therefore, by Corollary 5.3 we obtain that $R(\partial I_{K(v)} + \tilde{A}(v,\cdot)) = X^*$, where $I_{K(v)}$ is the indicator function of $K(v)$. This shows

there is $u \in K(v)$ such that $g^* \in \partial I_{K(v)}(u) + A(v,u)$, and u is a solution of (10.3). The uniqueness of the solution u of (10.3) follows from the strict monotonicity of $A(v,\cdot)$. Now we put $Sv = u$, and in this way S is a singlevalued mapping in K_0.

Next, we show that S is weakly continuous in K_0. Let $\{v_n\}$ be any sequence in K_0 such that $v_n \to v$ weakly in X, and put $u_n = Sv_n (\in K_0)$ for $n = 1, 2, \ldots$. Now, let $\{u_{n_k}\}$ be any weakly convergent subsequence of $\{u_n\}$ and denote by u the weak limit; note by condition (K2) that $u \in K(v)$. We are going to check that u is a unique solution of (10.3). To do so, first observe that there is $u_n^* \in A(v_n, u_n)$ such that

$$\langle u_n^* - g^*, u_n - w\rangle \leqslant 0, \quad \forall w \in K(v_n). \tag{10.4}$$

Using condition (K1), we find a sequence $\{\tilde{u}_k\}$ such that $\tilde{u}_k \in K(v_{n_k})$ and $\tilde{u}_k \to u$ in X (as $k \to +\infty$). By the boundedness of $\tilde{A}(\cdot,\cdot)$, we may assume that $u_{n_k}^* \to u^*$ weakly in X^* for some $u^* \in X^*$. Now, taking $n = n_k$ and $w = \tilde{u}_k$ in (10.4), we see that

$$\begin{aligned}
\limsup_{k\to+\infty} \langle u_{n_k}^*, u_{n_k}\rangle &= \limsup_{k\to+\infty} \big\{\langle u_{n_k}^*, u_{n_k} - \tilde{u}_k\rangle + \langle u_{n_k}^*, \tilde{u}_k\rangle\big\} \\
&\leqslant \limsup_{k\to+\infty} \big\{\langle g^*, u_{n_k} - \tilde{u}_k\rangle + \langle u_{n_k}^*, \tilde{u}_k\rangle\big\} \\
&= \langle u^*, u\rangle.
\end{aligned}$$

Now, given any $w \in X$ and any $w^* \in \tilde{A}(v,w)$, by using (SM2) choose a sequence $w_k^* \in \tilde{A}(v_{n_k}, w)$, $k = 1, 2, \ldots$, so that $w_k^* \to w^*$ in X^*. Then it follows from the monotonicity of $\tilde{A}(v_{n_k}, \cdot)$ that

$$\langle u_{n_k}^* - w_k^*, u_{n_k} - w\rangle \geqslant 0, \quad \forall k = 1, 2, \ldots. \tag{10.5}$$

By noting $\limsup_{k\to+\infty} \langle u_{n_k}^*, u_{n_k}\rangle \leqslant \langle u^*, u\rangle$ and passing to the limit as $k \to \infty$ in (10.5), we get

$$\langle u^* - w^*, u - w\rangle \geqslant 0,$$

which holds for any $w \in X$ and $w^* \in \tilde{A}(v,w)$. Therefore, we conclude by the maximal monotonicity of $\tilde{A}(v,\cdot)$ that $u^* \in \tilde{A}(v,u)$. Moreover, use the inequality (10.5) in the case of $w = u$ and $w^* = u^* \in \tilde{A}(v,u)$, choosing a sequence $w_k^* \in \tilde{A}(v_{n_k}, u)$ so that $w_k^* \to u^*$ in X^*. Then, we get

$$\begin{aligned}
\liminf_{k\to\infty} \langle u_{n_k}^*, u_{n_k}\rangle &\geqslant \liminf_{k\to\infty} \big\{\langle w_k^*, u_{n_k} - u\rangle + \langle u_{n_k}^*, u\rangle\big\} \\
&= \langle u^*, u\rangle.
\end{aligned}$$

Thus we have seen that

$$\lim_{k\to+\infty} \langle u_{n_k}^*, u_{n_k}\rangle = \langle u^*, u\rangle, \qquad u_{n_k}^* \to u^* \quad \text{weakly in } X^*. \tag{10.6}$$

Going back to (10.4) with $n = n_k$, taking $k \to +\infty$ and using (10.6) with condition (K1) again, we finally arrive at the variational inequality (10.3). Thus $u = Sv$, and S is weakly continuous in K_0.

Since K_0 is a weakly compact and convex set in X, we infer from the well-known fixed-point theorem for compact mappings (cf. Corollary 3.2) that S has at least one fixed point in K_0. This fixed point u is clearly a solution of our quasi-variational inequality (10.2).

In the case of (B). We approximate $\tilde{A}(v, u)$ by $\tilde{A}_\varepsilon(v, u) := \tilde{A}(v, u) + \varepsilon Ju$ for any $u, v \in X$ and with parameter $\varepsilon \in (0, 1]$; note that the duality mapping J from X into X^* is strictly monotone. By the result of the case (A), for each $g^* \in X^*$ there exists a solution $u_\varepsilon \in K_0$ of the quasi-variational inequality

$$u_\varepsilon \in K(u_\varepsilon),\ u_\varepsilon^* \in Au_\varepsilon; \quad \langle u_\varepsilon^* + \varepsilon Ju_\varepsilon - g^*, u_\varepsilon - w\rangle \leqslant 0, \quad \forall w \in K(u_\varepsilon). \tag{10.7}$$

Now, choose a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \downarrow 0$, such that $u_n := u_{\varepsilon_n} \to u$ weakly in X. Then, by conditions (K1) and (K2), we see that $u \in K(u)$ and there is a sequence $\{\tilde{u}_n\}$ such that $\tilde{u}_n \in K(u_n)$ and $\tilde{u}_n \to u$ in X. Moreover, by the boundedness of $\{u_n^* := u_{\varepsilon_n}^*\}$ in X^*, we may assume that $u_n^* \to u^*$ weakly in X^* for some $u^* \in X^*$. Substitute u_n and $\tilde{u}_n$ for u_ε and w in (10.7) and pass to the limit as $n \to \infty$ to get

$$\begin{aligned}
\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle &= \limsup_{n\to\infty} \big\{\langle u_n^* + \varepsilon_n Ju_n, u_n - \tilde{u}_n\rangle + \langle u_n^* + \varepsilon_n Ju_n, \tilde{u}_n - u\rangle\big\} \\
&\leqslant \limsup_{n\to\infty} \big\{\langle g^*, u_n - \tilde{u}_n\rangle + \langle u_n^*, \tilde{u}_n - u\rangle\big\} \\
&= 0.
\end{aligned}$$

Since A is pseudo-monotone from X into X^* (cf. Proposition 2.3(3)), it follows from the above inequality that

$$u^* \in Au, \quad \lim_{n\to\infty} \langle u_n^*, u_n\rangle = \langle u^*, u\rangle. \tag{10.8}$$

Now, for each $w \in K(u)$, we choose $\{\widetilde{w}_n\}$ such that $\widetilde{w}_n \in K(u_n)$ and $\widetilde{w}_n \to w$ in X, and then substitute them for w in (10.7) with $\varepsilon = \varepsilon_n$ to have

$$\langle u_n^* + \varepsilon_n Ju_n - g^*, u_n - \widetilde{w}_n\rangle \leqslant 0.$$

By (10.8), letting $n \to \infty$ in the inequality yields that $\langle u^* - g^*, u - w\rangle \leqslant 0$. Thus u is a solution of our quasi-variational inequality (10.2). □

REMARK 10.1. If condition (R) is satisfied and $\tilde{A}(v, u)$ is independent of $v \in X$, the weak continuity of the mapping S, which was shown in the proof of the case (A), is a direct consequence of Theorem 7.3; in fact, if $v_n \to v$ weakly in X, conditions (K1) and (K2) ensure that the convex function $I_{K(v_n)}$ converges to $I_{K(v)}$ as $n \to \infty$ in the sense of Mosco, and hence from Corollary 8.2 it follows that $\partial I_{K(v_n)} \to \partial I_{K(v)}$ in $X \times X^*$ in the graph sense.

The following theorem is a slightly general version of Theorem 10.1.

THEOREM 10.2. *Let $\tilde{A}: D(\tilde{A}) = X \times X \to X^*$ be as a bounded and semi-monotone mapping and define $A: D(A) = X \to X^*$ by $Au = \tilde{A}(u, u)$ for each $u \in X$. Suppose that to each $v \in X$ a non-empty, closed and convex subset $K(v)$ of X is assigned and there is a bounded, closed and convex subset G_0 of X such that*

$$K(v) \cap G_0 \neq \emptyset, \quad \forall v \in X, \tag{10.9}$$

and

$$\inf_{w^* \in Aw} \frac{\langle w^*, w - v\rangle}{|w|_X} \to +\infty \quad \text{as } |w|_X \to +\infty \text{ uniformly in } v \in G_0. \tag{10.10}$$

Furthermore, the mapping $v \to K(v)$ satisfies the following condition (K′1) *and the same condition* (K2) *as in Theorem* 10.1:

(K′1) *If $v_n \to v$ weakly in X, then for each $w \in K(v)$ there is a sequence $\{w_n\}$ in X such that $w_n \in K(v_n)$ and $w_n \to w$ in X.*

Then, for each $g^* \in X^*$ the quasi-variational inequality (10.2) *has at least one solution u.*

PROOF. Put $d_1 := \sup_{w \in G_0} |w|_X$ and

$$d_2 := \sup\left\{ |w|_X;\ w \in X,\ \inf_{w^* \in Aw} \frac{\langle w^*, w - v\rangle}{|w|_X} \leqslant |g^*|_{X^*}(1 + d_1),\ \forall v \in G_0 \right\}.$$

By condition (10.10), d_2 is finite. Also we put $M_0 := d_1 + d_2 + 1$, and for any number $M \geqslant M_0$ consider the closed ball $B_M := \{w \in X;\ |w|_X \leqslant M\}$ as well as bounded closed and convex sets $K_M(v) := K(v) \cap B_M$ for all $v \in X$. Since $G_0 \subset B_M$, (10.9) implies that $K_M(v)$ is non-empty for every $v \in X$.

We now show that conditions (K1) and (K2) in Theorem 10.1 with $K_0 = B_M$ and $K(\cdot) = K_M(\cdot)$ are satisfied. The verification of (K2) is easy. We check condition (K1) for $K_0 = B_M$ and $K(\cdot) = K_M(\cdot)$ as follows. Let w be any element in $K_M(v)$. Then, by condition (K′1) for $K(\cdot)$, for a sequence $\{v_n\} \subset B_M$ weakly converging to v there is a sequence $\{w_n\}$ such that $w_n \in K(v_n)$ and $w_n \to w$ in X. In the case of $|w|_X < M$, we see that $|w_n|_X < M$ and hence $w_n \in K_M(v_n)$ for all large n. In the case of $|w|_X = M$, choose an element $v_0 \in K(v) \cap G_0$ and put

$$w^m := \left(1 - \frac{1}{m}\right) w + \frac{1}{m} v_0, \quad m = 1, 2, \ldots.$$

Clearly $w^m \in K_M(v)$ and $|w^m|_X < M$. Therefore, according to the above argument, for each m there is a sequence $\{w_n^m\}_{n=1}^{\infty}$ such that $w_n^m \in K_M(v_n)$ and $w_n^m \to w^m$ in X as $n \to \infty$. For each m choose a number $n(m)$ so that $|w^m - w_n^m| \leqslant \frac{1}{m}$ for all $n \geqslant n(m)$. We may choose $\{n(m)\}_{m=1}^{\infty}$ so that $n(m-1) < n(m)$ for all $m = 0, 1, \ldots$, where $n(0) = 1$. We put

$$w_n = w_n^m \quad \text{if } n(m) \leqslant n < n(m+1),\ m = 0, 1, \ldots.$$

It is easy to see that $w_n \in K_M(v_n)$ and $w_n \to w$ in X.

By the above observation we can apply Theorem 10.1 to find an element u_M such that

$$u_M \in K_M(u_M),\ u_M^* \in Au_M; \quad \langle u_M^* - g^*, u_M - w\rangle \leqslant 0, \quad \forall w \in K_M(u_M). \tag{10.11}$$

Also, by condition (10.10), $\{u_M;\ M \geqslant M_0\}$ is bounded in X, so that there are a sequence $\{M_n\}$ with $M_n \uparrow +\infty$ and elements $u \in X$, $u^* \in X^*$ such that $u_n := u_{M_n} \to u$ weakly in X and $u_n^* := u_{M_n}^* \to u^*$ weakly in X^* as $n \to \infty$. We note $u \in K(u)$ by (K2). It follows from (K′1) that for each $w \in K(u)$ there is a sequence $\{\widetilde{w}_n\}$ such that $\widetilde{w}_n \in K(u_n)$ and $\widetilde{w}_n \to w$ in X. In particular, denote by $\{\tilde{u}_n\}$ the sequence $\{\widetilde{w}_n\}$ corresponding to $w = u$. Here, we substitute M_n and $\tilde{u}_n$ for M and w in (10.11) to obtain $\langle u_n^* - g^*, u_n - \tilde{u}_n\rangle \leqslant 0$. Hence it follows that

$$\begin{aligned}
&\limsup_{n\to+\infty}\langle u_n^*, u_n - u\rangle \\
&\quad = \limsup_{n\to+\infty}\big\{\langle u_n^* - g^*, u_n - \tilde{u}_n\rangle + \langle u_n^*, \tilde{u}_n - u\rangle + \langle g^*, u_n - \tilde{u}_n\rangle\big\} \\
&\quad \leqslant 0.
\end{aligned}$$

By the pseudo-monotonicity of A this implies that

$$u^* \in Au, \quad \lim_{n\to\infty}\langle u_n^*, u_n\rangle = \langle u^*, u\rangle.$$

By making use of these properties with (K1) and passing to the limit as $n \to \infty$ in (10.11) with $M = M_n$, we see that $u \in K(u)$ and $\langle u^* - g^*, u - w\rangle \leqslant 0$ for all $w \in K(u)$. Thus u is a solution of our problem (10.2). □

In the rest of this section we give some applications of Theorems 10.1 and 10.2 to quasi-variational inequalities.

APPLICATION 1 (Gradient obstacle problem). Let $X = W_0^{1,p}(\Omega)$, $1 < p < +\infty$, and let $a_0(x,\zeta), a_i(x,\zeta,\xi), i = 1, 2, \ldots, N$, be the same functions as in Section 9 and properties (a1)–(a4) are satisfied as well. We define a mapping $\tilde{A}(\cdot,\cdot) : X \times X \to X^*$ by putting

$$\big\langle \tilde{A}(v,u), w\big\rangle = \sum_{i=1}^{N}\int_\Omega a_i(x, v, \nabla u) w_{x_i}\,\mathrm{d}x + \int_\Omega a_0(x,v)w\,\mathrm{d}x, \quad \forall u, v, w \in X, \tag{10.12}$$

and Au by $\tilde{A}(u,u)$ for every $u \in X$. It should be noticed that $\tilde{A}$ is a singlevalued, bounded and semimonotone mapping from $X \times X$ into X^*, and by Proposition 9.1 the mapping A is pseudo-monotone from X into X^*.

Let k_c be a Lipschitz continuous real function on $\mathbf{R}$ such that

$$0 < k_c(r) \leqslant k^*, \quad \forall r \in \mathbf{R}, \tag{10.13}$$

where k^* is a positive constant, and put

$$K(v) := \{w \in X;\ |\nabla w| \leqslant k_c(v)\}, \quad \forall v \in X. \tag{10.14}$$

Also, we set

$$K_0 := \{w \in X;\ |\nabla w| \leqslant k^* \text{ a.e. on } \Omega\};$$

note from the Sobolev embedding theorem (cf. [1,16,19]) that K_0 is compact in $C(\overline{\Omega})$.

LEMMA 10.1. *The family* $\{K(v);\ v \in X\}$ *and the set* K_0 *satisfy conditions* (K1) *and* (K2).

PROOF. We prove (K2). Suppose that $v_n \in K_0$, $w_n \in K(v_n)$, $v_n \to v$ weakly in X and $w_n \to w$ weakly in X. Then, we note that $v_n \to v$ in $C(\overline{\Omega})$ and hence $k_c(v_n) \to k_c(v)$ in $C(\overline{\Omega})$. Therefore, given $\varepsilon > 0$, there exists a positive integer n_ε such that

$$k_c(v_n) \leqslant k_c(v) + \varepsilon \quad \text{on } \Omega,\ \forall n \geqslant n_\varepsilon.$$

This shows that

$$|\nabla w_n| \leqslant k_c(v) + \varepsilon \quad \text{a.e. on } \Omega,\ \forall n \geqslant n_\varepsilon. \tag{10.15}$$

Clearly the set $K_\varepsilon(v) := \{w \in X;\ |\nabla w| \leqslant k_c(v) + \varepsilon \text{ a.e. on } \Omega\}$ is bounded, closed and convex in X, so that $K_\varepsilon(v)$ is weakly compact in X. Now we derive by letting $n \to +\infty$ in (10.15) that $w \in K_\varepsilon(v)$. Since $\varepsilon > 0$ is arbitrary, we have $w \in K(v)$. Thus condition (K2) is satisfied.

Next we show (K1). Suppose that $v \in K_0$, $w \in K(v)$ and $\{v_n\} \subset K_0$ such that $v_n \to v$ weakly in X. By the compactness of K_0 in $C(\overline{\Omega})$ we have that $v_n \to v$ in $C(\overline{\Omega})$. Since $cw \in K(v)$ for all constant $c \in (0, 1)$ and $cw \to w$ as $c \uparrow 1$ in X, it is enough to show the existence of a sequence $\{\widetilde{w}_n\}$ such that $\widetilde{w}_n \in K(v_n)$ and $\widetilde{w}_n \to \widetilde{w}$ in X, when $\widetilde{w} = cw$ for any $c \in (0, 1)$. In such a case, by condition (10.13), we can take a small $\varepsilon > 0$ so that $|\nabla \widetilde{w}| \leqslant k_c(v) - \varepsilon$ a.e. on $\overline{\Omega}$. Furthermore, for this $\varepsilon > 0$ we can find a positive integer n_ε such that $k_c(v) \leqslant k_c(v_n) + \varepsilon$ for all $n \geqslant n_\varepsilon$. This implies that $|\nabla \widetilde{w}| \leqslant k_c(v_n)$ a.e. on Ω, namely $\widetilde{w} \in K(v_n)$ for all $n \geqslant n_\varepsilon$. Now we define $\{\widetilde{w}_n\}$ by putting

$$\widetilde{w}_n = \begin{cases} \widetilde{w} & \text{for } n \geqslant n_\varepsilon, \\ \text{some function in } K(v_n) & \text{for } 1 \leqslant n < n_\varepsilon. \end{cases}$$

Clearly this is a required sequence in condition (K1). □

According to Lemma 10.1, we can apply Theorem 10.1 to solve the following quasi-variational inequality:

$$\begin{cases} u \in X, \quad |\nabla u| \leqslant k_c(u) \quad \text{a.e. on } \Omega; \\ \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)(u_{x_i} - v_{x_i})\,\mathrm{d}x + \int_\Omega a_0(x, u)(u - v)\,\mathrm{d}x \\ \quad \leqslant \int_\Omega f(u - v)\,\mathrm{d}x, \quad \forall v \in X \text{ with } |\nabla v| \leqslant k_c(u) \text{ a.e. on } \Omega, \end{cases}$$

where f is given in $L^q(\Omega)$, $1/p + 1/q = 1$.

APPLICATION 2 (Interior obstacle problem). Let Ω be a one-dimensional bounded open interval, say $(0, 1)$, $X := W^{1,p}(0, 1)$, $1 < p < \infty$, and let $a_0(x, \zeta)$ and $a_1(x, \zeta, \xi)$ be two functions which satisfy (a1)–(a4) with $N = 1$ in Section 9.

Also, let $k_c(\cdot)$ be a Lipschitz continuous real function on $\mathbf{R}$ such that

$$k_c(r) \leqslant k^*, \quad \forall r \in \mathbf{R},$$

where k^* is a constant, and put

$$K(v) := \big\{w \in X;\ w \geqslant k_c(v) \text{ on } (0, 1)\big\}, \quad \forall v \in X.$$

Furthermore, let $f \in L^q(0, 1)$, $1/p + 1/q = 1$, and choose a large constant $M > 0$ such that

$$\int_0^1 \big\{a_1(x, v, v_x)v_x + a_0(x, v)(v - k^*)\big\}\,\mathrm{d}x > \int_0^1 f(v - k^*)\,\mathrm{d}x,$$
$$\forall v \in X \quad \text{with } |v|_{1,p} \geqslant M;$$

on account of condition (9.1) and (9.2) in condition (a3) we note that such a positive constant M exists. Now we take as K_0 the closed ball $B_M := \{v \in X;\ |v|_{1,p} \leqslant M\}$ of X.

LEMMA 10.2. *The family* $\{K(v);\ v \in X\}$ *and the set* K_0 *satisfy conditions* (K1) *and* (K2) *in Theorem* 10.1.

PROOF. We observe that B_M is a compact subset of $C([0, 1])$. Taking this fact into account, we obtain the lemma in a way similar to that in the proof of Lemma 10.1. □

Now, applying Theorem 10.1, we can find a function u such that

$$\begin{cases} u \in X,\ u \geqslant k_c(u) \quad \text{on } (0, 1); \\ \int_0^1 \{a_1(x, u, u_x)(u_x - v_x) + a_0(x, u)(u - v)\}\,\mathrm{d}x \leqslant \int_0^1 f(u - v)\,\mathrm{d}x, \\ \quad \forall v \in X,\ v \geqslant k_c(u) \text{ on } (0, 1). \end{cases} \tag{10.16}$$

Moreover, just as in Example 9.1, problem (10.16) is equivalent to the system:

$$\begin{cases} u \in X,\ u \geqslant k_c(u) \quad \text{on } (0,1); \\ \int_0^1 \{a_1(x,u,u_x)(u_x - k_c(u)_x) + a_0(x,u)(u - k_c(u))\}\,dx \\ \quad = \int_0^1 f(u - k_c(u))\,dx, \\ \int_0^1 \{a_1(x,u,u_x)\eta_x + a_0(x,u)\eta\}\,dx \geqslant \int_0^1 f\eta\,dx, \\ \quad \forall \text{ smooth } \eta \geqslant 0 \text{ on } (0,1). \end{cases} \tag{10.17}$$

By any solution u of problem (10.16), the interval $(0,1)$ is divided into the following two parts Ω_0 and Ω_1:

$$\Omega_0 := \big\{x \in (0,1);\ u(x) > k_c\big(u(x)\big)\big\},$$
$$\Omega_1 := \big\{x \in (0,1);\ u(x) = k_c\big(u(x)\big)\big\}.$$

Since Ω_0 is open set in $(0,1)$, we see that $u \pm \varepsilon\eta \geqslant k_c(u)$ on $(0,1)$ for any smooth η with compact support in Ω_0, if $\varepsilon > 0$ is sufficiently close to 0. Therefore, by substituting $u \pm \varepsilon\eta$ as v in (10.16), we have that $-a(\cdot,u,u_x)_x = f - a_0(x,u) \in L^q(0,1)$ and

$$-a_1(x,u,u_x)_x + a_0(x,u) = f \quad \text{a.e. on } \Omega_0.$$

Furthermore it follows from the last inequality in (10.17) that

$$-a_1(x,u,u_x)_x + a_0(x,u) \geqslant f \quad \text{in the sense of measures on } \mathit{Int}\,\Omega_1.$$

APPLICATION 3 (Boundary obstacle problem). We consider a quasi-variational inequality with constraint on the boundary. Let $X := W^{1,p}(0,1)$, $1 < p < +\infty$, and let $a_0(x,\zeta)$ and $a_1(x,\zeta,\xi)$ be two functions satisfying condition (a1)–(a4) with $N = 1$ in Section 9. Also, let $k_c^i(\cdot)$, $i = 0, 1$, be two Lipschitz continuous real functions on $\mathbf{R}$ such that

$$k_c^i(r) \leqslant k^*, \quad \forall r \in \mathbf{R},\ i = 0, 1,$$

where k^* is a constant. We define

$$K(v) := \big\{w \in X;\ w(i) \geqslant k_c^i\big(v(i)\big),\ i = 0, 1\big\}, \quad \forall v \in X,$$

and $G_0 = \{k^*\}$, being the singleton set of a constant function k^*. Then it is easy to see that $G_0 \cap K(v) \neq \emptyset$ and the family $\{K(v)\}$ satisfies conditions (K′1) and (K2) in the statement of Theorem 10.2. Therefore, applying Theorem 10.2, for each $f \in L^q(0,1)$, $\frac{1}{p} + \frac{1}{q} = 1$, we find a function u such that

$$\begin{cases} u \in X \quad \text{with } u(i) \geqslant k_c^i(u(i)),\ i = 0, 1; \\ \int_0^1 \{a_1(x,u,u_x)(u_x - v_x) + a_0(x,u)(u - v)\}\,dx \leqslant \int_0^1 f(u - v)\,dx, \\ \quad \forall v \in X \text{ with } v(i) \geqslant k_c^i(u(i)),\ i = 0, 1. \end{cases} \tag{10.18}$$

The solution u of (10.18) is characterized by the following system:

$$u \in X \quad \text{with } u(i) \geqslant k_c^i\big(u(i)\big),\ i = 0, 1, \tag{10.19-1}$$

$$-a_1(\cdot, u, u_x)_x + a_0(\cdot, u) = f \quad \text{a.e. on } (0, 1), \tag{10.19-2}$$

$$a_1(\cdot, u, u_x)(0) \leqslant 0, \quad a_1(\cdot, u, u_x)(0)\big(u(0) - k_c^0\big(u(0)\big)\big) = 0, \tag{10.19-3}$$

$$a_1(\cdot, u, u_x)(1) \geqslant 0, \quad a_1(\cdot, u, u_x)(1)\big(u(1) - k_c^1\big(u(1)\big)\big) = 0. \tag{10.19-4}$$

In fact, take $u \pm \eta$, with smooth η having compact support in $(0, 1)$, as v of (10.18). Then we have immediately (10.19-2) which implies $a_1(\cdot, u, u_x)$ is absolutely continuous on $[0, 1]$ and the boundary values $a_1(\cdot, u, u_x)(i), i = 0, 1$, exist. Therefore, by the integration by parts,

$$\begin{aligned}\int_0^1 a_1(x, u, u_x)(u_x - v_x)\,\mathrm{d}x &= -\int_0^1 a_1(x, u, u_x)_x(u - v)\,\mathrm{d}x \\ &\quad + a_1(\cdot, u, u_x)(1)\big(u(1) - v(1)\big) - a_1(\cdot, u, u_x)(0)\big(u(0) - v(0)\big)\end{aligned}$$

for all $v \in X$. Using this formula together with (10.19-2), we infer from (10.18) that

$$\begin{aligned}&a_1(\cdot, u, u_x)(1)\big(u(1) - v(1)\big) - a_1(\cdot, u, u_x)(0)\big(u(0) - v(0)\big) \leqslant 0, \\ &\quad \forall v \in X \text{ with } v(i) \geqslant k_c^i\big(u(i)\big),\ i = 0, 1.\end{aligned} \tag{10.20}$$

Now, choose v with $v(0) = u(0)$ and $v(1) > u(1)$ in (10.20) to get $a_1(\cdot, u, u_x)(1) \geqslant 0$. Next, choose v with $v(0) = u(0)$ and $v(1) = k_c^1(u(1))$ in (10.20) to get

$$a_1(\cdot, u, u_x)(1)\big(u(1) - k_c^1\big(u(1)\big)\big) \leqslant 0.$$

But, since $u(1) - k_c^1(u(1)) \geqslant 0$, it follows that $a_1(\cdot, u, u_x)(1)(u(1) - k_c^1(u(1))) \geqslant 0$. Consequently $a_1(\cdot, u, u_x)(1)(u(1) - k_c^1(u(1))) = 0$ must hold. Thus (10.19-4) is obtained. Similarly we have (10.19-3). Conversely, by the help of integration by parts we can derive (10.18) from (10.19).

APPLICATION 4 (Problem with non-local constraint). Let $X := W^{1,p}(\Omega)$, $1 < p < +\infty$, and let $a_0(x, \zeta), a_i(x, \zeta, \xi), i = 1, 2, \ldots, N$, be the same functions as in Section 9 and suppose that (a1)–(a4) are satisfied as well. Let $k_c(\cdot)$ be a C^1-function on $\mathbf{R}$ such that

$$k_c(r) \leqslant k^*, \quad \forall r \in \mathbf{R}.$$

For a given function $\rho \in C^1(\mathbf{R}^N \times \mathbf{R}^N)$, we consider

$$K(v) = \big\{w \in X;\ w \geqslant k_c(\Lambda v) \text{ a.e. on } \Omega\big\}, \quad \forall v \in X,$$

where

$$(\Lambda v)(x) = \int_\Omega \rho(x, y) v(y)\, \mathrm{d}y, \quad x \in \Omega.$$

Clearly, for every $v \in X$, $K(v)$ is closed and convex in X, and $k^* \in K(v)$.

LEMMA 10.3. *The family* $\{K(v);\ v \in X\}$ *and the set* $G_0 := \{k^*\}$ *satisfy conditions* (K′1) *and* (K2).

PROOF. Assume that $v_n \to v$ weakly in X and let w be any function in $K(v)$, namely $w \geqslant k_c(\Lambda v)$ a.e. on Ω. We note that $\Lambda v_n \to \Lambda v$ in $C^1(\overline{\Omega})$ and hence $k_c(\Lambda v_n) \to k_c(\Lambda v)$ in $C^1(\overline{\Omega})$. Putting $w_n = w - k_c(\Lambda v) + k_c(\Lambda v_n)$, we see that $w_n \in K(v_n)$ and $w_n \to w$ in X. Thus (K′1) is verified. Next, assume that $w_n \in K(v_n)$, $w_n \to w$ weakly in X and $v_n \to v$ weakly in X. Then $w_n \to w$ in $L^p(\Omega)$ and $v_n \to v$ in $L^p(\Omega)$ as well as $k_c(\Lambda v_n) \to k_c(\Lambda v)$ uniformly on Ω. Hence $w \geqslant k_c(\Lambda v)$ a.e. on Ω, that is, $w \in K(v)$. Thus (K2) is obtained. □

For the mapping $\tilde{A}(v, u)$ given by (10.12), all the conditions of Theorem 10.2 are verified. Therefore, given a function $f \in L^q(\Omega)$, we can find a function $u \in X$ such that

$$\begin{cases} u \in X,\ u \geqslant k_c(\Lambda u) \text{ a.e. on } \Omega, \\ \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)(u_{x_i} - v_{x_i})\, \mathrm{d}x + \int_\Omega a_0(x, u)(u - v)\, \mathrm{d}x \\ \quad \leqslant \int_\Omega f(u - v)\, \mathrm{d}x, \\ \quad v \in X \text{ with } v \geqslant k_c(\Lambda u) \text{ a.e. on } \Omega. \end{cases} \tag{10.21}$$

The solution u of the quasi-variational inequality (10.21) is characterized by the following system:

$$\begin{cases} u \in X,\ u \geqslant k_c(\Lambda u) \text{ a.e. on } \Omega, \\ \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)\eta_{x_i}\, \mathrm{d}x + \int_\Omega a_0(x, u)\eta\, \mathrm{d}x \geqslant \int_\Omega f\eta\, \mathrm{d}x, \\ \quad \forall \eta \in X \text{ with } \eta \geqslant 0 \text{ a.e. on } \Omega, \\ \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u)(u_{x_i} - k_c(\Lambda u)_{x_i})\, \mathrm{d}x \\ \quad + \int_\Omega a_0(x, u)(u - k_c(\Lambda u))\, \mathrm{d}x = \int_\Omega f(u - k_c(\Lambda u))\, \mathrm{d}x. \end{cases} \tag{10.22}$$

We consider two typical cases of the function $\rho(x, y)$.

EXAMPLE 1. Let $\rho(x, y)$ is identically $1/|\Omega|$ on $\mathbf{R}^2$, where $|\Omega|$ denotes the volume of Ω. Then

$$\Lambda v = \frac{1}{|\Omega|} \int_\Omega v\, \mathrm{d}x, \quad \forall v \in X,$$

which is the mean of v on Ω. The unknown constraint set $K(v)$ is given by

$$K(v) := \left\{ w \in X;\ w \geqslant k_c\left(\frac{1}{|\Omega|} \int_\Omega w\, \mathrm{d}x \right) \text{ a.e. on } \Omega \right\}.$$

EXAMPLE 2. Let us define as $\rho(x, y)$ a function ρ_ε with real parameter $0 < \varepsilon < 1$, by using the usual mollifiers as follows:

$$\rho_\varepsilon(x, y) := \varepsilon^{-N} \rho_0\left(\frac{x - y}{\varepsilon}\right), \quad \forall x, y \in \mathbf{R}^N,$$

where

$$\rho_0(x) := \begin{cases} c \cdot \exp\{-\frac{1}{1-|x|^2}\} & \text{if } |x| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

the constant $c > 0$ being chosen so that $\int_\Omega \rho_0 \, dx = 1$. Then, the mapping $\Lambda = \Lambda_\varepsilon$ is given by

$$(\Lambda_\varepsilon v)(x) = \int_\Omega \rho_\varepsilon(x - y) v(y) \, dy, \quad \forall v \in X,$$

as well as the constraint sets $K(v) = K_\varepsilon(v)$ is of the form:

$$K_\varepsilon(v) = \left\{ w \in X; \ w \geqslant k_c\left(\int_\Omega \rho_\varepsilon(\cdot - y) v(y) \, dy \right) \text{ a.e. on } \Omega \right\}.$$

According to the above existence result, for each $\varepsilon > 0$ problem (10.21) has at least one solution $u_\varepsilon (\in K_\varepsilon(u_\varepsilon))$, and we see by taking $v = k^*$ in the inequality of (10.21) that $\{u_\varepsilon\}_{0<\varepsilon<1}$ is bounded in X. It is quite interesting to discuss the convergence of $\{u_\varepsilon\}$ as $\varepsilon \downarrow 0$.

EXAMPLE 3. Consider $\Omega = (0, 1)$ and define $\rho(x, y)$ by

$$\rho(x, y) := \begin{cases} s(x - y) & \text{if } y \in [x, 1), \\ 0 & \text{if } y \in (0, x), \end{cases}$$

where $s(\cdot)$ is a non-negative smooth function on $\mathbf{R}$ such that $s(x) = 0$ for $x \leqslant 0$. In this case it is an interesting question to derive necessary condition of function u in order to satisfy the inequality

$$u(x) \geqslant k_c\left(\int_0^1 s(x - y) u(y) \, dy \right) \quad \text{a.e. } x \in (0, 1). \tag{10.23}$$

For instance, assume that k^* is a sufficient large positive number and $k_c(r) = r$ is non-negative on $(-\infty, k^*/2)$. Then, it is easily derived from (10.23) that $u(1) \geqslant 0$, because $s(1 - y) = 0$ for all $y \in (0, 1)$. This means that it may be possible to control the solution u of problem (10.21) by appropriate choice of convolution kernel $s(\cdot)$.

REMARK 10.2. So far as quasi-variational inequalities of elliptic type are concerned, we notice some abstract results in [3] (also, see its bibliography); in fact, a quasi-variational inequality of the form

$$u \in K(u); \quad \langle Au - f, u - v\rangle \leqslant 0, \quad \forall v \in K(u),$$

is discussed, where A is a continuous, coercive and monotone mapping from X into X^*, in the following two typical cases (i) and (ii):

(i) $K(v)$ is a compact convex set in a fixed compact convex set K_0 for each $v \in K_0$ and the mapping $v \to K(v)$ is continuous in the sense that the indicator function $I_{K(v_n)}$ of $K(v_n)$ converges to $I_{K(v)}$ in the sense of Mosco as $v_n \to v$ in X.

(ii) X is a Banach lattice and A satisfies

$$\left\langle Aw_1 - Aw_2, (w_1 - w_2)^+ \right\rangle \geqslant 0, \quad \forall w_1, w_2 \in X,$$

where $[\cdot]^+$ denotes the positive part of $[\cdot]$. Further, $K(v)$ is a closed, convex set in X and the mapping $v \to K(v)$ is monotone increasing in the sense that if $v \geqslant u$ a.e. on Ω, then

$$\min\{\tilde{v}, \tilde{u}\} \in K(v), \quad \max\{\tilde{v}, \tilde{u}\} \in K(u), \quad \forall \tilde{v} \in K(v),\ \forall \tilde{u} \in K(u).$$

Our results, Theorems 10.1 and 10.2, are obtained in a similar situation to the above (i). Actually in [3] an existence theorem is mentioned in a little bit more general setting than (i), but our abstract existence result obtained by Theorem 10.1 is not included in the result of [3] in respect that our construction of solutions for quasi-variational inequalities is done for multivalued semi-monotone mappings $\tilde{A}$. In the case of (ii), the construction of a solution of the quasi-variational inequality of the form (10.21) is done with the help of a fixed-point theorem in Banach lattices. Theorems 10.1 and 10.2 were obtained in a recent paper [20].

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York (1975).

[2] H. Attouch, *Variational Convergence for Functions and Operators*, Pitman Advanced Publishing Program, Pitman, Boston (1984).

[3] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities*, John Wiley and Sons, Chichester (1984).

[4] V. Barbu, *Nonlinear Semigroupes and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leiden (1976).

[5] H. Brézis, *Équations et inéquations non linéaires dans les espaces vectoriels en dualité*, Ann. Inst. Fourier (Grenoble) **18** (1968), 115–175.

[6] F. Browder, *The fixed point theory of multi-valued mappings in topological vector spaces*, Math. Ann. **177** (1968), 283–301.

[7] F. Browder and P. Hess, *Nonlinear mappings of monotone type in Banach spaces*, J. Funct. Anal. **11** (1972), 251–294.

[8] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin (1985).

[9] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

[10] I.L. Glicksberg, *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952), 170–174.
[11] S. Kakutani, *A generalization of Brouwer's fixed point theorem*, Duke Math. J. **8** (1941), 457–459.
[12] N. Kenmochi, *Nonlinear operators of monotone type in reflexive Banach spaces and nonlinear perturbations*, Hiroshima Math. J. **4** (1974), 229–263.
[13] N. Kenmochi, *Pseudomonotone operators and nonlinear elliptic boundary value problems*, J. Math. Soc. Japan **27** (1975), 121–149.
[14] J.L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I*, Springer-Verlag, Berlin (1972).
[15] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. in Math. **3** (1969), 510–585.
[16] J. Nečas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson et C^{ie} Édituers, Paris (1967).
[17] A.W. Roberts and D.E. Varberg, *Convex Functions*, Academic Press, New York (1973).
[18] E. Zeidler, *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, Heidelberg (1986).
[19] E. Zeidler, *Nonlinear Functional Analysis and its Applications IIB*, Springer-Verlag, Heidelberg (1990).
[20] R. Kano, Y. Murase and N. Kenmochi, *Existence theorems for elliptic quasi-variational inequalities in Banach spaces*, to appear.

CHAPTER 5

Stationary Navier–Stokes Flow in 2-D Channels Involving the General Outflow Condition

Hiroko Morimoto

Department of Mathematics, Meiji University, 1-1-1 Higashi-mita, Tanaka-ku, Kawasaki, 214 8571, Japan
E-mail: hiroko@math.meiji.ac.jp

Contents

1. Introduction . . . 301
2. Function spaces and the symmetry . . . 304
   2.1. Divergence free functions . . . 304
   2.2. Symmetry . . . 306
3. Channel with one outlet (semi-infinite channel) . . . 308
   3.1. Extension of the boundary value . . . 311
   3.2. Existence and regularity of the weak solution . . . 318
   3.3. Asymptotic behavior of the solution . . . 323
   3.4. A remark on the uniqueness of the solution . . . 335
4. Channel with J outlets ($J \geqslant 2$) . . . 336
   4.1. Connection of the Poiseuille flows . . . 338
   4.2. Extension of the boundary value . . . 340
   4.3. Existence of the solution . . . 341
   4.4. Regularity and asymptotic behavior of the solution . . . 345
5. Channel contained in T . . . 346
6. A characterization of Poiseuille flow . . . 350
Acknowledgement . . . 352
References . . . 352

Abstract

Let Ω be a two-dimensional multiply connected domain with unbounded smooth boundary. We suppose that Ω has several outlets. The boundary $\partial\Omega$ has, besides the infinite part Γ_0, several bounded connected components, $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$. We study the existence of the stationary Navier–Stokes flow in Ω satisfying prescribed non-zero flux on every Γ_i $(1 \leqslant i \leqslant N)$ and tending to the prescribed Poiseuille flow at the infinity. Assume that the domain and the boundary value are symmetric with respect to the x_1-axis and every Γ_i $(1 \leqslant i \leqslant N)$ intersects

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 4
Edited by M. Chipot
© 2007 Elsevier B.V. All rights reserved

the x_1-axis. Then, we can show the existence of weak solution for small Poiseuille flows and arbitrary large boundary value. The regularity of the solution and the asymptotic behavior at the infinity are also discussed.

Keywords: Stationary Navier–Stokes flow, General outflow condition, 2-D multiply connected infinite channel, Symmetry

MSC: 35Q30, 76D03, 76D05

1. Introduction

We study the two-dimensional stationary Navier–Stokes equations in infinite channels. Let Ω be a two-dimensional multiply connected domain with smooth boundary satisfying the following two conditions.

(Ω-1) Let $\Omega_1, \dots, \Omega_J$ be mutually disjoint semi-infinite channels in $\mathbb{R}^2$, that is, for every j $(1 \leqslant j \leqslant J)$, there exists a straight channel T_j such that Ω_j coincides with one of the connected component of $T_j \cap \{|x| > R_0\}$ for some $R_0 > 0$. Let $\Omega_0 = \Omega \setminus \bigcup_{j=1}^{J} \Omega_j$. We suppose that Ω_0 is a bounded set. We call Ω_j $(1 \leqslant j \leqslant J)$ outlet.

(Ω-2) The boundary $\partial\Omega$ is smooth and consists of infinite part Γ_0 and finite part $\Gamma_1, \Gamma_2, \dots, \Gamma_N$, the latter being simple closed curves. The outer boundary Γ_0 has J unbounded connected components $\Gamma_0^1, \Gamma_0^2, \dots, \Gamma_0^J$.

$$\partial\Omega = \bigcup_{i=0}^{N} \Gamma_i, \qquad \Gamma_0 = \bigcup_{j=1}^{J} \Gamma_0^j.$$

Here, a straight channel means an unbounded domain bounded by two parallel straight lines.

We study the stationary Navier–Stokes equations

$$\text{(NS)} \quad \begin{cases} (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu \Delta \boldsymbol{u} - \nabla p + \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \end{cases}$$

with the boundary condition

$$\text{(BC)} \quad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on } \partial\Omega, \\ \boldsymbol{u} \to \boldsymbol{V}_j & \text{as } |x| \to \infty \text{ in } \Omega_j \ (1 \leqslant j \leqslant J). \end{cases}$$

Here $\boldsymbol{u}$ is the velocity of the fluid, p is the pressure, $\boldsymbol{f}$ is the external force, ν is the kinematic viscosity(constant), $\boldsymbol{\beta}$ is given on the boundary $\partial\Omega$, $\boldsymbol{V}_j$ is a Poiseuille flow in T_j $(1 \leqslant j \leqslant J)$. Even if $T_i = T_j$ for some $i \neq j$, it may happen $\boldsymbol{V}_i \neq \boldsymbol{V}_j$. $\boldsymbol{\beta}$ and $\boldsymbol{V}_j$ $(1 \leqslant j \leqslant J)$ should satisfy an outflow condition:

$$\text{(OC)} \quad \int_{\partial\Omega} \boldsymbol{\beta} \cdot \boldsymbol{n} \, \mathrm{d}\sigma + \sum_{j=1}^{J} \int_{\Sigma_j} \boldsymbol{V}_j \cdot \boldsymbol{\nu}_j \, \mathrm{d}\sigma = 0$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$, Σ_j the section of Ω_j, $\boldsymbol{\nu}_j$ the unit normal vector to Σ_j, directing from Ω_0 to Ω_j.

This type of problem, called Leray's problem, has been studied by many authors for two or three-dimensional domain. Especially we refer to Amick [1,2]. He showed in [1] the existence result for two and three-dimensional domain as above mentioned type with $J = 2$, $N = 0$ and $\boldsymbol{\beta} = 0$. He mentioned also to the case $J \geqslant 3$. The method used in his proof

works well for the case $\boldsymbol{\beta} \neq 0$ satisfying on Γ_k $(k = 1, 2, \ldots, N)$ so called the stringent outflow condition:

$$\text{(SOC)} \quad \int_{\Gamma_k} \boldsymbol{\beta} \cdot \boldsymbol{n} \, d\sigma = 0 \quad (\forall k = 1, 2, \ldots, N), \quad \boldsymbol{\beta} = 0 \text{ on } \Gamma_0.$$

However, if we assume only the general outflow condition

$$\text{(GOC)} \quad \int_{\partial\Omega} \boldsymbol{\beta} \cdot \boldsymbol{n} \, d\sigma = 0,$$

Amick's method cannot give the existence result. In [2], he showed the asymptotic behavior, that is, the exponential decay of the solution to the Poiseuille flow at the infinity.

Recently, P. Rabier [17] proved the existence of symmetric solution for the two-dimensional straight channel, where $J = 2$, $N = 0$, for the symmetric boundary value $\boldsymbol{\beta}$ satisfying $\int_{\Gamma_0^j} \boldsymbol{\beta} \cdot \boldsymbol{n} \, d\sigma = 0$ $(j = 1, 2)$. It should be emphasized that there is no limitation for the magnitude of flux. For semi-infinite straight channel, the existence of the solution is established with some exceptional values of the flux. In [18], he discussed the problem for two-dimensional straight channel with non-symmetric data satisfying $\int_{\Gamma_0^j} \boldsymbol{\beta} \cdot \boldsymbol{n} \, d\sigma = 0$ $(j = 1, 2)$ and showed the existence of solution for small flux. He states that the upper bound for the flux is larger than the values ever known.

Our main concern is to study the case $N \geqslant 1$ and $\int_{\Gamma_j} \boldsymbol{\beta} \cdot \boldsymbol{n} \, d\sigma \neq 0$ for some j. We remark again that if $\boldsymbol{\beta} = 0$ on Γ_0 and $\int_{\Gamma_j} \boldsymbol{\beta} \cdot \boldsymbol{n} \, d\sigma = 0$ for all $j = 1, \ldots, N$, we can obtain an appropriate solenoidal extension of $\boldsymbol{\beta}$ and the problem can be reduced to Amick's case.

In order to state our results, we need some function spaces. Let $L^2(\Omega)$ be the set of all square summable functions defined in Ω. The inner product and the norm are denoted by $(\cdot, \cdot)_\Omega$ and $\| \cdot \|_\Omega$ or simply $(\cdot, \cdot)$ and $\| \cdot \|$. $H^1(\Omega)$ is the usual Sobolev space. For spaces of vector valued functions, we use the notation $\mathbb{L}^2(\Omega)$, $\mathbb{H}^1(\Omega)$ and so on. Let $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ be the set of all smooth solenoidal vector valued functions with compact support in Ω. $V(\Omega)$ is the completion of $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ in the Dirichlet norm $\|\nabla \cdot \|$. Since Poincaré's inequality holds for our domain, the Dirichlet norm is equivalent to the H^1 norm.

DEFINITION 1.1. $\boldsymbol{u}$ is called a weak solution to (NS), (BC) if

(1) $\boldsymbol{u} \in \mathbb{H}^1_{\text{loc}}(\Omega)$ and satisfies

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}\big) = (\boldsymbol{f}, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in \mathbb{C}^\infty_{0,\sigma}(\Omega)). \tag{1.1}$$

(2) $\boldsymbol{u} = \boldsymbol{\beta}$ on $\partial\Omega$ in the trace sense.

(3) $\boldsymbol{u} - \boldsymbol{V}_j \in \mathbb{H}^1(\Omega_j)$ for every $1 \leqslant j \leqslant J$.

DEFINITION 1.2. The set $\mathcal{S}$ consists of all $\boldsymbol{u}$ of the form

$$\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{W} + \boldsymbol{b}_0$$

such that

(1) $\boldsymbol{w} \in V(\Omega)$.
(2) $\boldsymbol{W}$ belongs to $\mathbb{H}^1_{\mathrm{loc}}(\Omega)$, is solenoidal in Ω. $\boldsymbol{W}$ coincides with $\boldsymbol{V}_j$ in $\Omega_j \cap \{|x| > R_0 + 1\}$ for every $1 \leqslant j \leqslant J$.
(3) $\boldsymbol{b}_0$ is a solenoidal extension of $\boldsymbol{\beta} - \boldsymbol{W}|_{\partial\Omega}$ and of compact support.

We look for $\boldsymbol{u} \in \mathcal{S}$ satisfying (1.1).

Our results are as follows. Let Ω be a 2-dimensional multiply connected channel, satisfying the conditions (Ω-1), (Ω-2). We suppose Ω is symmetric with respect to the x_1-axis. Therefore there may be some pairs of outlets which are symmetric with respect to the x_1-axis. If an outlet Ω_i does not find another outlet symmetric to it, Ω_i itself is symmetric with respect to the x_1-axis. We call this outlet a self-symmetric outlet. It may happen three cases.

(a) There is no self-symmetric outlet.
(b) There is only one self-symmetric outlet.
(c) There are two self-symmetric outlets.

In the case (a), the x_1-axis intersects the outer boundary Γ_0 at two points. In the case (b), the x_1-axis intersects the outer boundary Γ_0 at one point. In the case (c), the x_1-axis does not intersect the outer boundary Γ_0.

We give some examples of geometry. The case (a): wedge shaped channels [15]. The case (b): wedge shaped channels with an infinite stem [14], semi infinite channels [13]. The case (c): ordinary straight channels [16].

We consider the case where the boundary value $\boldsymbol{\beta}$ has, in general, non-zero flux on Γ_i. Suppose that $\boldsymbol{\beta}$ is symmetric with respect to the x_1-axis. Here the vector field $\boldsymbol{u} = (u_1, u_2)$ is called symmetric with respect to the x_1-axis if u_1 is even in x_2 and u_2 odd in x_2, that is,

$$u_1(x_1, x_2) = u_1(x_1, -x_2), \qquad u_2(x_1, x_2) = -u_2(x_1, -x_2)$$

holds. If Ω_j and Ω_k are symmetric with respect to the x_1-axis, we suppose the Poiseuille flows $\boldsymbol{V}_j$ and $\boldsymbol{V}_k$ are symmetric with respect to the x_1-axis.

$\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$ is the set of all $\boldsymbol{\varphi}$ in $\mathbb{C}^{\infty}_{0,\sigma}(\Omega)$ which is symmetric with respect to the x_1-axis. $V^S(\Omega)$ is the set of all $\boldsymbol{u}$ in $V(\Omega)$ which is symmetric with respect to the x_1-axis.

Next definition is concerned with a characterization of the Poiseuille flow.

DEFINITION 1.3. For $1 \leqslant j \leqslant J$, σ_j is defined as follows.

$$\sigma_j \equiv \sup_{\boldsymbol{\varphi} \in V(T_j)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}_j)_{T_j}}{\|\nabla \boldsymbol{\varphi}\|^2_{T_j}}. \tag{1.2}$$

It is known $\sigma_j > 0$. Cf. [13].

If the straight channel T_j is symmetric with respect to the x_1-axis, i.e., is self-symmetric, we define σ_j^S as follows.

$$\sigma_j^S \equiv \sup_{\boldsymbol{\varphi} \in V^S(T_j)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V}_j)_{T_j}}{\|\nabla \boldsymbol{\varphi}\|^2_{T_j}}. \tag{1.3}$$

It is known that $0 < \sigma_j^S \leqslant \sigma_j$. See [13].

Now we can state our results. Suppose that every Γ_i $(1 \leqslant i \leqslant N)$ intersects the axis of symmetry, that $\boldsymbol{\beta}$ vanishes on Γ_0 and satisfies (OC). If $\nu > \max(\sigma_1, \sigma_2, \ldots, \sigma_J)$, then there exists a symmetric weak solution $\boldsymbol{u}$ (Theorems 3.1, 4.1, and 4.2). In case T_j is symmetric with respect to the x_1-axis, σ_j is replaced by σ_j^S. The proof is similar to that of Amick [1], except the extension of the boundary values with (GOC). Furthermore, $\boldsymbol{u}$ tends to the Poiseuille flow $\boldsymbol{V}_j$ exponentially as $|\boldsymbol{x}| \to \infty$ in Ω_j (Theorems 3.3 and 4.4). We use totally different method from Amick [2].

2. Function spaces and the symmetry

2.1. *Divergence free functions*

Let $\Omega \subset \mathbb{R}^2$ be a domain satisfying the conditions (Ω-1) and (Ω-2). $\mathbb{L}^2(\Omega)$ is a Hilbert space consisting of vector valued functions, every component of which belongs to $L^2(\Omega)$. We introduce a space of the divergence free test functions.

$$\mathbb{C}^\infty_{0,\sigma}(\Omega) = \left\{\boldsymbol{u} \in \mathbb{C}^\infty_0(\Omega) \mid \operatorname{div} \boldsymbol{u} = 0\right\}.$$

DEFINITION 2.1. $H = H(\Omega)$ is the closure of $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ in $\mathbb{L}^2(\Omega)$. H is a Hilbert space with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_H = (\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{v}(x)\, \mathrm{d}x = \sum_{i=1}^2 \int_\Omega u_i(x) v_i(x)\, \mathrm{d}x.$$

The norm is denoted by $\|\boldsymbol{u}\|_{L^2(\Omega)}$, or simply $\|\boldsymbol{u}\|_\Omega$, $\|\boldsymbol{u}\|$.

$V = V(\Omega)$ is the completion of $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ under the Dirichlet norm $\|\nabla \cdot\|$. V is a Hilbert space with the inner product

$$((\boldsymbol{u}, \boldsymbol{v}))_V = ((\boldsymbol{u}, \boldsymbol{v})) = \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v}\, \mathrm{d}x = \sum_{i,j=1}^2 \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}\, \mathrm{d}x.$$

The norm is denoted by $\|\boldsymbol{u}\|_V$ which is equivalent to $\|\nabla \boldsymbol{u}\|$.

DEFINITION 2.2. $G(\Omega) = \{w \in \mathbb{L}^2(\Omega) \mid \exists p \in H^1_{\mathrm{loc}}(\Omega),\ w = \nabla p\}$.

The next theorem is well known (Ladyzhenskaya [10], Temam [19]).

THEOREM 2.1 (Helmholtz decomposition).

$$\mathbb{L}^2(\Omega) = H(\Omega) \oplus G(\Omega).$$

That is $H(\Omega)^\perp = G(\Omega)$. For every $\boldsymbol{u} \in \mathbb{L}^2(\Omega)$ there exist $\boldsymbol{v} \in H(\Omega)$ and $\boldsymbol{w} \in G(\Omega)$ such that $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ holds. The decomposition is unique.

Now we present some useful formula for vector functions:

LEMMA 2.1. *For vector valued functions* $\boldsymbol{u}$, $\boldsymbol{v}$, *we have*

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}=\nabla(\boldsymbol{u}\cdot\boldsymbol{v})-\boldsymbol{u}\times\operatorname{rot}\boldsymbol{v}-\boldsymbol{v}\times\operatorname{rot}\boldsymbol{u},$$

where (*for* 2*-dimensional case*)

$$\boldsymbol{u}\times\operatorname{rot}\boldsymbol{v}=\omega(\boldsymbol{v})\boldsymbol{u}^{\perp},\quad \omega(\boldsymbol{v})=D_1v_2-D_2v_1,\quad \boldsymbol{u}^{\perp}=(u_2,-u_1).$$

In particular,

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\frac{1}{2}\nabla|\boldsymbol{u}|^2-\omega(\boldsymbol{u})\boldsymbol{u}^{\perp}.$$

Let us consider a trilinear form for smooth functions $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ defined in $\overline{\Omega}$.

$$B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}):=\big((\boldsymbol{u}\cdot\nabla)\boldsymbol{v},\ \boldsymbol{w}\big)_{L^2(\Omega)}=\int_\Omega\sum_{i,j=1}^{2}u_i\frac{\partial v_j}{\partial x_i}w_j\,\mathrm{d}x. \tag{2.1}$$

Integration by parts yields

$$B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=\int_{\partial\Omega}(\boldsymbol{u}\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\boldsymbol{w})\,\mathrm{d}\sigma-\int_\Omega(\boldsymbol{w}\cdot\boldsymbol{v})\operatorname{div}\boldsymbol{u}\,\mathrm{d}x-B(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v}) \tag{2.2}$$

where $\boldsymbol{n}$ is the unit outward normal vector to the boundary $\partial\Omega$.

LEMMA 2.2. *The trilinear functional* $B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})$ *is extended continuously to* $V\times V\times V$. *There exists a positive constant* $C_0>0$ *such that*

$$\big|B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})\big|\leqslant C_0\|\boldsymbol{u}\|_V\|\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V\quad(\forall\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V) \tag{2.3}$$

holds true. Furthermore,

$$B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-B(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})\quad(\forall\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in V), \tag{2.4}$$

$$B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v})=0\quad(\forall\boldsymbol{u},\boldsymbol{v}\in V). \tag{2.5}$$

REMARK 2.1. Let $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathbb{H}^1(\Omega)$. Because of (2.2), it is immediate that if $\operatorname{div}\boldsymbol{u}=0$ and if one of the trace to $\partial\Omega$ of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ vanishes, then $B(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w})=-B(\boldsymbol{u},\boldsymbol{w},\boldsymbol{v})$ holds.

We need the following result for divergence operator, which is due to Bogovskii [4]. See also Galdi [8], Babuska and Aziz [3].

LEMMA 2.3. *Let Q be a bounded domain, star-like with respect to every point of a ball $B \Subset Q$. Let $f(x)$ be in $L^2(Q)$ satisfying*

$$\int_Q f(x)\,dx = 0.$$

Then there exists $\boldsymbol{v} \in \mathbb{H}_0^1(Q)$ such that

$$\begin{cases} \operatorname{div} \boldsymbol{v} = f & \text{in } Q, \\ \boldsymbol{v} = 0 & \text{on } \partial Q \end{cases} \tag{2.6}$$

satisfying

$$\|\boldsymbol{v}\|_{H^1(Q)} \leqslant c\|f\|_{L^2(Q)},$$

where c is a constant dependent only on the diameter of Q and B.

Furthermore, if $f(x)$ is in $H_0^1(Q)$, the solution $\boldsymbol{v}$ to (2.6) *belongs to $\mathbb{H}_0^2(Q)$ and satisfies*

$$\|\boldsymbol{v}\|_{H^2(Q)} \leqslant c\|f\|_{H^1(Q)}.$$

REMARK 2.2. It is to be remarked that the constant c is the same for domains of similar figure and also for domains of congruent figure.

LEMMA 2.4. *Let Q be a bounded domain or a straight channel and $\Sigma(t) = \{(x_1, x_2) \in Q \mid x_1 = t\}$. Then, for every $\boldsymbol{w} \in V(Q)$, we have*

$$\int_{\Sigma(t)} w_1(t, x_2)\,dx_2 = 0.$$

OUTLINE OF THE PROOF. Let Q be bounded, $\omega = \{(x_1, x_2) \in Q \mid x_1 < t\}$. By Gauss' Theorem, we have

$$0 = \int_\omega \operatorname{div} \boldsymbol{w}\,dx = \int_{\partial\omega} \boldsymbol{w} \cdot \boldsymbol{n}\,d\sigma = \int_{\Sigma(t)} w_1\,dx_2.$$

For the case Q straight channel, using the density argument, we can prove the result similarly.

2.2. *Symmetry*

We suppose that the domain Ω is symmetric with respect to the x_1-axis. For the vector valued function $\boldsymbol{\varphi}(x) = \boldsymbol{\varphi}(x_1, x_2)$ defined in Ω, put

$$\boldsymbol{\varphi}^s(x_1, x_2) = \frac{1}{2}\big(\varphi_1(x_1, x_2) + \varphi_1(x_1, -x_2),\ \varphi_2(x_1, x_2) - \varphi_2(x_1, -x_2)\big),$$

$$\boldsymbol{\varphi}^a(x_1, x_2) = \frac{1}{2}\big(\varphi_1(x_1, x_2) - \varphi_1(x_1, -x_2),\ \varphi_2(x_1, x_2) + \varphi_2(x_1, -x_2)\big).$$

$\boldsymbol{\varphi}^s$ is called symmetric part of $\boldsymbol{\varphi}$, and $\boldsymbol{\varphi}^a$ antisymmetric part of $\boldsymbol{\varphi}$. It holds

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}^s + \boldsymbol{\varphi}^a.$$

DEFINITION 2.3. A vector valued function $\boldsymbol{u}(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2))$ is called symmetric with respect to the x_1-axis, or simply, symmetric, if and only if $\boldsymbol{u} = \boldsymbol{u}^s$, that is,

$$u_1(x_1, -x_2) = u_1(x_1, x_2), \qquad u_2(x_1, -x_2) = -u_2(x_1, x_2).$$

$u_1(x_1, x_2)$ is even in x_2 and $u_2(x_1, x_2)$ is odd in x_2.

A vector valued function $\boldsymbol{u}(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2))$ is antisymmetric, if and only if $\boldsymbol{u} = \boldsymbol{u}^a$, that is,

$$u_1(x_1, -x_2) = -u_1(x_1, x_2), \qquad u_2(x_1, -x_2) = u_2(x_1, x_2).$$

$u_1(x_1, x_2)$ is odd in x_2 and $u_2(x_1, x_2)$ is even in x_2.

Straightforward calculation shows

LEMMA 2.5. *Let Ω be a symmetric domain with respect to the x_1-axis. Let $\boldsymbol{u}$ be smooth and symmetric with respect to the x_1-axis, $p(x) = p(x_1, x_2)$ even in x_2. Then,*

(i) *$\Delta\boldsymbol{u}$ and $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ are symmetric with respect to the x_1-axis.*
(ii) *$\operatorname{div} \boldsymbol{u}$ is even in x_2.*
(iii) *$\nabla p(x)$ is symmetric with respect to the x_1-axis.*

LEMMA 2.6. *Let Ω be a symmetric domain with respect to the x_1-axis. Let $\boldsymbol{u} \in \mathbb{L}^2(\Omega)$ be symmetric, and $\boldsymbol{v} \in \mathbb{L}^2(\Omega)$ antisymmetric. Then,*

$$(\boldsymbol{u}, \boldsymbol{v})_{L^2(\Omega)} = \int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{v}(x)\, dx = 0. \tag{2.7}$$

Furthermore, if $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{H}^1(\Omega)$, then,

$$(\nabla\boldsymbol{u}, \nabla\boldsymbol{v})_{L^2(\Omega)} = 0. \tag{2.8}$$

PROOF. Put

$$\Omega_+ = \{x = (x_1, x_2) \in \Omega \mid x_2 > 0\}, \qquad \Omega_- = \{x = (x_1, x_2) \in \Omega \mid x_2 < 0\}.$$

Let $\boldsymbol{u} = (u_1, u_2)$, $\boldsymbol{v} = (v_1, v_2)$. Then

$$\boldsymbol{u}(x) \cdot \boldsymbol{v}(x) = u_1(x_1, x_2)v_1(x_1, x_2) + u_2(x_1, x_2)v_2(x_1, x_2).$$

Since $u_1(x_1, x_2)$ is even in x_2 and $v_1(x_1, x_2)$ is odd in x_2, their product is odd in x_2.

$$\int_{\Omega} u_1(x)v_1(x)\,dx = \int_{\Omega_+} u_1(x)v_1(x)\,dx + \int_{\Omega_-} u_1(x)v_1(x)\,dx$$
$$= \int_{\Omega_+} u_1(x)v_1(x)\,dx - \int_{\Omega_+} u_1(x)v_1(x)\,dx = 0,$$

because the domain of integration is symmetric with respect to the x_1-axis. Similarly, we have

$$\int_{\Omega} u_2(x)v_2(x)\,dx = 0$$

and (2.7) is proved. In the same manner, (2.8) is obtained. □

LEMMA 2.7. *Let Ω be a symmetric domain with respect to the x_1-axis. Let $\boldsymbol{u} \in \mathbb{H}^1(\Omega)$ be symmetric with respect to the x_1-axis. Then the trace to the x_1-axis exists and the second component vanishes there, that is,*

$$\boldsymbol{u}(x_1, 0) = \big(u_1(x_1, 0), u_2(x_1, 0)\big) = \big(u_1(x_1, 0), 0\big) \quad ((x_1, 0) \in \Omega).$$

LEMMA 2.8. *Let Ω be a symmetric domain with respect to the x_1-axis. Put*

$$\Omega_+ = \big\{x = (x_1, x_2) \in \Omega \mid x_2 > 0\big\}, \qquad \Omega_- = \big\{x = (x_1, x_2) \in \Omega \mid x_2 < 0\big\}$$

and

$$\Gamma_{i,+} = \big\{x = (x_1, x_2) \in \Gamma_i \mid x_2 > 0\big\}, \quad \Gamma_{i,-} = \big\{x = (x_1, x_2) \in \Gamma_i \mid x_2 < 0\big\}$$
$$(i = 0, 1, \ldots, N).$$

Let $\boldsymbol{u} \in \mathbb{H}^1(\Omega)$ be symmetric with respect to the x_1-axis. Then, $\boldsymbol{u} \cdot \boldsymbol{n}$ is even in x_2 and

$$\int_{\Gamma_{i,+}} \boldsymbol{u} \cdot \boldsymbol{n}\,d\sigma = \int_{\Gamma_{i,-}} \boldsymbol{u} \cdot \boldsymbol{n}\,d\sigma = \frac{1}{2}\int_{\Gamma_i} \boldsymbol{u} \cdot \boldsymbol{n}\,d\sigma \quad (i = 0, 1, \ldots, N),$$

$$\int_{\partial\Omega_+} \boldsymbol{u} \cdot \boldsymbol{n}\,d\sigma = \sum_{i=0}^{N} \int_{\Gamma_{i,+}} \boldsymbol{u} \cdot \boldsymbol{n}\,d\sigma.$$

3. Channel with one outlet (semi-infinite channel)

Let Ω be an unbounded domain in $\mathbb{R}^2$, x_1x_2 plane, satisfying the following two conditions. Let $T = \mathbb{R} \times (-1, 1)$.

(I) Let $\Omega = \Omega_0 \cup \Omega_1$ where

$$\Omega_0 := \{x \in \Omega\colon\ x_1 \leqslant 0\},$$
$$\Omega_1 := \{x \in \Omega\colon\ x_1 > 0\} = T \cap \{x_1 > 0\}.$$

Ω_0 is a bounded set, symmetric with respect to the x_1-axis.

(II) The boundary of Ω is smooth and is composed of the infinite component Γ_0 and the finite components $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$, the latter being simple closed curves. Namely,

$$\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_N.$$

Γ_0 can be regarded as the outer boundary of Ω while Γ_i $(1 \leqslant i \leqslant N)$ as the inner boundary.

REMARK 3.1. In this case, the x_1-axis intersects Γ_0 at one point (type (b)).

We consider the stationary Navier–Stokes equations

$$(\mathrm{NS})_1 \quad \begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \end{cases}$$

with the boundary condition

$$(\mathrm{BC})_1 \quad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on } \partial\Omega, \\ \boldsymbol{u} \to \boldsymbol{V} & \text{as } x_1 \to \infty \text{ in } \Omega, \end{cases}$$

where $\boldsymbol{\beta}$ is a given function on $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_N$, satisfying $\boldsymbol{\beta} = 0$ on Γ_0, and $\boldsymbol{V} = \mu\boldsymbol{U}$, $\boldsymbol{U}$ being the standard Poiseuille flow in $T = \mathbb{R}\times(-1,1)$:

$$(\mathrm{U}) \quad \boldsymbol{U} = \frac{3}{4}\bigl(1 - x_2^2, 0\bigr)$$

and μ a non-zero constant. Put $P = \frac{3}{2}\nu\mu x_1$. $\boldsymbol{V}$ and P satisfy the Navier–Stokes equations $(\mathrm{NS})_1$. Note that $(\boldsymbol{V}\cdot\nabla)\boldsymbol{V} = 0$.

For the boundary value $\boldsymbol{\beta}$, we suppose

$$(\mathrm{OC})_1 \quad \int_{\partial\Omega} \boldsymbol{\beta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \int_{-1}^{1} \boldsymbol{V}\cdot\boldsymbol{v}\,\mathrm{d}x_2 = \int_{\partial\Omega} \boldsymbol{\beta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \mu = 0$$

where $\boldsymbol{n}$ is the unit outward normal vector to the boundary $\partial\Omega$, and $\boldsymbol{v} = {}^t(1,0)$.

We treated this problem in [13] and proved the existence of the solution for $\boldsymbol{f} \in \mathbb{L}^2(\Omega)$ and a decay result for $\boldsymbol{f} = 0$. In this note, we state a little more general case $\boldsymbol{f} \neq 0$.

DEFINITION 3.1. $\boldsymbol{u}$ is called a weak solution to $(\mathrm{NS})_1$, $(\mathrm{BC})_1$ if

(1) $\boldsymbol{u} \in \mathbb{H}^1_{\mathrm{loc}}(\Omega)$ and satisfies

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}\big) = (\boldsymbol{f}, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in \mathbb{C}^\infty_{0,\sigma}(\Omega)). \tag{3.1}$$

(2) $\boldsymbol{u} = \boldsymbol{\beta}$ on $\partial\Omega$ in the trace sense.
(3) $\boldsymbol{u} - \boldsymbol{V} \in \mathbb{H}^1(\Omega)$.

Let

$$\sigma^S \equiv \sup_{\boldsymbol{\varphi} \in V^S(T)} \frac{((\boldsymbol{\varphi} \cdot \nabla)\boldsymbol{\varphi}, \boldsymbol{V})_T}{\|\nabla \boldsymbol{\varphi}\|^2_T}. \tag{3.2}$$

It is known that $\sigma^S > 0$. Cf. Morimoto and Fujita [13].

THEOREM 3.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is symmetric with respect to the x_1-axis satisfying* (I) *and* (II), *and that every Γ_i $(1 \leqslant i \leqslant N)$ intersects the x_1-axis. Assume that $\boldsymbol{f} \in \mathbb{L}^2(\Omega)$ is symmetric with respect to the x_1-axis and that the boundary value $\boldsymbol{\beta}$ is smooth on $\partial\Omega$ and symmetric, $\boldsymbol{\beta} = 0$ on Γ_0 and satisfies* $(\mathrm{OC})_1$. *If $\nu - \sigma^S > 0$, then there exists a symmetric weak solution $\boldsymbol{u} \in \mathcal{S}$ to* $(\mathrm{NS})_1$, $(\mathrm{BC})_1$.

REMARK 3.2. Since we look for the symmetric solution, the test functions $\mathbb{C}^\infty_{0,\sigma}(\Omega)$ in the equation (3.1), can be replaced by $\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$. It is easy to prove this fact by using Lemmas 2.6 and 2.7.

REMARK 3.3. It is well known (e.g. Ladyzhenskaya [10], Temam [19], Galdi [8]) that for the weak solution $\boldsymbol{u}$ to $(\mathrm{NS})_1$, $(\mathrm{BC})_1$, there exists a scalar function $p \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \big((\boldsymbol{u} \cdot \nabla \boldsymbol{u}), \boldsymbol{v}\big) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in \mathbb{C}^\infty_0(\Omega)).$$

p is called an associated pressure of $\boldsymbol{u}$. We call $\{\boldsymbol{u}, p\}$ solution pair to $(\mathrm{NS})_1$, $(\mathrm{BC})_1$.

For the regularity and the asymptotic behavior of the flow, we obtain the following results.

THEOREM 3.2. *Under the assumption of Theorem* 3.1, *suppose further $\boldsymbol{f} \in \mathbb{C}^\infty(\overline{\Omega})$. The solution pair $\{\boldsymbol{u}, p\}$ to* $(\mathrm{NS})_1$, $(\mathrm{BC})_1$ *obtained in Theorem* 3.1 *is smooth in the closure of Ω.*

Let α be multi-index (α_1, α_2) and $|\alpha| = \alpha_1 + \alpha_2$.

$$D^\alpha u = D_1^{\alpha_1} D_2^{\alpha_2} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}.$$

We use the notation $\Omega^{t,\infty} = \{(x_1, x_2) \in \Omega \mid x_1 > t\}$.

THEOREM 3.3. *Under the assumption of Theorem* 3.2, *suppose that there exist positive constants* C *and* λ_0 *such that*

$$\|\boldsymbol{f}\|_{\Omega^t,\infty} < C\mathrm{e}^{-\lambda_0 t}.$$

Let $\{\boldsymbol{u}, p\}$ *be the solution pair to* $(\mathrm{NS})_1$, $(\mathrm{BC})_1$ *obtained in Theorem* 3.1. *Then* $\{\boldsymbol{u}, \nabla p\}$ *tends to* $\{\boldsymbol{V}, \nabla P\}$ *exponentially, as* $x_1 \to \infty$ *in* Ω, *that is, there exists a positive constant* λ *such that for every multi-index* α, *the estimates*

$$\left|D^\alpha\left\{\boldsymbol{u}(x) - \boldsymbol{V}(x)\right\}\right| \leqslant C_\alpha \mathrm{e}^{-\lambda x_1} \quad (x = (x_1, x_2) \in \Omega_1, x_1 > 0),$$
$$\left|D^\alpha \nabla\left\{p(x) - P(x)\right\}\right| \leqslant C_\alpha \mathrm{e}^{-\lambda x_1} \quad (x = (x_1, x_2) \in \Omega_1, x_1 > 0)$$

hold true for some positive constant C_α.

3.1. *Extension of the boundary value*

We need some suitable extension of the boundary value. The following lemma holds true even for non-symmetric domain and non-symmetric boundary value (e.g. Galdi [8]). For the convenience of the readers, we prove the lemma.

LEMMA 3.1. *Let* $\boldsymbol{\beta}_0$ *be defined on* $\partial\Omega$, *smooth and have a compact support. Suppose*

$$\text{(SOC)} \quad \int_{\Gamma_i} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, \mathrm{d}\sigma = 0 \quad (i = 0, 1, 2, \ldots, N).$$

Then for every $\varepsilon > 0$ *there exists a solenoidal extension* $\boldsymbol{b}_0$ *of* $\boldsymbol{\beta}_0$, *having a compact support and satisfying*

$$\text{(L)} \quad \left|\left((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_0\right)\right| \leqslant \varepsilon\|\nabla\boldsymbol{v}\|^2 \quad (\forall \boldsymbol{v} \in V(\Omega)).$$

If $\boldsymbol{\beta}_0$ does not enjoy the condition (SOC), we cannot expect the existence of its solenoidal extension satisfying the inequality (L). Nevertheless, if it is symmetric, we have

LEMMA 3.2. *Suppose that* $\boldsymbol{\beta}_0$ *is smooth on* $\partial\Omega$, *symmetric with respect to the* x_1*-axis, has compact support, and that every connected component of* $\partial\Omega$ *intersects the* x_1*-axis. Suppose further*

$$\text{(GOC)} \quad \int_{\partial\Omega} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, \mathrm{d}\sigma = \sum_{i=0}^{N} \int_{\Gamma_i} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, \mathrm{d}\sigma = 0.$$

Then, for every $\varepsilon > 0$, *there exists a solenoidal symmetric extension* $\boldsymbol{b}_0$ *of* $\boldsymbol{\beta}_0$, *having a compact support and satisfying the inequality*

$$\text{(LF)} \quad \left|\left((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_0\right)\right| \leqslant \varepsilon\|\nabla\boldsymbol{v}\|^2 \quad (\forall \boldsymbol{v} \in V^S(\Omega)).$$

REMARK 3.4. We call (L) Leray's inequality and (LF) Leray–Fujita's inequality. For 2-dimensional bounded symmetric domain, Fujita [7] proved this type of inequality by the virtual drain method, and showed an a priori estimate for the solution.

PROOF OF LEMMA 3.1. Since the support of $\boldsymbol{\beta}$ is compact, we can find a bounded domain Ω_{00} such that

$$\Omega_0 \subset \Omega_{00} \subset \Omega, \qquad \partial\Omega_{00} \cap \partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_N \cup (\Gamma_0 \cap \operatorname{supp}\boldsymbol{\beta}).$$

Put

$$\tilde{\boldsymbol{\beta}}(x) = \begin{cases} \boldsymbol{\beta}(x) & (x \in \partial\Omega_{00} \cap \partial\Omega), \\ 0 & (x \in \partial\Omega_{00} \setminus \partial\Omega). \end{cases}$$

There exists a function $\tilde{\varphi}(x)$ defined in Ω_{00} such that

$$\tilde{\boldsymbol{\beta}}(x) = \nabla^{\perp}\tilde{\varphi} = \big(D_2\tilde{\varphi}(x), -D_1\tilde{\varphi}(x)\big) \quad (x \in \partial\Omega_{00}).$$

Since $\tilde{\varphi}(x)$ is constant on $\partial\Omega_{00} \cap \Omega$, we choose the constant 0. Put

$$\varphi(x) = \begin{cases} \tilde{\varphi}(x) & (x \in \Omega_{00}), \\ 0 & (x \in \Omega \setminus \Omega_{00}) \end{cases}$$

$\varphi(x)$ is of compact support, and $\boldsymbol{\beta}(x) = \nabla^{\perp}\varphi(x)$ $(x \in \partial\Omega)$ holds.

Let δ_0, κ_0 be real numbers such that $\delta_0 > 0$, $1/4 > \kappa_0 > 0$. We take a function $j(t) \in C_0^{\infty}[0, \infty)$ having the following properties (Fujita [6]).

$$\begin{aligned} &0 \leqslant j(t) \leqslant \frac{1}{t}, \\ &j(t) = 0 \quad (0 \leqslant t \leqslant \kappa_0\delta_0,\ (1-\kappa_0)\delta_0 \leqslant t), \\ &j(t) = \frac{1}{t} \quad (2\kappa_0\delta_0 \leqslant t \leqslant (1-2\kappa_0)\delta_0). \end{aligned}$$

Put $h(t) = 1 - \int_0^t j(s)\,\mathrm{d}s \big/ \int_0^{\infty} j(s)\,\mathrm{d}s$. Then, it holds that

$$\begin{aligned} &0 \leqslant h(t) \leqslant 1 \quad (0 \leqslant t), \qquad h(t) \equiv 1 \quad (0 \leqslant t \leqslant \kappa_0\delta_0), \\ &h(t) \equiv 0 \quad (\delta_0 \leqslant t), \end{aligned} \tag{3.3}$$

$$\sup_{0 \leqslant t \leqslant \delta_0} |t \cdot h'(t)| \leqslant \left(\log \frac{1-2\kappa_0}{2\kappa_0}\right)^{-1} \to 0 \quad \text{as } \kappa_0 \to 0. \tag{3.4}$$

Let $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$ and put

$$\boldsymbol{b}_0(x) = \nabla^{\perp}\big(h\big(\rho(x)\big)\varphi(x)\big) \quad (x \in \Omega).$$

Note that $\boldsymbol{b}_0(x) = \boldsymbol{\beta}(x)$ $(x \in \partial\Omega)$ and

$$\boldsymbol{b}_0(x) = h\big(\rho(x)\big)\nabla^\perp\varphi(x) + \varphi(x)h'\big(\rho(x)\big)\nabla^\perp\rho(x). \tag{3.5}$$

Furthermore the support of $\boldsymbol{b}_0$ is contained in $\Omega_{\delta_0} \cap \Omega_{00}$ where

$$\Omega_{\delta_0} := \{x \in \Omega \mid \rho(x) < \delta_0\}.$$

Using the formula

$$(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \frac{1}{2}\nabla|\boldsymbol{v}|^2 - \omega\boldsymbol{v}^\perp = \frac{1}{2}\nabla|\boldsymbol{v}|^2 - (D_1v_2 - D_2v_1)(v_2, -v_1)$$

and $\operatorname{div}\boldsymbol{b}_0 = 0$, we have

$$\big|\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_0\big)\big| = \big|(\omega\boldsymbol{v}^\perp, \boldsymbol{b}_0)\big| \leqslant \int_\Omega |\omega||\boldsymbol{v}^\perp||\boldsymbol{b}_0|\,\mathrm{d}x \leqslant \|\nabla\boldsymbol{v}\|\big\||\boldsymbol{v}||\boldsymbol{b}_0|\big\|.$$

Since the support of $\boldsymbol{b}_0$ is contained in Ω_{δ_0},

$$\big\||\boldsymbol{v}||\boldsymbol{b}_0|\big\|^2 = \int_{\Omega_{\delta_0}} |\boldsymbol{v}|^2|\boldsymbol{b}_0|^2\,\mathrm{d}x.$$

According to (3.5),

$$|\boldsymbol{v}||\boldsymbol{b}_0| \leqslant |\boldsymbol{v}|\big|h\big(\rho(x)\big)\big|\big|\nabla^\perp\varphi(x)\big| + \left|\frac{\boldsymbol{v}(x)}{\rho(x)}\right|\rho(x)\big|h'\big(\rho(x)\big)\big||\nabla\rho|\big|\varphi(x)\big|. \tag{3.6}$$

Therefore

$$\big\||\boldsymbol{v}||\boldsymbol{b}_0|\big\|_{L^2(\Omega_{\delta_0})} \leqslant \big\||\boldsymbol{v}||\nabla^\perp\varphi|\big\|_{L^2(\Omega_{\delta_0})} + \left\|\frac{|\boldsymbol{v}|}{\rho}\rho h'(\rho)\varphi|\nabla\rho|\right\|_{L^2(\Omega_{\delta_0})}. \tag{3.7}$$

Using Hölder's inequality and the Sobolev embeddings, we have

$$\begin{aligned}\big\||\boldsymbol{v}||\nabla^\perp\varphi|\big\|_{L^2(\Omega_{\delta_0})} &\leqslant \big\||\boldsymbol{v}|\big\|_{L^4(\Omega_{\delta_0})}\big\||\nabla^\perp\varphi|\big\|_{L^4(\Omega_{\delta_0})}\\ &\leqslant C\big\||\nabla^\perp\varphi|\big\|_{L^4(\Omega_{\delta_0})}\|\nabla\boldsymbol{v}\|.\end{aligned}$$

If we choose δ_0 sufficiently small, the first term of the right-hand side of (3.7) can be less than $\varepsilon\|\nabla\boldsymbol{v}\|$. On the other hand,

$$\begin{aligned}\left\|\frac{|\boldsymbol{v}|}{\rho}\rho h'(\rho)\varphi|\nabla\rho|\right\|_{L^2(\Omega_{\delta_0})} &\leqslant C\sup_{0\leqslant\rho\leqslant\delta_0}\big|\rho h'(\rho)\big|\|\varphi\|_{L^\infty(\Omega_{\delta_0})}\left\|\frac{|\boldsymbol{v}|}{\rho}\right\|\\ &\leqslant C'\sup_{0\leqslant\rho\leqslant\delta_0}\big|\rho h'(\rho)\big|\|\varphi\|_{L^\infty(\Omega_{\delta_0})}\|\nabla\boldsymbol{v}\|\end{aligned}$$

with some constant C and C'. Using (3.4), we can estimate the above quantity as small as $\varepsilon\|\nabla \boldsymbol{v}\|$ if κ_0 is sufficiently small. Therefore, we obtain the extension of $\boldsymbol{\beta}$ satisfying the inequality (L). □

PROOF OF LEMMA 3.2. This lemma can be proved by the virtual drain method of [7]. However, we present here the proof using the idea of [11]. Put

$$\Omega_+ = \big\{(x_1, x_2) \in \Omega \mid x_2 > 0\big\}.$$

Since we suppose that every bounded connected component of the boundary intersects the axis of symmetry, the x_1-axis, the set Ω_+ is simply connected.

Let $\widetilde{\Omega}$ be a bounded symmetric domain with smooth boundary, contained in Ω and containing Ω_0. We suppose further $\partial\widetilde{\Omega}$ contains the support of $\boldsymbol{\beta}_0$. Put

$$\tilde{\boldsymbol{\beta}} = \begin{cases} \boldsymbol{\beta}_0 & \text{in } \partial\widetilde{\Omega} \cap \partial\Omega, \\ 0 & \text{in } \partial\widetilde{\Omega} \setminus \partial\Omega. \end{cases}$$

Let $\tilde{\boldsymbol{b}}$ be the symmetric solution of the Stokes boundary value problem [10].

$$\begin{cases} -\Delta\tilde{\boldsymbol{b}} + \nabla q = 0 & \text{in } \widetilde{\Omega}, \\ \operatorname{div}\tilde{\boldsymbol{b}} = 0 & \text{in } \widetilde{\Omega}, \\ \tilde{\boldsymbol{b}} = \tilde{\boldsymbol{\beta}} & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

We extend this $\tilde{\boldsymbol{b}}$ to Ω by putting 0 outside of $\widetilde{\Omega}$, that is,

$$\boldsymbol{b} = \begin{cases} \tilde{\boldsymbol{b}} & \text{in } \widetilde{\Omega}, \\ 0 & \text{in } \Omega \setminus \widetilde{\Omega}. \end{cases}$$

It is to be remarked the second component of $\boldsymbol{b}$ vanishes on the axis of symmetry, the x_1-axis. Let $P_0 \in \Omega_+$ be fixed. For $P \in \Omega_+$, we define

$$\varphi(P) = \int_{P_0}^{P} b_1\,\mathrm{d}x_2 - b_2\,\mathrm{d}x_1,$$

where the path of integration remains in Ω_+. The integral does not depend on the path but depends only on the end point of the path. Thus, the function φ is a single valued function. It is easy to see that φ is of constant value in $\Omega_+ \setminus \widetilde{\Omega}_+$. We choose the constant as zero. Therefore φ is of compact support and

$$\boldsymbol{b} = \nabla^{\perp}\varphi = (D_2\varphi, -D_1\varphi) \quad \text{in } \Omega_+.$$

Put

$$\boldsymbol{b}^*(x) = \nabla^{\perp}\big\{h\big(\rho(x)\big)\varphi(x)\big\} \quad (x \in \Omega_+), \tag{3.8}$$

where $h(t)$ is the function introduced in the proof of Lemma 3.1 and $\rho(x) = \operatorname{dist}(x, \partial\Omega_+)$. Put

$$\boldsymbol{b}_\varepsilon(x_1, x_2) = \begin{cases} (b_1^*(x_1, x_2), b_2^*(x_1, x_2)) & (x_1, x_2) \in \Omega_+, \\ (b_1^*(x_1, -x_2), -b_2^*(x_1, -x_2)) & (x_1, x_2) \in \Omega_-. \end{cases}$$

For a given ϵ, choose κ_0 and δ_0 sufficiently small. Then, $\boldsymbol{b}_\varepsilon$ is a desired extension of $\boldsymbol{\beta}$, symmetric with respect to the x_1 axis and satisfies the inequality (LF). See Morimoto [11] for details. □

Now we construct "an approximation" $\boldsymbol{s} \in V^S(\Omega)$ of the Poiseuille flow $\boldsymbol{V}$, which has a compact support. We follow the argument of Amick [1]. Let $\Phi(x_2) = 3\mu(x_2 - x_2^3/3)/4$ be a stream function of $\boldsymbol{V}$,

$$\boldsymbol{V} = \nabla^\perp \Phi(x_2) = (D_2\Phi, -D_1\Phi) = (\Phi'(x_2), 0).$$

Let $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$, and $h(t)$ defined in the proof of Lemma 3.1.

Put $\widetilde{\boldsymbol{V}} = \nabla^\perp\{h(\rho(x))\Phi(x_2)\}$. Then, $\operatorname{div}\widetilde{\boldsymbol{V}} = 0$ in Ω, $\widetilde{\boldsymbol{V}}|_{\partial\Omega} = \boldsymbol{V}|_{\partial\Omega}$ and

$$\widetilde{\boldsymbol{V}}(x) = h\big(\rho(x)\big)\boldsymbol{V} + \Phi(x_2)h'\big(\rho(x)\big)\nabla^\perp\rho.$$

For x in Ω_1

$$\widetilde{\boldsymbol{V}}(x) = h\big(\rho(x)\big)\boldsymbol{V} + \Phi(x_2)h'\big(\rho(x)\big)(-\operatorname{sgn} x_2, 0)$$

hold, where

$$\operatorname{sgn} t = \begin{cases} 1 & (t > 0), \\ -1 & (t < 0). \end{cases}$$

Let $\theta(t)$ be a smooth function such that

$$0 \leqslant \theta(t) \leqslant 1 \quad (t \in \mathbb{R}),$$
$$\theta(t) \equiv 0 \quad \text{for } t \leqslant \frac{1}{2},$$
$$\theta(t) \equiv 1 \quad \text{for } t \geqslant 1.$$

For $n = 1, 2, \ldots$, put

$$\theta_n(x) = \theta\left(\frac{x_1}{2n}\right).$$

We introduce the following function.

$$\boldsymbol{s}(x) = \begin{cases} \boldsymbol{V} - \widetilde{\boldsymbol{V}} & \text{in } \Omega_0, \\ \nabla^\perp\{(1 - h(\rho(x)))(1 - \theta_n(x)^2)\Phi(x_2)\} & \text{in } \Omega_1. \end{cases} \tag{3.9}$$

It is easy to see that the support of $\boldsymbol{s}$ is contained in $\Omega \cap \{x_1 < 2n\}$ and $\boldsymbol{s} \in V^S(\Omega)$. The function $\boldsymbol{s}$ is "an approximation" of $\boldsymbol{V}$ in the following sense.

LEMMA 3.3. *There exist a positive constant C_0 such that for every positive ε and $n = 1, 2, \ldots$, there exists $\boldsymbol{s} \in V^S(\Omega)$ with compact support contained in $\Omega \cap \{x_1 < 2n\}$, satisfying the following estimate.*

$$\begin{aligned}\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_\Omega &\leqslant \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{s}\big)_\Omega + \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_n^2\boldsymbol{V}\big)_{\Omega_1} \\ &\quad + \left(\varepsilon + \frac{C_0}{n}\right)\|\nabla\boldsymbol{w}\|_\Omega^2 \quad (\forall \boldsymbol{w} \in V(\Omega)).\end{aligned} \tag{3.10}$$

PROOF. It is sufficient to show the following two estimates.

$$\big|\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_{\Omega_0} - \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{s}\big)_{\Omega_0}\big| \leqslant \frac{\varepsilon}{2}\|\nabla\boldsymbol{w}\|^2, \quad \forall \boldsymbol{w} \in V(\Omega) \tag{3.11}$$

and

$$\begin{aligned}&\big|\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_{\Omega_1} - \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{s}\big)_{\Omega_1} - \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_n^2\boldsymbol{V}\big)_{\Omega_1}\big| \\ &\quad \leqslant \left(\frac{\varepsilon}{2} + \frac{C_0}{n}\right)\|\nabla\boldsymbol{w}\|_\Omega^2, \quad \forall \boldsymbol{w} \in V(\Omega).\end{aligned} \tag{3.12}$$

Firstly we show (3.11).

$$\begin{aligned}&\big|\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_{\Omega_0} - \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{s}\big)_{\Omega_0}\big| \\ &\quad = \big|\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \widetilde{\boldsymbol{V}}\big)_{\Omega_0}\big| \\ &\quad \leqslant \int_{\Omega_0}\big|(\boldsymbol{w}\cdot\nabla)w_1 h\big(\rho(x)\big)V_1\big|\,\mathrm{d}x + \int_{\Omega_0}\big|(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}\cdot\Phi(x_2)h'\big(\rho(x)\big)\nabla^\perp\rho\big|\,\mathrm{d}x.\end{aligned}$$

If we choose δ_0 sufficiently small, the first term in the right-hand side is less than $\varepsilon/2\|\nabla\boldsymbol{w}\|^2$. The second term is

$$\begin{aligned}&\int_{\Omega_0}\big|(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}\cdot\Phi(x_2)h'\big(\rho(x)\big)\nabla^\perp\rho\big|\,\mathrm{d}x \\ &\quad \leqslant \int_{\Omega_0}\left|\frac{\boldsymbol{w}}{\rho(x)}\right||\nabla\boldsymbol{w}|\big|\Phi(x_2)\big|\big|\rho(x)h'\big(\rho(x)\big)\big||\nabla^\perp\rho|\,\mathrm{d}x \\ &\quad \leqslant C\sup\big|\rho(x)h'\big(\rho(x)\big)\big|\int_{\Omega_0}\left|\frac{\boldsymbol{w}}{\rho(x)}\right||\nabla\boldsymbol{w}|\,\mathrm{d}x \\ &\quad \leqslant C\sup\big|\rho(x)h'\big(\rho(x)\big)\big|\|\nabla\boldsymbol{w}\|^2,\end{aligned} \tag{3.13}$$

where we used the Hardy type inequality. As $\kappa_0 \to 0$, $C\sup|\rho(x)h'(\rho(x))|$ becomes less than $\varepsilon/2$, which proves (3.11).

Now we show (3.12). According to the definition of $\boldsymbol{s}$, we have

$$\begin{aligned}&\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{V}\big)_{\Omega_1}-\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{s}\big)_{\Omega_1}-\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\theta_n^2\boldsymbol{V}\big)_{\Omega_1}\\&\quad=\int_{\Omega_1}(\boldsymbol{w}\cdot\nabla)w_1\big\{1-\theta_n^2(x_1)\big\}\big\{h(\rho)V_1-\Phi(x_2)D_2h(\rho)\big\}\,\mathrm{d}x\\&\qquad-\int_{\Omega_1}(\boldsymbol{w}\cdot\nabla)w_2s_2\,\mathrm{d}x.\end{aligned}$$

Therefore, the left-hand side of (3.12) is less than

$$\begin{aligned}&\int_{\Omega_1}\big|(\boldsymbol{w}\cdot\nabla)w_1h(\rho)V_1\big(1-\theta_n^2\big)\big|\,\mathrm{d}x+\int_{\Omega_1}\big|(\boldsymbol{w}\cdot\nabla)w_1h'(\rho)\big(1-\theta_n^2\big)\Phi(x_2)\big|\,\mathrm{d}x\\&\quad+\int_{\Omega_1}\big|(\boldsymbol{w}\cdot\nabla)w_2s_2\big|\,\mathrm{d}x.\end{aligned}$$

If δ_0 is sufficiently small, then $|h(\rho(x))V_1|$ is small in Ω_1, and the first term is less than $\frac{\varepsilon}{2}\|\nabla\boldsymbol{w}\|^2$. The second term is estimated as in (3.13) and, if we take κ_0 sufficiently small, then this term is less than $\frac{\varepsilon}{2}\|\nabla\boldsymbol{w}\|^2$. On the other hand, since

$$s_2(x)=-D_1\left[\big\{1-h\big(\rho(x)\big)\big\}\left\{1-\theta\left(\frac{x_1}{2n}\right)^2\right\}\Phi(x_2)\right],$$

the estimate $|s_2|\leqslant C/n$ holds, where $C=\sup|\theta'(x_1/(2n))\Phi(x_2)|$. Therefore the third term is estimated as

$$\int_{\Omega_1}\big|(\boldsymbol{w}\cdot\nabla)w_2s_2\big|\,\mathrm{d}x\leqslant\frac{C}{n}\int_{\Omega_1}\big|(\boldsymbol{w}\cdot\nabla)w_2\big|\,\mathrm{d}x,$$

and we obtain (3.12). Thus Lemma 3.3 is proved. □

We need the following theorem for the proof of Theorem 3.1. The constant σ^S is defined in (3.2).

THEOREM A. *Put*

$$\text{(G)}\quad G(n)=\sup_{\boldsymbol{v}\in V^S(\Omega)}\frac{((\boldsymbol{v}\cdot\nabla)\boldsymbol{v},\theta_n^2\boldsymbol{V})_{\Omega_1}}{\|\nabla\boldsymbol{v}\|_{\Omega}^2}.$$

Then

$$\lim_{n\to\infty}G(n)=\sigma^S.$$

This theorem was proved by Amick for more general situation. See [1].

3.2. *Existence and regularity of the weak solution*

PROOF OF THEOREM 3.1. First, we prove Theorem 3.1. By the assumption $\nu - \sigma^S > 0$ and Theorem A, we can choose $\varepsilon > 0$ and $m \in \mathbf{N}$ so that

$$G(m) - \sigma^S \leqslant 4^{-1}\big(\nu - \sigma^S\big), \qquad 2\varepsilon + \frac{C_0}{m} \leqslant 4^{-1}\big(\nu - \sigma^S\big) \tag{3.14}$$

hold where C_0 is the constant given in Lemma 3.3. We fix these ε and m.

Let Ω^n, $n = 1, 2, \ldots$, be a sequence of bounded symmetric domains with smooth boundary such that

$$\Omega^n \subset \Omega^{n+1} \subset \Omega, \qquad \partial\Omega^n \cap \partial\Omega \subset \partial\Omega^{n+1} \cap \partial\Omega$$

and

$$\Omega^n \to \Omega \quad \text{as } n \to \infty.$$

We suppose that the domain Ω^1 contains $\Omega \cap \{x_1 < 2m\}$.

$$\Omega \cap \{x_1 < 2m\} \subset \Omega^1 \subset \Omega^2 \subset \cdots.$$

We consider the stationary Navier–Stokes equations in Ω^n.

$$(\mathrm{NS})^n \quad \begin{cases} (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} & \text{in } \Omega^n, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega^n, \end{cases}$$

with the boundary condition

$$(\mathrm{BC})^n \quad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on } \partial\Omega^n \cap \partial\Omega, \\ \boldsymbol{u} = \boldsymbol{V} & \text{on } \partial\Omega^n \setminus \partial\Omega. \end{cases}$$

A function $\boldsymbol{u}$ is called a weak solution to $(\mathrm{NS})^n$ $(\mathrm{BC})^n$, if $\boldsymbol{u} \in \mathbb{H}^1(\Omega^n)$, $\operatorname{div}\boldsymbol{u} = 0$,

$$\nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}\big) = (\boldsymbol{f}, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in V(\Omega^n)),$$

and $\boldsymbol{u}$ satisfies the boundary condition $(\mathrm{BC})^n$ in the trace sense.

Let $\boldsymbol{u}_n$ be a symmetric weak solution to $(\mathrm{NS})^n$, $(\mathrm{BC})^n$, the existence of which was established by Fujita [7]. Put $\boldsymbol{u}_n = \boldsymbol{w}_n + \boldsymbol{b} + \boldsymbol{V}$, where $\boldsymbol{b} \in \mathbb{H}^2(\Omega)$ is the solenoidal symmetric extension (in Ω) of $\boldsymbol{\beta} - \boldsymbol{V}|_{\partial\Omega}$. As $\boldsymbol{\beta} - \boldsymbol{V}|_{\partial\Omega}$ satisfies the hypothesis of Lemma 3.2, there exists its solenoidal extension $\boldsymbol{b} \in \mathbb{H}^2(\Omega)$ for which the following inequality holds.

$$\big|\big((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}\big)_\Omega\big| \leqslant \varepsilon\|\nabla\boldsymbol{v}\|_\Omega^2 \quad (\forall \boldsymbol{v} \in V^S(\Omega)).$$

Since $\boldsymbol{w}_n \in V^S(\Omega)$, we have

$$\left|\left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{b}\right)_\Omega\right| \leqslant \varepsilon \|\nabla \boldsymbol{w}_n\|_\Omega^2 \quad (\forall n).$$

It should be noted the support of this extension is compact. Without loss of generality, we can suppose that the support is contained in $\{x_1 < 2m\}$. Then $\operatorname{div} \boldsymbol{w}_n = 0$ and $\boldsymbol{w}_n|_{\partial\Omega^n} = 0$. Therefore $\boldsymbol{w}_n$ belongs to $V^S(\Omega^n)$ and satisfies the following equation.

$$\begin{aligned}
&\nu(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{v}) + \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}\right) + \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}\right) \\
&\qquad + \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}\right) \\
&\quad = (\boldsymbol{F}, \boldsymbol{v}) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in V^S(\Omega^n)),
\end{aligned} \tag{3.15}$$

where $\boldsymbol{F} = \boldsymbol{f} - \{(\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{V}\}$. We have used the relation

$$\nu(\nabla \boldsymbol{V}, \nabla \boldsymbol{v}) = 0 \quad (\forall \boldsymbol{v} \in V^S(\Omega)).$$

Since $\boldsymbol{w}_n \in V^S(\Omega^n)$, we can substitute $\boldsymbol{v} = \boldsymbol{w}_n$ into (3.15), and obtain

$$\begin{aligned}
&\nu\|\nabla \boldsymbol{w}_n\|^2 \\
&\quad = \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{V}\right) + \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{b}\right) + (\boldsymbol{F}, \boldsymbol{w}_n) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{w}_n).
\end{aligned} \tag{3.16}$$

Because $V^S(\Omega^n)$ is a subset of $V^S(\Omega)$ for all n, the following inequality holds true from the definition of $G(m)$.

$$\left((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \theta_m^2 \boldsymbol{V}\right)_{\Omega_1} \leqslant G(m)\|\nabla \boldsymbol{v}\|^2 \quad (\forall \boldsymbol{v} \in V^S(\Omega^n)).$$

Let $\boldsymbol{s}$ be fixed as in Lemma 3.3. By (3.10), we have

$$\begin{aligned}
&((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{V})_\Omega \\
&\quad \leqslant \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\right)_\Omega + \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \theta_m^2 \boldsymbol{V}\right)_{\Omega_1} + \left(\varepsilon + \frac{C_0}{m}\right)\|\nabla \boldsymbol{w}_n\|^2.
\end{aligned} \tag{3.17}$$

Since $\boldsymbol{s} \in V^S(\Omega^n)$, we substitute $\boldsymbol{v} = \boldsymbol{s}$ in (3.15), and obtain

$$\begin{aligned}
\left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\right)_\Omega = &-\nu(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{s}) - \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\right) \\
&- \left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\right) + (\boldsymbol{F}, \boldsymbol{s}) \\
&- \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{s}).
\end{aligned} \tag{3.18}$$

The right-hand side is linear in $\boldsymbol{w}_n$. Therefore the following inequality holds true with some positive constants k', k'' which do not depend on $\boldsymbol{w}_n$.

$$\left((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\right)_\Omega \leqslant k' + k''\|\nabla \boldsymbol{w}_n\|. \tag{3.19}$$

Hence we have

$$\begin{aligned}
&\nu\|\nabla \boldsymbol{w}_n\|^2 \\
&\quad = \big((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n, \boldsymbol{V}\big)_{\Omega} + \big((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n, \boldsymbol{b}\big)_{\Omega} + (\boldsymbol{F}, \boldsymbol{w}_n)_{\Omega} - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{w}_n)_{\Omega} \\
&\quad \leqslant \big((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n, \boldsymbol{s}\big)_{\Omega} + \big((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n, \theta_m^2\boldsymbol{V}\big)_{\Omega_1} + \left(\varepsilon + \frac{C_0}{m}\right)\|\nabla \boldsymbol{w}_n\|^2 \\
&\qquad + \big((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n, \boldsymbol{b}\big)_{\Omega} + (\boldsymbol{F}, \boldsymbol{w}_n)_{\Omega} - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{w}_n)_{\Omega} \\
&\quad \leqslant k' + k''\|\nabla \boldsymbol{w}_n\| + \big((\boldsymbol{w}_n\cdot\nabla)\boldsymbol{w}_n, \theta_m^2\boldsymbol{V}\big)_{\Omega_1} + \left(\varepsilon + \frac{C_0}{m}\right)\|\nabla \boldsymbol{w}_n\|^2 \\
&\qquad + \varepsilon\|\nabla \boldsymbol{w}_n\|^2 + k'''\|\nabla \boldsymbol{w}_n\|, \qquad (3.20)
\end{aligned}$$

where k''' is a positive constant independent of $\boldsymbol{w}_n$. Hence,

$$\nu\|\nabla \boldsymbol{w}_n\|^2 \leqslant k_1 + k_2\|\nabla \boldsymbol{w}_n\| + \left(2\varepsilon + \frac{C_0}{m}\right)\|\nabla \boldsymbol{w}_n\|^2 + G(m)\|\nabla \boldsymbol{w}_n\|^2, \qquad (3.21)$$

where $k_1 = k'$ and $k_2 = k'' + k'''$. Since ε and m are chosen as in (3.14), we obtain

$$2^{-1}\big(\nu - \sigma^S\big)\|\nabla \boldsymbol{w}_n\|^2 \leqslant k_1 + k_2\|\nabla \boldsymbol{w}_n\|.$$

Therefore, there exists a positive constant M such that the following estimate holds for all n.

$$\|\nabla \boldsymbol{w}_n\| \leqslant M.$$

The sequence $\boldsymbol{w}_n$ being bounded in $V^S(\Omega)$, we can choose a subsequence $\boldsymbol{w}_{n'}$ which converges weakly in $V^S(\Omega)$. Let $\boldsymbol{w}$ be the limit. Let $\boldsymbol{\varphi}$ be an arbitrary element in $\mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$. There exists an integer n_0 such that the support of $\boldsymbol{\varphi}$ is contained in Ω^{n_0}. Let $n' \geqslant n_0$.

$$\begin{aligned}
&\nu(\nabla \boldsymbol{w}_{n'}, \nabla \boldsymbol{\varphi}) + \big((\boldsymbol{w}_{n'}\cdot\nabla)\boldsymbol{w}_{n'}, \boldsymbol{\varphi}\big) + \big((\boldsymbol{w}_{n'}\cdot\nabla)\boldsymbol{V}, \boldsymbol{\varphi}\big) \\
&\qquad + \big((\boldsymbol{V}\cdot\nabla)\boldsymbol{w}_{n'}, \boldsymbol{\varphi}\big) + \big((\boldsymbol{w}_{n'}\cdot\nabla)\boldsymbol{b}, \boldsymbol{\varphi}\big) + \big((\boldsymbol{b}\cdot\nabla)\boldsymbol{w}_{n'}, \boldsymbol{\varphi}\big) \\
&\quad = (\boldsymbol{F}, \boldsymbol{\varphi}) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{\varphi}). \qquad (3.22)
\end{aligned}$$

We can select a subsequence which converges strongly in $\mathbb{L}^4(\Omega^{n_0})$. We denote this subsequence with the same symbol $\boldsymbol{w}_{n'}$. Letting $n' \to \infty$, we obtain

$$\begin{aligned}
&\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) + \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{\varphi}\big) + \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{V}, \boldsymbol{\varphi}\big) + \big((\boldsymbol{V}\cdot\nabla)\boldsymbol{w}, \boldsymbol{\varphi}\big) \\
&\qquad + \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{b}, \boldsymbol{\varphi}\big) + \big((\boldsymbol{b}\cdot\nabla)\boldsymbol{w}, \boldsymbol{\varphi}\big) \\
&\quad = (\boldsymbol{F}, \boldsymbol{\varphi}) - \nu(\nabla \boldsymbol{b}, \nabla \boldsymbol{\varphi}). \qquad (3.23)
\end{aligned}$$

Therefore $\boldsymbol{u} := \boldsymbol{w} + \boldsymbol{b} + \boldsymbol{V}$ is a symmetric weak solution to $(\mathrm{NS})_1$, $(\mathrm{BC})_1$. And Theorem 3.1 is proved. □

PROOF OF THEOREM 3.2. Regularity of the solution can be shown by the standard argument. For the convenience of the readers, we note the outline of the proof of Theorem 3.2. Let $\{\boldsymbol{u}, p\}$ be solution pair to $(\mathrm{NS})_1$, $(\mathrm{BC})_1$. Put $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{V} - \boldsymbol{b}$ and $q = p - P$. Then $\boldsymbol{w} \in V(\Omega)$ and $q \in L^2_{\mathrm{loc}}(\Omega)$ satisfy the following equation.

$$\begin{aligned}&\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) - (q, \operatorname{div} \boldsymbol{v}) + \big((\boldsymbol{w} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v}\big) + \big((\boldsymbol{w} \cdot \nabla)\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v}\big)\\ &\quad + \big((\boldsymbol{w} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}, \boldsymbol{v}\big) = (\boldsymbol{F}', \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in \mathbb{C}_0^\infty(\Omega)),\end{aligned}$$

where

$$\boldsymbol{F}' \equiv \boldsymbol{F} + \nu \Delta \boldsymbol{b} = \boldsymbol{f} - \big\{(\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{V}\big\} + \nu \Delta \boldsymbol{b}$$

is a smooth function. Note that $\boldsymbol{b}$ is of compact support. Let us show that $\boldsymbol{w}, q \in C^\infty(\overline{\omega})$ for any bounded domain $\omega \subset \Omega$. We can find $a > 0$ and a bounded domain Q with smooth boundary such that

$$\omega \subset \{x \in \Omega \mid x_2 < a\} \subset Q.$$

Let $\psi(x)$ be a smooth function defined in the closure of Ω such that

$$\psi(x) \equiv 1 \quad (x \in \overline{\omega}), \qquad \psi(x) \equiv 0 \quad (x \in \overline{\Omega} \setminus \overline{Q}).$$

Put

$$\boldsymbol{W} \equiv \psi \boldsymbol{w}, \qquad \pi \equiv \psi q.$$

Then, it is easy to see that $\boldsymbol{W} \in \mathbb{H}^1_0(Q)$, $\pi \in L^2(Q)$ and they satisfy

$$\begin{aligned}&\nu(\nabla \boldsymbol{W}, \nabla \boldsymbol{v}) - (\pi, \operatorname{div} \boldsymbol{v}) = -(\psi \boldsymbol{G}_0 + \boldsymbol{G}_1, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in \mathbb{C}_0^\infty(\Omega))\\ &\operatorname{div} \boldsymbol{W} = \operatorname{div}(\psi \boldsymbol{w}) = \nabla \psi \cdot \boldsymbol{w}, \qquad \boldsymbol{W}|_{\partial Q} = 0,\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}&\boldsymbol{G}_0 = (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w} - \boldsymbol{F}',\\ &\boldsymbol{G}_1 = 2\nu \nabla \psi \nabla \boldsymbol{w} + \nu(\Delta \psi)\boldsymbol{w} - (\nabla \psi) q.\end{aligned}$$

Note

$$\operatorname{div} \boldsymbol{W} = \operatorname{div}(\psi \boldsymbol{w}) = \nabla \psi \cdot \boldsymbol{w} + \psi \operatorname{div} \boldsymbol{w} = \nabla \psi \cdot \boldsymbol{w} \in H^1_0(Q).$$

According to Hölder's inequality and the Sobolev embedding theorem, we have

$$\left\|(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}\right\|_{L^{4/3}(Q)} \leqslant \|\boldsymbol{w}\|_{L^4(Q)}\|\nabla\boldsymbol{w}\|_{L^2(Q)} \leqslant C\|\boldsymbol{w}\|_{H^1(Q)}\|\nabla\boldsymbol{w}\|_{L^2(Q)}$$

and it is easy to show $\boldsymbol{G}_0 \in \mathbb{L}^{4/3}(Q)$. On the other hand, we can easily check $\boldsymbol{G}_1 \in \mathbb{L}^2(Q)$. Therefore $\psi\boldsymbol{G}_0 + \boldsymbol{G}_1 \in \mathbb{L}^{4/3}(Q)$. Using the well known result of Cattabriga [5] for the Stokes boundary value problem, we have

$$\boldsymbol{W} \in \mathbb{W}^{2,4/3}(Q), \qquad \nabla\pi \in \mathbb{L}^{4/3}(Q).$$

Since

$$\boldsymbol{w} = \boldsymbol{W} \quad \text{and} \quad q = \pi \quad \text{in } \omega,$$

it holds that

$$\boldsymbol{w} \in \mathbb{W}^{2,4/3}(\omega) \subset \mathbb{C}(\overline{\omega}), \qquad \nabla q \in \mathbb{L}^{4/3}(\omega).$$

According to the above estimate, it is easy to check $\boldsymbol{G}_0 \in \mathbb{L}^2(\omega)$. Repeating the previous argument, we see that

$$\boldsymbol{w} \in \mathbb{H}^2(\omega), \qquad \nabla q \in \mathbb{L}^2(\omega).$$

Now, let us show further regularity of the solution $\boldsymbol{w}$, q. Let ω and Q be as before. It is easy to check that $\nabla\boldsymbol{G}_0, \nabla\boldsymbol{G}_1 \in \mathbb{L}^2(Q)$, that is, $\boldsymbol{G}_0, \boldsymbol{G}_1 \in \mathbb{H}^1(Q)$. We apply the argument as before and obtain

$$\boldsymbol{W} \in \mathbb{H}^3(Q), \qquad \nabla\pi \in \mathbb{H}^1(Q).$$

This means that

$$\boldsymbol{w} \in \mathbb{H}^3(\omega), \qquad \nabla q \in \mathbb{H}^1(\omega).$$

We continue in this fashion to show that

$$\boldsymbol{w} \in \mathbb{H}^m(\omega), \quad \nabla q \in \mathbb{H}^{m-2}(\omega), \quad m = 2, 3, \ldots.$$

And we see

$$\boldsymbol{w} \in \mathbb{C}^\infty(\overline{\omega}), \qquad \nabla q \in \mathbb{C}^\infty(\overline{\omega}).$$

This completes the proof of Theorem 3.2. □

3.3. *Asymptotic behavior of the solution*

Before proving Theorem 3.3, we begin with the next lemma. We use the notation

$$\Omega^{t,\infty} = \big\{(x_1, x_2) \in \Omega \mid x_1 > t\big\}.$$

LEMMA 3.4. *Let* $\{\boldsymbol{u}, p\}$ *be the solution pair to* $(\mathrm{NS})_1$, $(\mathrm{BC})_1$ *obtained in Theorem* 3.1. *Then*, $\{\boldsymbol{u}, \nabla p\}$ *tends to* $\{\boldsymbol{V}, \nabla P\}$ *uniformly at the infinity, that is, if* R *is sufficiently large, then*,

$$\sup_{\Omega^{R,\infty}} \big|D^\alpha\big\{\boldsymbol{u}(x) - \boldsymbol{V}(x)\big\}\big| \leqslant C_\alpha\big(\big\|\nabla\{\boldsymbol{u} - \boldsymbol{V}\}\big\|_{\Omega^{R-2-|\alpha|,\infty}} + \|\boldsymbol{f}\|_{\Omega^{R-2-|\alpha|,\infty}}\big), \tag{3.25}$$

$$\sup_{\Omega^{R,\infty}} \big|D^\alpha\nabla\big\{p(x) - P(x)\big\}\big| \leqslant C_\alpha\big(\big\|\nabla\{\boldsymbol{u} - \boldsymbol{V}\}\big\|_{\Omega^{R-2-|\alpha|,\infty}} + \|\boldsymbol{f}\|_{\Omega^{R-2-|\alpha|,\infty}}\big) \tag{3.26}$$

hold true with a positive constant C_α.

PROOF OF LEMMA 3.4. We choose R sufficiently large so that $\boldsymbol{b} \equiv 0$ in $\Omega^{R,\infty}$. Since $\boldsymbol{w}$ and ∇q are smooth in $\Omega^{R,\infty}$ (Theorem 3.2), we see that

$$\begin{cases} -\nu\Delta\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{V} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{w} + \nabla q = \boldsymbol{f} & \text{in } \Omega^{R,\infty}, \\ \operatorname{div}\boldsymbol{w} = 0 & \text{in } \Omega^{R,\infty} \end{cases} \tag{3.27}$$

hold. We fix this R. Let

$$Q_0 = \big\{(x_1, x_2) \mid -5/8 < x_1 < 5/8, -1 < x_2 < 1\big\},$$
$$Q_1 = \big\{(x_1, x_2) \mid -1 < x_1 < 1, -1 < x_2 < 1\big\}.$$

Let ω_0 and Ω^0 be bounded symmetric domains with smooth boundary satisfying

$$Q_0 \subset \omega_0 \subset \Omega^0 \subset Q_1,$$

such that the right (resp. left) component of $\partial\omega_0 \setminus \partial T$ is congruent with the right (resp. left) component of $\partial\Omega^0 \setminus \partial T$. Put

$$\Omega^{(k)} = \big\{(x_1, x_2) \mid (x_1 - R - k, x_2) \in \Omega^0\big\}, \quad k = 1, 2, \ldots,$$
$$\omega^{(k)} = \big\{(x_1, x_2) \mid (x_1 - R - k, x_2) \in \omega_0\big\}, \quad k = 1, 2, \ldots.$$

$\Omega^{(k)}$'s (resp. $\omega^{(k)}$'s) are congruent figures. Let $\psi(x)$ be a scalar function in $C^\infty(\overline{T}\,)$ such that

$$\psi(x_1, -x_2) = \psi(x_1, x_2), \quad 0 \leqslant \psi(x) \leqslant 1, \quad \psi(x) \equiv \begin{cases} 1 & (x \in \overline{\omega}_0), \\ 0 & (x \in \overline{T} \setminus \overline{\Omega^0}\,) \end{cases}$$

and put

$$\psi_k(x_1, x_2) \equiv \psi(x_1 - R - k, x_2).$$

Let m_k be the mean value of q over $\Omega^{(k)}$, i.e., $m_k = (1/|\Omega^{(k)}|) \iint_{\Omega^{(k)}} q(x)\,\mathrm{d}x$, where $|\Omega^{(k)}|$ is the measure of $\Omega^{(k)}$. Since the pressure q is free by additive constant, we take $q - m_k$ instead of q in (3.27). Put

$$\begin{cases} \boldsymbol{W}^{(k)} = \psi_k \boldsymbol{w}, \\ \pi^{(k)} = \psi_k (q - m_k). \end{cases}$$

Using the equation (3.27), we can write

$$\nu \Delta \boldsymbol{W}^{(k)} - \nabla \pi^{(k)} = \psi_k \boldsymbol{G}_0 + \boldsymbol{G}_k, \quad \operatorname{div} \boldsymbol{W}^{(k)} = \boldsymbol{w} \cdot \nabla \psi_k \quad \text{in } \Omega^{(k)}, \tag{3.28}$$

$$\boldsymbol{W}^{(k)} = \boldsymbol{0} \quad \text{on } \partial\Omega^{(k)}, \tag{3.29}$$

with

$$\boldsymbol{G}_0 = \boldsymbol{G}_0(\boldsymbol{w}) = (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{V} - \boldsymbol{f},$$

$$\boldsymbol{G}_k = \boldsymbol{G}_k(\boldsymbol{w}, q) = 2\nu \nabla \boldsymbol{w} \nabla \psi_k + \nu(\Delta \psi_k)\boldsymbol{w} - (\nabla \psi_k)(q - m_k).$$

Let us show $\boldsymbol{G}_0, \boldsymbol{G}_k \in \mathbb{L}^{4/3}(\Omega^{(k)})$. The next lemma is easy to prove.

LEMMA 3.5. *Let $Q = (-1, 1)^2 \subset \mathbb{R}^2$. There exist constants κ_0, κ_1 such that for every $f \in H^1(Q)$ with $f(x_1, \pm 1) = 0$, the inequalities hold.*

$$\|f\|_{L^2(Q)} \leqslant \kappa_0 \|\nabla f\|_{L^2(Q)}, \tag{3.30}$$

$$\|f\|_{L^4(Q)} \leqslant \kappa_1 \|\nabla f\|_{L^2(Q)}. \tag{3.31}$$

By the inequality (3.31), we obtain

$$\big\|(\boldsymbol{w} \cdot \nabla)\boldsymbol{w}\big\|_{L^{4/3}(\Omega^{(k)})} \leqslant \|\nabla \boldsymbol{w}\| \|\boldsymbol{w}\|_{L^4(\Omega^{(k)})} \leqslant \kappa_1 \|\nabla \boldsymbol{w}\|^2. \tag{3.32}$$

On the other hand,

$$\begin{aligned} \big\|(\boldsymbol{V} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{V}\big\|_{L^{4/3}(\Omega^{(k)})} &\leqslant a \big\|(\boldsymbol{V} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{V}\big\|_{\Omega^{(k)}} \\ &\leqslant a|\mu|(1 + 2\kappa_0)\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}}, \end{aligned}$$

where $a = |\Omega^{(k)}|^{1/4}$ and κ_0 is a positive constant in the inequality (3.30). Note that κ_0, κ_1 and a are constants independent of k. Therefore

$$\begin{aligned} \|\psi_k \boldsymbol{G}_0\|_{L^{4/3}(\Omega^{(k)})} &\leqslant \|\boldsymbol{G}_0\|_{L^{4/3}(\Omega^{(k)})} \\ &\leqslant \kappa_1 \|\nabla \boldsymbol{w}\|^2_{\Omega^{(k)}} + a(1 + 2\kappa_0)|\mu| \|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + c\|\boldsymbol{f}\|_{\Omega^{(k)}} \\ &\leqslant C_1\big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big), \end{aligned} \tag{3.33}$$

where $C_1 = \kappa_1 b + a(1+2\kappa_0)|\mu| + c$ and $b = \|\nabla \boldsymbol{w}\|_{\Omega}$. Since $q \in L^2_{\mathrm{loc}}(\Omega)$, we have

$$\|\boldsymbol{G}_k\|_{L^{4/3}(\Omega^{(k)})} \leqslant a\|\boldsymbol{G}_k\|_{\Omega^{(k)}} \leqslant C_2\big\{\nu\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|q - m_k\|_{\Omega^{(k)}}\big\}, \tag{3.34}$$

where $C_2 = a(2\sup|\nabla\psi| + \kappa_0 \sup|\Delta\psi|)$. Therefore, $\psi_k \boldsymbol{G}_0 + \boldsymbol{G}_k$ is in $\mathbb{L}^{4/3}(\Omega^{(k)})$. It is to be noted that

$$\begin{aligned}
&\operatorname{div}(\psi_k \boldsymbol{w}) = (\nabla\psi_k)\cdot \boldsymbol{w} \in H^1_0\big(\Omega^{(k)}\big),\\
&\int_{\Omega^{(k)}} \operatorname{div}(\psi_k \boldsymbol{w})\,\mathrm{d}x = \int_{\partial\Omega^{(k)}} (\psi_k \boldsymbol{w})\cdot \boldsymbol{n}\,\mathrm{d}\sigma = 0,
\end{aligned}$$

because $\operatorname{div} \boldsymbol{w} = 0$ in $\Omega^{(k)}$, and $\boldsymbol{w} = 0$ on $\partial\Omega \cap \partial\Omega^{(k)}$. According to the well-known estimate for solutions to the Stokes inhomogeneous boundary value problem (3.28), (3.29) (Cattabriga [5]), we have

$$\boldsymbol{W}^{(k)} \in \mathbb{W}^{2,4/3}\big(\Omega^{(k)}\big), \qquad \nabla\pi^{(k)} \in \mathbb{L}^{4/3}\big(\Omega^{(k)}\big).$$

Furthermore

$$\begin{aligned}
&\nu\big\|\boldsymbol{W}^{(k)}\big\|_{W^{2,4/3}(\Omega^{(k)})} + \big\|\nabla\pi^{(k)}\big\|_{L^{4/3}(\Omega^{(k)})}\\
&\quad\leqslant C_3\big\{\|\psi_k \boldsymbol{G}_0 + \boldsymbol{G}_k\|_{L^{4/3}(\Omega^{(k)})} + \|\boldsymbol{w}\cdot\nabla\psi_k\|_{W^{1,4/3}(\Omega^{(k)})}\big\}\\
&\quad\leqslant C_4\big\{\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|q - m_k\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big\},
\end{aligned} \tag{3.35}$$

where the constants C_3 and C_4 do not depend on k. Now we proceed to the estimation for the pressure q.

LEMMA 3.6. *There is a constant C_5 independent of k such that the estimate*

$$\|q - m_k\|_{\Omega^{(k)}} \leqslant C_5\big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big) \tag{3.36}$$

holds.

PROOF OF LEMMA 3.6. From (3.27) we have

$$-\nu\Delta \boldsymbol{w} + \nabla q = \boldsymbol{f} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} - (\boldsymbol{w}\cdot\nabla)\boldsymbol{V} - (\boldsymbol{V}\cdot\nabla)\boldsymbol{w} = -\boldsymbol{G}_0(\boldsymbol{w}).$$

Multiplying this equation by $\boldsymbol{v} \in \mathbb{H}^1_0(\Omega^{(k)})$, integrating over $\Omega^{(k)}$, and also considering

$$\int_{\Omega^{(k)}} \operatorname{div} \boldsymbol{v}\,\mathrm{d}x = 0,$$

we obtain

$$\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{v})_{\Omega^{(k)}} - (q - m_k, \operatorname{div} \boldsymbol{v})_{\Omega^{(k)}} = -\big(\boldsymbol{G}_0(\boldsymbol{w}), \boldsymbol{v}\big)_{\Omega^{(k)}}. \tag{3.37}$$

We look for a function $\boldsymbol{v}$ satisfying the following equation.

$$\begin{cases} \operatorname{div} \boldsymbol{v} = q - m_k & \text{in } \Omega^{(k)}, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \partial\Omega^{(k)}. \end{cases} \tag{3.38}$$

Since $q - m_k \in L^2(\Omega^{(k)})$ and the integral $\iint_{\Omega^{(k)}} (q - m_k)\,\mathrm{d}x$ vanishes, we can apply Lemma 2.3 and find the solution $\boldsymbol{v} \in \mathbb{H}^1_0(\Omega^{(k)})$ to (3.38) such that the estimate

$$\|\boldsymbol{v}\|_{H^1_0(\Omega^{(k)})} \leqslant C_0 \|q - m_k\|_{\Omega^{(k)}}$$

holds true where C_0 is a constant independent of k. Substituting this $\boldsymbol{v}$ into (3.37), we obtain

$$\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{v})_{\Omega^{(k)}} - \|q - m_k\|^2_{\Omega^{(k)}} = -(\boldsymbol{G}_0(\boldsymbol{w}), \boldsymbol{v})_{\Omega^{(k)}}.$$

Therefore

$$\begin{aligned} \|q - m_k\|^2_{\Omega^{(k)}} &= \nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{v})_{\Omega^{(k)}} + (\boldsymbol{G}_0(\boldsymbol{w}), \boldsymbol{v})_{\Omega^{(k)}} \\ &\leqslant \nu \|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} \|\nabla \boldsymbol{v}\|_{\Omega^{(k)}} + \|\boldsymbol{G}_0\|_{L^{4/3}(\Omega^{(k)})} \|\boldsymbol{v}\|_{L^4(\Omega^{(k)})} \\ &\leqslant \nu \|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} \|\nabla \boldsymbol{v}\|_{\Omega^{(k)}} + \kappa_1 \|\boldsymbol{G}_0\|_{L^{4/3}(\Omega^{(k)})} \|\nabla \boldsymbol{v}\|_{\Omega^{(k)}} \\ &\leqslant C_0 \|q - m_k\|_{\Omega^{(k)}} \big(\nu \|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \kappa_1 \|\boldsymbol{G}_0\|_{L^{4/3}(\Omega^{(k)})}\big). \end{aligned}$$

According to the estimate (3.33) for $\boldsymbol{G}_0$, we obtain

$$\|q - m_k\|_{\Omega^{(k)}} \leqslant C_5 \big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big),$$

where $C_5 = C_0(\nu + \kappa_1 C_1)$. The constant C_5 does not depend on k and the proof of Lemma 3.6 is completed. □

Substituting (3.36) into (3.35), we obtain

$$\nu \big\|\boldsymbol{W}^{(k)}\big\|_{W^{2,4/3}(\Omega^{(k)})} + \big\|\nabla \pi^{(k)}\big\|_{L^{4/3}(\Omega^{(k)})} \leqslant C_6 \big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big), \tag{3.39}$$

where $C_6 = C_4(1 + C_5)$. Since $\psi_k \equiv 1$ on the set $\omega^{(k)}$,

$$\boldsymbol{w} = \boldsymbol{W}^{(k)} \quad \text{and} \quad q = \pi^{(k)} + m_k \quad \text{in } \omega^{(k)}.$$

Therefore

$$\boldsymbol{w} \in \mathbb{W}^{2,4/3}\big(\omega^{(k)}\big), \qquad \nabla q \in \mathbb{L}^{4/3}\big(\omega^{(k)}\big)$$

and

$$\nu \|\boldsymbol{w}\|_{W^{2,4/3}(\omega^{(k)})} + \|\nabla q\|_{L^{4/3}(\omega^{(k)})} \leqslant C_6 \big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big). \tag{3.40}$$

According to the Sobolev embedding theorem, the inclusion

$$W^{2,4/3}\big(\omega^{(k)}\big) \subset C\big(\overline{\omega^{(k)}}\big)$$

holds. Therefore $\boldsymbol{w}$ is bounded and continuous in $\omega^{(k)}$ and

$$\|\boldsymbol{w}\|_{C(\overline{\omega^{(k)}})} \leqslant C\|\boldsymbol{w}\|_{W^{2,4/3}(\omega^{(k)})} \leqslant C_7\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}},$$

where the constants C and $C_7 = CC_6$ do not depend on k. Since

$$\Omega^{(k)} \subset \Omega^{k-1,k+1} \subset \omega^{(k-1)} \cup \omega^{(k)} \cup \omega^{(k+1)}, \tag{3.41}$$

it holds that

$$\begin{aligned}\|\boldsymbol{w}\|_{C(\Omega^{(k)})} &\leqslant \sup_{k-1\leqslant j\leqslant k+1} \|\boldsymbol{w}\|_{C(\omega^{(j)})} \\ &\leqslant \sup_{k-1\leqslant j\leqslant k+1} C_7\|\nabla \boldsymbol{w}\|_{\Omega^{(j)}} \leqslant C_7 b \quad (\forall k),\end{aligned} \tag{3.42}$$

where $b = \|\nabla \boldsymbol{w}\|_{\Omega}$. Furthermore, by (3.42), we see $\boldsymbol{G}_0 \in \mathbb{L}^2(\Omega^{(k)})$ and

$$\begin{aligned}\|\boldsymbol{G}_0\|_{\Omega^{(k)}} &\leqslant \|\boldsymbol{w}\|_{C(\Omega^{(k)})}\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + |\mu|(1+2\kappa_0)\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}} \\ &\leqslant C\big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big),\end{aligned}$$

where $C = C_7 b + |\mu|(1+2\kappa_0)$ is a constant independent of k. As for $\boldsymbol{G}_k$, using (3.34) and (3.36), we obtain

$$\|\boldsymbol{G}_k\|_{\Omega^{(k)}} \leqslant C\big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big),$$

where $C = C_2(\nu + C_5)/a$ is a constant independent of k. Here and after C denotes various positive constant independent of k. Repeating the preceding procedure, we conclude that $\boldsymbol{W}^{(k)} \in \mathbb{H}^2(\Omega^{(k)})$, $\nabla \pi^{(k)} \in \mathbb{L}^2(\Omega^{(k)})$ and

$$\nu\big\|\boldsymbol{W}^{(k)}\big\|_{H^2(\Omega^{(k)})} + \big\|\nabla \pi^{(k)}\big\|_{L^2(\Omega^{(k)})} \leqslant C\big(\|\nabla \boldsymbol{w}\|_{\Omega^{(k)}} + \|\boldsymbol{f}\|_{\Omega^{(k)}}\big).$$

Therefore, $\boldsymbol{w} \in \mathbb{H}^2(\omega^{(k)})$, $\nabla q \in \mathbb{L}^2(\omega^{(k)})$. Thanks to the inclusion (3.41), we have

LEMMA 3.7. *The inequality*

$$\nu\|\boldsymbol{w}\|_{H^2(\Omega^{(k)})} + \|\nabla q\|_{L^2(\Omega^{(k)})} \leqslant C\sum_{j=k-1}^{k+1}\big(\|\nabla \boldsymbol{w}\|_{\Omega^{(j)}} + \|\boldsymbol{f}\|_{\Omega^{(j)}}\big), \tag{3.43}$$

holds for a constant C independent of k.

Furthermore $\boldsymbol{w} \in \mathbb{H}^2(\Omega^{k,\infty})$ and $\nabla q \in \mathbb{L}^2(\Omega^{k,\infty})$.

Now we can show easily

$$\operatorname{div}(\psi_k \boldsymbol{w}) \in H^2\big(\Omega^{(k)}\big) \cap H_0^1\big(\Omega^{(k)}\big), \qquad \nabla \boldsymbol{G}_0, \nabla \boldsymbol{G}_k \in \mathbb{L}^2\big(\Omega^{(k)}\big).$$

We apply the argument as before and obtain

$$\boldsymbol{W} \in \mathbb{H}^3\big(\Omega^{(k)}\big), \qquad \nabla \pi \in \mathbb{H}^1\big(\Omega^{(k)}\big).$$

This means that

$$\boldsymbol{w} \in \mathbb{H}^3\big(\omega^{(k)}\big), \qquad \nabla q \in \mathbb{H}^1\big(\omega^{(k)}\big).$$

As before we can show easily

$$\nu \|\boldsymbol{w}\|_{H^3(\Omega^{(k)})} + \|\nabla q\|_{H^1(\Omega^{(k)})} \leqslant C \sum_{j=k-2}^{k+2} \big(\|\nabla \boldsymbol{w}\|_{\Omega^{(j)}} + \|\boldsymbol{f}\|_{\Omega^{(j)}}\big). \tag{3.44}$$

Similarly,

$$\nu \|\boldsymbol{w}\|_{H^{2+\ell}(\Omega^{(k)})} + \|\nabla q\|_{H^{\ell}(\Omega^{(k)})} \leqslant C \sum_{j=k-1-\ell}^{k+1+\ell} \big(\|\nabla \boldsymbol{w}\|_{\Omega^{(j)}} + \|\boldsymbol{f}\|_{\Omega^{(j)}}\big) \tag{3.45}$$

for every $\ell = 2, 3, \ldots$. Now we estimate $\boldsymbol{w}$ in $\Omega^{k,\infty}$. Using (3.43) and the inclusion

$$\omega^{(j)} \subset \Omega^{(j)}, \qquad \Omega^{k,\infty} \subset \bigcup_{j \geqslant k} \omega^{(j)} \subset \Omega^{k-1,\infty},$$

we obtain

$$\begin{aligned}
\|\boldsymbol{w}\|_{C(\overline{\Omega^{k,\infty}})} &\leqslant \sup_{j \geqslant k} \|\boldsymbol{w}\|_{C(\overline{\omega^{(j)}})} \leqslant \sup_{j \geqslant k} \|\boldsymbol{w}\|_{C(\overline{\Omega^{(j)}})} \\
&\leqslant C \sup_{j \geqslant k} \|\boldsymbol{w}\|_{H^2(\Omega^{(j)})} \leqslant C \sup_{j \geqslant k} \sum_{j-1}^{j+1} \big(\|\nabla \boldsymbol{w}\|_{\Omega^{(i)}} + \|\boldsymbol{f}\|_{\Omega^{(j)}}\big) \\
&\leqslant 3C\big(\|\nabla \boldsymbol{w}\|_{\Omega^{k-2,\infty}} + \|\boldsymbol{f}\|_{\Omega^{k-2,\infty}}\big).
\end{aligned} \tag{3.46}$$

The right-hand side tends to 0 as $k \to \infty$, because the constant C does not depend on k and $\nabla \boldsymbol{w} \in \mathbb{L}^2(\Omega)$. Repeating the above argument, we have the estimate for $|\alpha| > 0$:

$$\begin{aligned}
\big\|D^\alpha \boldsymbol{w}\big\|_{C(\overline{\Omega^{k,\infty}})} &\leqslant \sup_{j \geqslant k} \big\|D^\alpha \boldsymbol{w}\big\|_{C(\overline{\omega^{(j)}})} \leqslant \sup_{j \geqslant k} \big\|D^\alpha \boldsymbol{w}\big\|_{C(\overline{\Omega^{(j)}})} \\
&\leqslant C \sup_{j \geqslant k} \|\boldsymbol{w}\|_{H^{|\alpha|+2}(\Omega^{(j)})} \leqslant C \sup_{j \geqslant k} \sum_{j-1-|\alpha|}^{j+1+|\alpha|} \big(\|\nabla \boldsymbol{w}\|_{\Omega^{(i)}} + \|\boldsymbol{f}\|_{\Omega^{(i)}}\big)
\end{aligned}$$

$$\leqslant \big(2|\alpha|+3\big)C\big(\|\nabla \boldsymbol{w}\|_{\Omega^{k-2-|\alpha|,\infty}} + \|\boldsymbol{f}\|_{\Omega^{k-2-|\alpha|,\infty}}\big). \tag{3.47}$$

∇q can be similarly estimated by (3.44) and (3.45).

This completes the proof of Lemma 3.4. □

Now let us show the exponential decay of the solution to the Poiseuille flow at the infinity. We need the next lemma. This is a variation of the result obtained by Horgan and Wheeler [9]. Cf. Galdi [8], Morimoto [12].

LEMMA 3.8. *Suppose* $y(t) \in C[R-1,\infty) \cap C^1(R-1,\infty)$, $y(t) > 0$ *and*

$$\sup_{t \geqslant R} \frac{y(t-1)-y(t)}{y(t)} < \infty \tag{3.48}$$

for some R. *Suppose further that* $y(t)$ *satisfies the following inequality*

$$y'(t) + \xi \int_t^\infty y(s)\,\mathrm{d}s \leqslant \eta y(t-1) + \zeta \mathrm{e}^{-\lambda t}, \quad \forall t > R \tag{3.49}$$

for some constants $\xi > 0$, $\eta > 0$, $\zeta \geqslant 0$ *and* $\lambda > 0$. *Then, there exist positive constants* λ_0 *and* ζ_0 *such that*

$$y(t) \leqslant \zeta_0 \mathrm{e}^{-\lambda_0 t}, \quad \forall t > R \tag{3.50}$$

holds true.

PROOF. Put

$$\gamma := \sup_{t \geqslant R} \frac{y(t-1)-y(t)}{y(t)}.$$

We can find $a > 0$, $b > 0$, $c \geqslant 0$ satisfying the following inequalities:

$$\begin{cases} b - a + \eta\gamma \leqslant 0, \\ ab - a\eta - \xi \leqslant 0, \\ \zeta + (b - \lambda - \eta)c \leqslant 0. \end{cases} \tag{3.51}$$

Let a_0, b_0 be positive solution of $b - a + \eta\gamma = 0$, $ab - a\eta - \xi = 0$. Note that $b_0 > \eta (> 0)$. If $b_0 - \eta < \lambda$, we take

$$a = a_0, \quad b = b_0, \quad c = \frac{\zeta}{\lambda - b_0 + \eta},$$

and if $b_0 - \eta \geqslant \lambda$, we take

$$a = a_0, \quad b = \eta + \frac{\lambda}{2}, \quad c = \frac{2\zeta}{\lambda}.$$

We fix these numbers and put

$$F(t) := \mathrm{e}^{-\eta(t-R)}\left\{y(t) + a\int_t^\infty y(s)\,\mathrm{d}s + c\,\mathrm{e}^{-\lambda t}\right\}. \tag{3.52}$$

Since we have chosen a, b, c satisfying (3.51) and $y(t)$ enjoys the inequality (3.49), we have

$$F'(t) + bF(t) \leqslant 0.$$

Therefore it holds true

$$F(t) \leqslant \mathrm{e}^{-b(t-R)}F(R) \quad (t \geqslant R). \tag{3.53}$$

According to (3.52), we see that

$$y(t) \leqslant \mathrm{e}^{-(b-\eta)(t-R)}F(R). \tag{3.54}$$

By straightforward calculation, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathrm{e}^{-a(t-R)}\int_t^\infty y(s)\,\mathrm{d}s\right\} \leqslant \mathrm{e}^{-(a-\eta)(t-R)}F(t) \leqslant \mathrm{e}^{-(a+b-\eta)(t-R)}F(R).$$

Integrate this inequality from R to ∞,

$$\int_R^\infty y(s)\,\mathrm{d}s \leqslant \frac{F(R)}{a+b-\eta}.$$

According to the definition (3.52) of F,

$$F(R) = y(R) + a\int_R^\infty y(s)\,\mathrm{d}s + c\mathrm{e}^{-\lambda R} \leqslant y(R) + a\frac{F(R)}{a+b-\eta} + c\mathrm{e}^{-\lambda R}.$$

Therefore

$$y(t) \leqslant \frac{a+b-\eta}{b-\eta}\left\{y(R) + c\mathrm{e}^{-\lambda R}\right\}\mathrm{e}^{-(b-\eta)(t-R)}.$$

We have proved lemma putting

$$\lambda_0 = b-\eta, \qquad \zeta_0 = \frac{a+b-\eta}{b-\eta}\left\{y(R) + c\mathrm{e}^{-\lambda R}\right\}\mathrm{e}^{(b-\eta)R}. \qquad \square$$

Next lemma is crucial in order to show the exponential decay of the solution. We use the following notation.

$$\Omega^{t,s} = \left\{(x_1, x_2) \in \Omega \mid t < x_1 < s\right\}, \qquad \Sigma(t) = \left\{(t, x_2) \in \Omega \mid -1 < x_2 < 1\right\}.$$

LEMMA 3.9. *There exists a constant $C > 0$ independent of t such that*

$$\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_{\Omega^{t,\infty}} \leqslant \sigma^S \|\nabla \boldsymbol{w}\|^2_{\Omega^{t,\infty}} + C\|\nabla \boldsymbol{w}\|^2_{\Omega^{t-1,t}} \quad (\forall t > 1,\ \forall \boldsymbol{w} \in V^S(\Omega)). \tag{3.55}$$

PROOF. Let $\boldsymbol{w} \in V^S(\Omega)$. Firstly, we show that there exist a function $\boldsymbol{\varphi} \in V^S(\Omega^{t-1,\infty})$ and a constant $C > 0$ independent of t such that

$$\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_{\Omega^{t,\infty}} \leqslant \big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}\big)_{\Omega^{t-1,\infty}} + C\|\nabla \boldsymbol{w}\|^2_{\Omega^{t-1,t}}, \tag{3.56}$$

$$\|\nabla \boldsymbol{\varphi}\|^2_{\Omega} \leqslant \|\nabla \boldsymbol{w}\|^2_{\Omega^{t,\infty}} + C\|\nabla \boldsymbol{w}\|^2_{\Omega^{t-1,t}} \tag{3.57}$$

hold. Let $\theta(s)$ be a smooth function such that

$$0 \leqslant \theta(s) \leqslant 1, \qquad \theta(s) = \begin{cases} 1 & (s \geqslant 1), \\ 0 & (s \leqslant 0). \end{cases}$$

Then $\theta(x_1 - t + 1)\boldsymbol{w}(x_1, x_2)$ is in $\mathbb{H}^1_0(\Omega^{t-1,\infty})$. Note that

$$\operatorname{div}\{\theta(x_1 - t + 1)\boldsymbol{w}\} = \nabla\theta \cdot \boldsymbol{w} + \theta \operatorname{div} \boldsymbol{w} = \theta' w_1 \in H^1_0(\Omega^{t-1,t})$$

and

$$\int_{\Sigma(t)} w_1 \,\mathrm{d}x_2 = 0.$$

Consider the boundary value problem

$$\begin{cases} \operatorname{div} \boldsymbol{v} = \operatorname{div}\{\theta(x_1 - t + 1)\boldsymbol{w}\} & \text{in } \Omega^{t-1,t}, \\ \boldsymbol{v} = 0 & \text{on } \partial\Omega^{t-1,t}. \end{cases} \tag{3.58}$$

According to Lemma 2.3, there exists a solution $\boldsymbol{v} \in \mathbb{H}^2_0(\Omega^{t-1,t})$ to (3.58) satisfying

$$\|\boldsymbol{v}\|_{H^2(\Omega^{t-1,t})} \leqslant C\|\nabla \boldsymbol{w}\|_{\Omega^{t-1,t}}, \tag{3.59}$$

where C is a constant independent of t. Moreover it is easy to see that $\boldsymbol{v}$ is symmetric with respect to the x_1-axis. Define

$$\boldsymbol{\varphi}(x) = \begin{cases} \boldsymbol{w} & \text{in } \Omega^{t,\infty}, \\ \theta(x_1 - t + 1)\boldsymbol{w} - \boldsymbol{v} & \text{in } \Omega^{t-1,t}, \\ 0 & \text{in } T \cap \{x_1 < t - 1\}. \end{cases}$$

It is clear that $\boldsymbol{\varphi} \in V^S(\Omega^{t-1,\infty}) \subset V^S(T)$. Furthermore, using (3.59), we can show

$$\|\nabla \boldsymbol{\varphi}\|_{\Omega^{t-1,t}} \leqslant C\|\nabla \boldsymbol{w}\|_{\Omega^{t-1,t}}, \tag{3.60}$$

where C is a constant independent of t.

Using the boundedness of $\boldsymbol{V}$, Poincaré's inequality and (3.60), we can obtain the estimate as follows.

$$\begin{aligned}\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{V}\big)_{\Omega^{t,\infty}}-\big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi},\boldsymbol{V}\big)_{\Omega^{t-1,\infty}}&=-\big((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi},\boldsymbol{V}\big)_{\Omega^{t-1,t}}\\&\leqslant C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t-1,t}},\end{aligned}$$

where C is a constant independent of t, and (3.56) follows. On the other hand,

$$\begin{aligned}\|\nabla\boldsymbol{\varphi}\|^2_{\Omega}&=\|\nabla\boldsymbol{\varphi}\|^2_{\Omega^{t-1,\infty}}=\|\nabla\boldsymbol{\varphi}\|^2_{\Omega^{t-1,t}}+\|\nabla\boldsymbol{\varphi}\|^2_{\Omega^{t,\infty}}\\&=\|\nabla\boldsymbol{\varphi}\|^2_{\Omega^{t-1,t}}+\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,\infty}}\end{aligned}$$

holds. According to the estimate (3.60), we prove (3.57).

Combining (3.56) and (3.57), and using the definition of σ^S, we obtain (3.55) and complete the proof of lemma. □

REMARK 3.5. (3.55) holds true without symmetry assumption, that is,

$$\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{V}\big)_{\Omega^{t,\infty}}\leqslant\sigma\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,\infty}}+C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t-1,t}}\quad(\forall\boldsymbol{w}\in V(\Omega)).$$

Now we proceed to the proof of Theorem 3.3. We use the similar argument used in Galdi [8]. According to Theorem 3.2, $\boldsymbol{w}=\boldsymbol{u}-\boldsymbol{V}$ is smooth, and satisfies the following equations in $\Omega^{R,\infty}$.

$$\begin{cases}-\nu\Delta\boldsymbol{w}+(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}+(\boldsymbol{w}\cdot\nabla)\boldsymbol{V}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{w}+\nabla q=\boldsymbol{f},\\ \operatorname{div}\boldsymbol{w}=0.\end{cases}\tag{3.61}$$

Let $t>R$. Multiplying the first equation of (3.61) by $\boldsymbol{w}$ and integrating it over $\Omega^{t,s}$ $(t<s)$, we have

$$\begin{aligned}&-\nu(\Delta\boldsymbol{w},\boldsymbol{w})_{\Omega^{t,s}}+\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{w}\big)_{\Omega^{t,s}}\\&\quad+\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{V}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{w},\boldsymbol{w}\big)_{\Omega^{t,s}}+(\nabla q,\boldsymbol{w})_{\Omega^{t,s}}=(\boldsymbol{f},\boldsymbol{w})_{\Omega^{t,s}}.\end{aligned}\tag{3.62}$$

Integration by parts yields

$$\begin{aligned}-\nu(\Delta\boldsymbol{w},\boldsymbol{w})_{\Omega^{t,s}}&=\nu\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,s}}+\frac{\nu}{2}\int_{\Sigma(t)}\frac{\partial|\boldsymbol{w}|^2}{\partial x_1}\,\mathrm{d}x_2-\frac{\nu}{2}\int_{\Sigma(s)}\frac{\partial|\boldsymbol{w}|^2}{\partial x_1}\,\mathrm{d}x_2,\\\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{w}\big)_{\Omega^{t,s}}&=\frac{1}{2}\left\{\int_{\Sigma(s)}w_1|\boldsymbol{w}|^2\,\mathrm{d}x_2-\int_{\Sigma(t)}w_1|\boldsymbol{w}|^2\,\mathrm{d}x_2\right\},\\\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{V},\boldsymbol{w}\big)_{\Omega^{t,s}}&=-\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{V}\big)_{\Omega^{t,s}}+\int_{\Sigma(s)}V_1w_1^2\,\mathrm{d}x_2-\int_{\Sigma(t)}V_1w_1^2\,\mathrm{d}x_2,\end{aligned}$$

$$\big((\boldsymbol{V}\cdot\nabla)\boldsymbol{w},\boldsymbol{w}\big)_{\Omega^{t,s}}=\frac{1}{2}\left\{\int_{\Sigma(s)}V_1|\boldsymbol{w}|^2\,\mathrm{d}x_2-\int_{\Sigma(t)}V_1|\boldsymbol{w}|^2\,\mathrm{d}x_2\right\},$$

$$(\nabla q,\boldsymbol{w})_{\Omega^{t,s}}=\int_{\partial\Omega^{t,s}}q\boldsymbol{w}\cdot\boldsymbol{n}\,\mathrm{d}\sigma=\int_{\Sigma(s)}qw_1\,\mathrm{d}x_2-\int_{\Sigma(t)}qw_1\,\mathrm{d}x_2,$$

where we have used (2.2). Substitute these expressions into (3.62) and let $s\to\infty$. According to Lemma 3.4, the integrals on $\Sigma(s)$ tend to 0, and we obtain

$$\begin{aligned}\nu\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,\infty}}=&-\int_{\Sigma(t)}\left\{\frac{\nu}{2}\frac{\partial|\boldsymbol{w}|^2}{\partial x_1}-\frac{1}{2}w_1|\boldsymbol{w}|^2-V_1w_1^2-\frac{1}{2}V_1|\boldsymbol{w}|^2-qw_1\right\}\mathrm{d}x_2\\&+\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w},\boldsymbol{V}\big)_{\Omega^{t,\infty}}+(\boldsymbol{f},\boldsymbol{w})_{\Omega^{t,\infty}}.\end{aligned}\tag{3.63}$$

Using (3.55), we have

$$\begin{aligned}\nu\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,\infty}}\leqslant&-\int_{\Sigma(t)}\left\{\frac{\nu}{2}\frac{\partial|\boldsymbol{w}|^2}{\partial x_1}-\frac{1}{2}w_1|\boldsymbol{w}|^2-V_1w_1^2-\frac{1}{2}V_1|\boldsymbol{w}|^2-qw_1\right\}\mathrm{d}x_2\\&+\sigma^S\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,\infty}}+C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t-1,t}}+(\boldsymbol{f},\boldsymbol{w})_{\Omega^{t,\infty}}.\end{aligned}\tag{3.64}$$

Here and after, C denotes various constants independent of t. Put

$$\mathcal{H}(t)\equiv\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,\infty}}.$$

Then

$$\mathcal{H}'(t)=-\int_{\Sigma(t)}\big|\nabla\boldsymbol{w}(t,x_2)\big|^2\,\mathrm{d}x_2.$$

By this notation, (3.64) is written as

$$\begin{aligned}\frac{\nu-\sigma^S}{2}\mathcal{H}(t)\leqslant&-\int_{\Sigma(t)}\left\{\frac{\nu}{2}\frac{\partial|\boldsymbol{w}|^2}{\partial x_1}-\frac{1}{2}w_1|\boldsymbol{w}|^2-V_1w_1^2-\frac{1}{2}V_1|\boldsymbol{w}|^2-qw_1\right\}\mathrm{d}x_2\\&+C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t-1,t}}+C\|\boldsymbol{f}\|^2_{\Omega^{t,\infty}},\end{aligned}$$

where we have used the estimate

$$(\boldsymbol{f},\boldsymbol{w})_{\Omega^{t,\infty}}\leqslant\|\boldsymbol{f}\|_{\Omega^{t,\infty}}\|\boldsymbol{w}\|_{\Omega^{t,\infty}}\leqslant\frac{\nu-\sigma^S}{2}\|\boldsymbol{w}\|^2_{\Omega^{t,\infty}}+C\|\boldsymbol{f}\|^2_{\Omega^{t,\infty}}.$$

Integrate the both side with respect to t from t to $t+2$:

$$\begin{aligned}
&\frac{\nu-\sigma^S}{2}\int_t^{t+2}\mathcal{H}(s)\,\mathrm{d}s\\
&\leqslant -\int_{\Omega^{t,t+2}}\left\{\frac{\nu}{2}\frac{\partial|\boldsymbol{w}|^2}{\partial x_1}-\frac{1}{2}w_1|\boldsymbol{w}|^2-V_1w_1^2-\frac{1}{2}V_1|\boldsymbol{w}|^2-qw_1\right\}\mathrm{d}x\\
&\quad+C\int_t^{t+2}\|\nabla\boldsymbol{w}\|^2_{\Omega^{s-1,s}}\,\mathrm{d}s+C\mathrm{e}^{-2\lambda_0 t},
\end{aligned}$$

where we have used the assumption for the norm of $\boldsymbol{f}$. Note that

$$\int_{\Omega^{t,t+2}}\frac{\nu}{2}\frac{\partial|\boldsymbol{w}|^2}{\partial x_1}\,\mathrm{d}x=\int_{\Sigma(t+2)}\frac{\nu}{2}|\boldsymbol{w}|^2\,\mathrm{d}x_2-\int_{\Sigma(t)}\frac{\nu}{2}|\boldsymbol{w}|^2\,\mathrm{d}x_2.$$

Let $m_{t+1}=\int_{\Omega^{t,t+2}}q\,\mathrm{d}x/|\Omega^{t,t+2}|$, where $|\Omega^{t,t+2}|$ is the measure of $\Omega^{t,t+2}$. From Lemma 3.6, the estimate

$$\begin{aligned}
\int_{\Omega^{t,t+2}}qw_1\,\mathrm{d}x&=\int_{\Omega^{t,t+2}}(q-m_{t+1})w_1\,\mathrm{d}x\leqslant\|q-m_{t+1}\|_{\Omega^{t,t+2}}\|w_1\|_{\Omega^{t,t+2}}\\
&\leqslant C\big(\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,t+2}}+\|\boldsymbol{f}\|^2_{\Omega^{t,t+2}}\big)
\end{aligned}$$

follows with a constant C independent of t. Therefore,

$$\begin{aligned}
\frac{\nu-\sigma^S}{2}\int_t^{t+2}\mathcal{H}(s)\,\mathrm{d}s&\leqslant C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,t+2}}-\frac{\nu}{2}\left\{\int_{\Sigma(t+2)}|\boldsymbol{w}|^2\,\mathrm{d}x_2-\int_{\Sigma(t)}|\boldsymbol{w}|^2\,\mathrm{d}x_2\right\}\\
&\quad+C\int_t^{t+2}\|\nabla\boldsymbol{w}\|^2_{\Omega^{s-1,s}}\,\mathrm{d}s+C\mathrm{e}^{-2\lambda_0 t}.
\end{aligned}$$

Repeating this process ℓ times, and summing up, we obtain

$$\begin{aligned}
&\frac{\nu-\sigma^S}{2}\int_t^{t+2\ell}\mathcal{H}(s)\,\mathrm{d}s\\
&\leqslant C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,t+2\ell}}-\frac{\nu}{2}\left\{\int_{\Sigma(t+2\ell)}|\boldsymbol{w}|^2\,\mathrm{d}x_2-\int_{\Sigma(t)}|\boldsymbol{w}|^2\,\mathrm{d}x_2\right\}\\
&\quad+C\int_t^{t+2\ell}\|\nabla\boldsymbol{w}\|^2_{\Omega^{s-1,s}}\,\mathrm{d}s+C\mathrm{e}^{-2\lambda_0 t}\\
&\leqslant C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t,t+2\ell}}-\frac{\nu}{2}\left\{\int_{\Sigma(t+2\ell)}|\boldsymbol{w}|^2\,\mathrm{d}x_2-\int_{\Sigma(t)}|\boldsymbol{w}|^2\,\mathrm{d}x_2\right\}\\
&\quad+C\|\nabla\boldsymbol{w}\|^2_{\Omega^{t-1,t+2\ell+1}}+C\mathrm{e}^{-2\lambda_0 t}.
\end{aligned}$$

Letting $\ell \to \infty$, we have

$$\frac{\nu - \sigma^S}{2} \int_t^\infty \mathcal{H}(s)\,\mathrm{d}s \leqslant C\mathcal{H}(t) + \frac{\nu}{2}\int_{\Sigma(t)} |\boldsymbol{w}|^2\,\mathrm{d}x_2 + C\mathcal{H}(t-1) + C\mathrm{e}^{-2\lambda_0 t}. \tag{3.65}$$

According to Poincaré's inequality, it holds that

$$\int_{\Sigma(t)} |\boldsymbol{w}|^2\,\mathrm{d}x_2 \leqslant C\int_{\Sigma(t)} |\nabla \boldsymbol{w}|^2\,\mathrm{d}x_2 = -C\mathcal{H}'(t).$$

Finally we obtain

$$\frac{\nu - \sigma^S}{2}\int_t^\infty \mathcal{H}(s)\,\mathrm{d}s + \frac{\nu}{2}C\mathcal{H}'(t) \leqslant 2C\mathcal{H}(t-1) + C\mathrm{e}^{-2\lambda_0 t}. \tag{3.66}$$

Since $\nu - \sigma^S > 0$, $\mathcal{H}(t)$ enjoys the assumptions for $y(t)$ of Lemma 3.8. Therefore, there exist constants C and λ such that

$$\|\nabla \boldsymbol{w}\|_{\Omega^t,\infty} \leqslant C\mathrm{e}^{-\lambda t}.$$

Combining the estimates (3.25) and (3.26) of Lemma 3.4, we obtain Theorem 3.3.

3.4. *A remark on the uniqueness of the solution*

In this subsection, we report a uniqueness result for special case. Let Ω be a semi-infinite channel as in Theorem 3.1. We suppose further $\Omega \subset T$.

Let $\{\boldsymbol{u}_1, p_1\}$ $\{\boldsymbol{u}_2, p_2\}$ be two solutions of $(\mathrm{NS})_1$, $(\mathrm{BC})_1$. Put $\boldsymbol{w}_i = \boldsymbol{u}_i - \boldsymbol{V} - \boldsymbol{b}$ and $\boldsymbol{w} = \boldsymbol{w}_1 - \boldsymbol{w}_2$, $q = p_1 - p_2$. Then, $\{\boldsymbol{w}, q\}$ satisfies the following:

$$\begin{aligned}
&\boldsymbol{w} \in V^S(\Omega)\\
&-\nu\Delta\boldsymbol{w} + (\boldsymbol{w}_1\cdot\nabla)\boldsymbol{w}_1 - (\boldsymbol{w}_2\cdot\nabla)\boldsymbol{w}_2\\
&\quad + (\boldsymbol{V}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{V} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{b} + \nabla q = 0.
\end{aligned}$$

Taking the inner product with $\boldsymbol{w}$, we obtain

$$\nu\|\boldsymbol{w}\|^2 + \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{V}, \boldsymbol{w}\big) + \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{b}, \boldsymbol{w}\big) = \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{w}_2\big).$$

Since $\Omega \subset T$, $\boldsymbol{w}$ can be considered as an element of $V^S(T)$ if we extend it 0 outside of Ω, and we obtain

$$\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_\Omega = \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{V}\big)_T \leqslant \sigma^S\|\nabla\boldsymbol{w}\|_T^2 = \sigma^S\|\nabla\boldsymbol{w}\|_\Omega^2.$$

Therefore,

$$\nu\|\boldsymbol{w}\|^2 \leqslant \sigma^S\|\nabla\boldsymbol{w}\|_\Omega^2 + \varepsilon\|\nabla\boldsymbol{w}\|_\Omega^2 + \|\boldsymbol{w}_2\|_4\|\boldsymbol{w}\|_4\|\nabla\boldsymbol{w}\|.$$

Sobolev's inequality $\|\boldsymbol{w}\|_4 \leqslant c\|\nabla\boldsymbol{w}\|$ yields

$$\nu\|\nabla\boldsymbol{w}\|^2 \leqslant \big(\sigma^S + \varepsilon + c\|\boldsymbol{w}_2\|_4\big)\|\nabla\boldsymbol{w}\|^2.$$

If $\boldsymbol{w}_2$ is sufficiently small, i.e., $\nu > \sigma^S + \varepsilon + c\|\boldsymbol{w}_2\|_4$, then we can deduce $\boldsymbol{w} = 0$, i.e., $\boldsymbol{w}_1 = \boldsymbol{w}_2$, and we have proved the following theorem.

THEOREM 3.4. *Suppose Ω, $\boldsymbol{f}$ and $\boldsymbol{\beta}$ satisfy the same assumptions as in Theorem* 3.1, *and $\Omega \subset T$. Suppose further $\nu - \sigma^S > 0$. Then, the solution to* $(\mathrm{NS})_1$, $(\mathrm{BC})_1$ *is unique if it is small.*

4. Channel with J outlets ($J \geqslant 2$)

We study the case where Ω has $J(\geqslant 2)$ outlets. We recall that Ω is a two-dimensional symmetric and multiply connected domain with smooth boundary satisfying the two conditions (Ω-1), (Ω-2) in Introduction. Remember

$$\partial\Omega = \bigcup_{i=0}^{N} \Gamma_i, \qquad \Gamma_0 = \bigcup_{j=1}^{J} \Gamma_0^j.$$

We state the existence results according to the type of channel.

(a) There is no self-symmetric outlet.
(b) There is only one self-symmetric outlet.
(c) There are two self-symmetric outlets.

Let $\boldsymbol{V}_j$ be a Poiseuille flow in T_j $(1 \leqslant j \leqslant J)$, and P_j associated pressure. Let $\boldsymbol{\beta}$ be given on the boundary $\partial\Omega$. We study the following equations.

$$(\mathrm{NS}) \quad \begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } \Omega, \end{cases}$$

with the boundary condition

$$(\mathrm{BC}) \quad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} \quad \text{on } \partial\Omega, \\ \boldsymbol{u} \to \boldsymbol{V}_j \quad \text{as } |x| \to \infty \text{ in } \Omega_j \ (1 \leqslant j \leqslant J). \end{cases}$$

Here $\boldsymbol{\beta}$ and $\boldsymbol{V}_j (1 \leqslant j \leqslant J)$ should satisfy an outflow condition:

$$(\mathrm{OC}) \quad \int_{\partial\Omega} \boldsymbol{\beta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \sum_{j=1}^{J}\int_{\Sigma_j} \boldsymbol{V}_j\cdot\boldsymbol{\nu}_j\,\mathrm{d}\sigma = 0,$$

where $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$, Σ_j the section of Ω_j, $\boldsymbol{v}_j$ the unit normal vector to Σ_j, directing from Ω_0 to Ω_j. We say that the Poiseuille flows $\{\boldsymbol{V}_j \mid 1 \leqslant j \leqslant J\}$ are symmetric if, for the semi-infinite channels Ω_i and Ω_j symmetric with respect to the x_1-axis, the Poiseuille flows $\boldsymbol{V}_i$ and $\boldsymbol{V}_j$ are symmetric with respect to the x_1-axis.

First, for the type (a) and (b), we have:

THEOREM 4.1. *Let Ω be a domain symmetric with respect to the x_1-axis with the properties (Ω-1), (Ω-2). Suppose that Γ_0 intersects the x_1-axis, $\boldsymbol{f} \in \mathbb{L}^2(\Omega)$ is symmetric with respect to the x_1 axis, $\boldsymbol{\beta}$ is smooth, symmetric and of compact support, and the Poiseuille flows $\{\boldsymbol{V}_j \mid 1 \leqslant j \leqslant J\}$ are symmetric. Suppose also $\boldsymbol{\beta}$ and $\{\boldsymbol{V}_j \mid 1 \leqslant j \leqslant J\}$ satisfy* (OC). *If the boundary component Γ_i or Γ_0^k does not intersect the x_1-axis, we suppose further*

$$\int_{\Gamma_i} \boldsymbol{\beta} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0, \qquad \int_{\Gamma_0^k} \boldsymbol{\beta} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0.$$

If $\nu > \max(\sigma_1, \sigma_2, \ldots, \sigma_J)$, then there exists a symmetric weak solution $\boldsymbol{u} \in \mathcal{S}$ to (NS), (BC). *In case T_j is self-symmetric, σ_j should be replaced by σ_j^S.*

For the type (c) channel, we have:

THEOREM 4.2. *Let Ω be a domain symmetric with respect to the x_1-axis with the properties (Ω-1), (Ω-2). Suppose that Γ_0 does not intersect the x_1-axis, $\boldsymbol{f} \in \mathbb{L}^2(\Omega)$ is symmetric with respect to the x_1 axis, $\boldsymbol{\beta}$ is smooth, symmetric and of compact support, and $\{\boldsymbol{V}_j \mid 1 \leqslant j \leqslant J\}$ are symmetric. Suppose also $\boldsymbol{\beta}$ and $\boldsymbol{V}_j$ satisfy the following conditions.*

$$\sum_{i=1}^{N} \int_{\Gamma_i} \boldsymbol{\beta} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0, \ \int_{\Gamma_0^j} \boldsymbol{\beta} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0 \quad (1 \leqslant j \leqslant J),$$

$$\sum_{j=1}^{J} \int_{\Sigma_j} \boldsymbol{V}_j \cdot \boldsymbol{v}_j \, \mathrm{d}\sigma = 0.$$

If the boundary component Γ_i does not intersect the x_1-axis, we suppose further

$$\int_{\Gamma_i} \boldsymbol{\beta} \cdot \boldsymbol{n} \, \mathrm{d}\sigma = 0.$$

If $\nu > \max(\sigma_1, \sigma_2, \ldots, \sigma_J)$, then there exists a symmetric weak solution $\boldsymbol{u} \in \mathcal{S}$ to (NS), (BC). *For the two self-symmetric Ω_j, σ_j should be replaced by σ_j^S.*

REMARK 4.1. If $\boldsymbol{\beta}$ and $\{\boldsymbol{V}_j | 1 \leqslant j \leqslant J\}$ satisfy only (OC) and not satisfy the condition stated in Theorem 4.2, we do not know if the solution exists or not.

REMARK 4.2. It is well known that for the weak solution $\boldsymbol{u}$ to (NS), (BC), there exists a scalar function $p \in L^2_{\mathrm{loc}}(\Omega)$ such that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \big((\boldsymbol{u} \cdot \nabla \boldsymbol{u}), \boldsymbol{v}\big) - (p, \operatorname{div} \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in \mathbb{C}_0^\infty(\Omega)).$$

p is called an associated pressure of $\boldsymbol{u}$. We call $\{\boldsymbol{u}, p\}$ solution pair to (NS), (BC).

As for the regularity of the solutions, we have

THEOREM 4.3. *If $\boldsymbol{f} \in \mathbb{C}^\infty(\overline{\Omega})$, then the solution pair $\{\boldsymbol{u}, p\}$ obtained in Theorems* 4.1 *and* 4.2 *is smooth in Ω.*

Now we state the asymptotic behavior of the solution. The solutions obtained above tend to the Poiseuille flow as $|x| \to \infty$. We use the notation

$$\Omega_{j,r} := \Omega_j \cap \big\{|x| > r\big\}.$$

LEMMA 4.1. *Let $\{\boldsymbol{u}, p\}$ be the solution pair obtained in Theorems* 4.1 *and* 4.2. *Then $\{\boldsymbol{u}, \nabla p\}$ tends to $\{\boldsymbol{V}_j, \nabla P_j\}$ uniformly as $|x| \to \infty$ in Ω_j, that is,*

$$\sup_{\Omega_{j,R}} \big|D^\alpha \big\{\boldsymbol{u}(x) - \boldsymbol{V}_j(x)\big\}\big| \leqslant C_\alpha \big(\big\|\nabla\{\boldsymbol{u} - \boldsymbol{V}_j\}\big\|_{\Omega_{j,R-2-|\alpha|}} + \|\boldsymbol{f}\|_{\Omega_{j,R-2-|\alpha|}}\big),$$

$$\sup_{\Omega_{j,R}} \big|D^\alpha \big\{\nabla p(x) - \nabla P_j(x)\big\}\big| \leqslant C_\alpha \big(\big\|\nabla\{\boldsymbol{u} - \boldsymbol{V}_j\}\big\|_{\Omega_{j,R-2-|\alpha|}} + \|\boldsymbol{f}\|_{\Omega_{j,R-2-|\alpha|}}\big)$$

hold true for sufficiently large R with a positive constant C_α.

THEOREM 4.4. *Under the assumption of Theorem* 4.3, *we assume further that there are positive constants C_j, λ_j such that*

$$\|\boldsymbol{f}\|_{\Omega_{j,t}} \leqslant C_j \mathrm{e}^{-\lambda_j t}$$

holds. Let $\{\boldsymbol{u}, p\}$ be the solution pair obtained in Theorems 4.1 *and* 4.2. *Then $\{\boldsymbol{u}, \nabla p\}$ tends to $\{\boldsymbol{V}_j, \nabla P_j\}$ exponentially, as $|x| \to \infty$ in Ω_j, that is, there exists a positive constant λ such that for every multi-index α, the estimates*

$$\big|D^\alpha \big\{\boldsymbol{u}(x) - \boldsymbol{V}_j(x)\big\}\big| \leqslant C_\alpha \mathrm{e}^{-\lambda|x|} \quad (x \in \Omega_j),$$

$$\big|D^\alpha \big\{\nabla p(x) - \nabla P_j(x)\big\}\big| \leqslant C_\alpha \mathrm{e}^{-\lambda|x|} \quad (x \in \Omega_j)$$

hold true with some positive constant C_α.

4.1. *Connection of the Poiseuille flows*

Let us "connect" Poiseuille flows $\boldsymbol{V}_j$ $(1 \leqslant j \leqslant J)$ in Ω, that is, construct a solenoidal vector field $\boldsymbol{W}$ which coincides with the Poiseuille flow $\boldsymbol{V}_j$ in every $\Omega_j \cap \{|x| > R_0 + 1\}$.

In this subsection, it is not necessary to suppose that Ω is symmetric. Let $\widetilde{\Omega}_0$ be a bounded domain with smooth boundary such that

$$\Omega_0 \subset \widetilde{\Omega}_0 \subset \Omega \cap \{|x| < R_0 + 1\}.$$

(P-1) The case $\sum_{j=1}^{J} \int_{\Sigma_j} \boldsymbol{V}_j \cdot \boldsymbol{\nu}_j \, \mathrm{d}\sigma = 0$.

Let $\tilde{\gamma}_i = \Omega_i \cap \partial\widetilde{\Omega}_0$ $(i = 1, 2, \ldots, J)$. Consider the following Stokes problem.

$$-\Delta \boldsymbol{v} + \nabla q = 0, \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \widetilde{\Omega}_0$$

with the boundary condition

$$\begin{aligned}
&\boldsymbol{v}|_{\tilde{\gamma}_j} = \boldsymbol{V}_j, \quad 1 \leqslant j \leqslant J, \\
&\boldsymbol{v}|_{\Gamma_i} = 0, \quad 1 \leqslant i \leqslant N, \\
&\boldsymbol{v}|_{\partial\widetilde{\Omega}_0 \setminus (\Gamma_1 \cup \cdots \cup \Gamma_N \cup \tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_J)} = 0.
\end{aligned}$$

According to the condition (P-1), the solution exists. Let $\boldsymbol{v}_0$ be the solution. Put

$$\boldsymbol{W} = \begin{cases} \boldsymbol{v}_0 & \text{in } \widetilde{\Omega}_0, \\ \boldsymbol{V}_j & \text{in } \Omega_j \setminus \widetilde{\Omega}_0 \ (1 \leqslant j \leqslant J). \end{cases} \tag{4.1}$$

(P-2) The case $\sum_{j=1}^{J} \int_{\Sigma_j} \boldsymbol{V}_j \cdot \boldsymbol{\nu}_j \, \mathrm{d}\sigma \equiv \mu \neq 0$.

$\Gamma_0 \cap \partial\widetilde{\Omega}_0$ consists of J curves. We choose one of these curves, say γ_0, a part of which is represented by a smooth function f as follows.

$$\gamma_0 : x_1 = f(x_2), \quad a < x_2 < b,$$

for appropriate a, b. Let $\eta(t)$ be a smooth function of $t \in (-\infty, \infty)$ with compact support in the interval (a, b) and $\int_{-\infty}^{\infty} \eta(t) \, \mathrm{d}t = 1$. We consider the Stokes problem:

$$-\Delta \boldsymbol{v} + \nabla q = 0, \quad \operatorname{div} \boldsymbol{v} = 0 \quad \text{in } \widetilde{\Omega}_0$$

with the boundary condition

$$\begin{aligned}
&\boldsymbol{v}|_{\tilde{\gamma}_j} = \boldsymbol{V}_j, \quad 1 \leqslant j \leqslant J, \\
&\boldsymbol{v}|_{\Gamma_i} = 0, \quad 1 \leqslant i \leqslant N, \\
&\boldsymbol{v}|_{\gamma_0} = (-\mu\eta(x_2), 0), \qquad \boldsymbol{v}|_{\partial\widetilde{\Omega}_0 \setminus (\Gamma_1 \cup \cdots \cup \Gamma_N \cup \gamma_0 \cup \tilde{\gamma}_1 \cup \cdots \cup \tilde{\gamma}_J)} = 0.
\end{aligned}$$

Thanks to (P-2), the solution exists. Let $\boldsymbol{v}_0$ be the solution. We define a connection of the Poiseuille flows as follows.

$$\boldsymbol{W} = \begin{cases} \boldsymbol{v}_0 & \text{in } \widetilde{\Omega}_0, \\ \boldsymbol{V}_j & \text{in } \Omega_j \setminus \widetilde{\Omega}_0 \ (1 \leqslant j \leqslant J). \end{cases} \tag{4.2}$$

The vector field $\boldsymbol{W}$ defined by (4.1) and (4.2) satisfies the condition (2) of Definition 1.2. Since $\boldsymbol{v}_0$ satisfies the stringent outflow condition, that is, for every connected component γ of the boundary of $\widetilde{\Omega}_0$ the integral $\int_\gamma \boldsymbol{v}_0 \cdot \boldsymbol{n}\, d\sigma = 0$, there exists its stream function φ_0, and $\boldsymbol{v}_0 = \nabla^\perp \varphi_0 = (D_2\varphi_0, -D_1\varphi_0)$ holds. Therefore, there exists a stream function $\Phi(x)$ of $\boldsymbol{W}$:

$$\boldsymbol{W} = \nabla^\perp \Phi.$$

We can choose $\Phi(x)$ such that $\Phi(x)|_{\Omega_0}$ belongs to $H^2(\Omega_0)$, and is continuous and bounded in $\overline{\Omega}_j$ for $1 \leqslant j \leqslant J$.

REMARK 4.3. Suppose that Ω is symmetric with respect to the x_1-axis. Then, we can choose $\widetilde{\Omega}_0$ symmetric. Suppose further the Poiseuille flows $\boldsymbol{V}_j$ $(1 \leqslant j \leqslant J)$ are also symmetric. If at least one of the connected components of the outer boundary Γ_0 intersects the x_1-axis, we can use it for γ_0. If we choose an even function $\eta(t)$ having compact support in a neighborhood of 0, the vector field $\boldsymbol{W}$ constructed as above is symmetric. We make use of this $\boldsymbol{W}$ in the proof of Theorems 4.1 and 4.2.

4.2. *Extension of the boundary value*

Next lemma holds true even for non-symmetric domain.

LEMMA 4.2. *Let $\boldsymbol{\beta}_0$ be defined on $\partial\Omega$, smooth and have a compact support. Suppose*

$$\text{(SOC)} \quad \int_{\Gamma_i} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, d\sigma = 0 \quad (1 \leqslant i \leqslant N), \quad \int_{\Gamma_0^j} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, d\sigma = 0 \quad (1 \leqslant j \leqslant J).$$

Then for every $\varepsilon > 0$ there exists a solenoidal extension $\boldsymbol{b}_0$ of $\boldsymbol{\beta}_0$ satisfying

$$\text{(L)} \quad \left|\left((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}_0\right)\right| \leqslant \varepsilon \|\nabla \boldsymbol{v}\|^2 \quad (\forall \boldsymbol{v} \in V(\Omega)).$$

Furthermore $\boldsymbol{b}_0$ is of compact support.

If $\boldsymbol{\beta}_0$ does not enjoy the condition (SOC), we cannot expect the existence of its solenoidal extension satisfying the above inequality (L). Nevertheless, if the domain and the boundary value are symmetric, we have

LEMMA 4.3. *Let Ω be symmetric with respect to the x_1-axis. Let $\boldsymbol{\beta}_0$ be smooth on $\partial\Omega$, symmetric with respect to the x_1-axis and have a compact support. Suppose $\boldsymbol{\beta}_0$ satisfy*

$$\text{(GOC)} \quad \int_{\partial\Omega} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, d\sigma = 0.$$

If the boundary component Γ_i or Γ_0^k does not intersect the x_1-axis, we suppose further

$$\int_{\Gamma_i} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, d\sigma = 0, \qquad \int_{\Gamma_0^k} \boldsymbol{\beta}_0 \cdot \boldsymbol{n}\, d\sigma = 0.$$

Then, for every positive ε, there exists a solenoidal symmetric extension $\boldsymbol{b}_0$ of $\boldsymbol{\beta}_0$, such that

$$\text{(LF)} \quad \big|\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b}_0\big)\big| \leqslant \varepsilon \|\nabla \boldsymbol{v}\|^2 \quad (\forall \boldsymbol{v} \in V^S(\Omega))$$

holds true. Furthermore $\boldsymbol{b}_0$ is of compact support.

The proof is similar to that of Lemma 3.2 and is omitted.

4.3. *Existence of the solution*

Now, we prove the existence of the solution. Firstly, we show the proof of Theorem 4.1. Theorem 4.2 can similarly be proved. Let $\boldsymbol{W}$ be the connection of $\boldsymbol{V}_j$ in Ω obtained in Section 4.1. According to Remark 4.3, we can choose $\boldsymbol{W}$ symmetric.

We construct an element $\boldsymbol{s} \in V^S(\Omega)$ which approximates $\boldsymbol{W}$, using an auxiliary function $\theta_{j,n}$. Let $\theta(t)$ be a smooth function of $t \geqslant 0$ such that

$$0 \leqslant \theta(t) \leqslant 1, \qquad \theta(t) = 0 \quad (0 \leqslant t \leqslant 1/2), \qquad \theta(t) = 1 \quad (1 \leqslant t).$$

Put $\theta_n(t) = \theta(t/(2n))$. Let $a_j x_1 + b_j x_2 = c_j$ be the equation of the axis of T_j $(1 \leqslant j \leqslant J)$. If $b_j \neq 0$, we define

$$\theta_{j,n}(x) = \theta_n\big(g_j(x)\big), \quad \text{where } g_j(x) = \left\{x_1^2 + \left(\frac{c_j - a_j x_1}{b_j}\right)^2\right\}^{1/2}.$$

If $b_j = 0$, then

$$\theta_{j,n}(x) = \theta_n\left(\left\{x_2^2 + \left(\frac{c_j}{a_j}\right)^2\right\}^{1/2}\right).$$

LEMMA 4.4. *There exists a positive constant C_0 such that for every positive ε and $n > R_0$, there exists a function $\boldsymbol{s} \in V^S(\Omega)$ with compact support, satisfying the following estimates.*

$$\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{W}\big)_\Omega \leqslant \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{s}\big)_\Omega + \sum_{j=1}^{J} \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_{j,n}^2 \boldsymbol{W}\big)_{\Omega_j} + \big(\varepsilon + C_0 n^{-1}\big)\|\nabla \boldsymbol{w}\|^2 \quad (\forall \boldsymbol{w} \in V(\Omega)). \tag{4.3}$$

PROOF. The proof is similar to that of Lemma 3.3. We have only to show the existence of a function $\boldsymbol{s} \in V^S(\Omega)$ with compact support satisfying the next two inequalities.

$$\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{W}\big)_{\Omega_0} - \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{s}\big)_{\Omega_0} \leqslant \frac{\varepsilon}{J+1}\|\nabla\boldsymbol{w}\|^2 \quad (\forall \boldsymbol{w} \in V(\Omega)), \tag{4.4}$$

$$\begin{aligned}&\big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{W}\big)_{\Omega_j} - \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \boldsymbol{s}\big)_{\Omega_j} - \big((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_{j,n}^2\boldsymbol{W}\big)_{\Omega_j}\\ &\quad\leqslant \left(\frac{\varepsilon}{J+1} + \frac{C_0}{Jn}\right)\|\nabla\boldsymbol{w}\|^2 \quad (\forall \boldsymbol{w} \in V(\Omega),\ 1 \leqslant j \leqslant J).\end{aligned} \tag{4.5}$$

Let Φ be the stream function of $\boldsymbol{W}$, that is, $\boldsymbol{W} = \nabla^\perp\Phi$. We use the function $h(t)$ defined in the proof of Lemma 3.1 and $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. Put $\widetilde{\boldsymbol{W}} = \nabla^\perp\{h(\rho(x))\Phi(x)\}$ and define $\boldsymbol{s}$ as follows.

$$\boldsymbol{s}(x) = \begin{cases} \boldsymbol{W} - \widetilde{\boldsymbol{W}} & \text{in } \Omega_0,\\ \nabla^\perp\{(1-h(\rho(x)))(1-\theta_{j,n}(x)^2)\Phi(x)\} & \text{in } \Omega_j\ (1 \leqslant j \leqslant J).\end{cases} \tag{4.6}$$

Let n be sufficiently large. It is easy to see that $\boldsymbol{s} \in V^S(\Omega)$ and the support of $\boldsymbol{s}$ is contained in $\{|x| < 2n\} \cap \Omega$. If we choose the parameters κ_0 and δ_0 sufficiently small, we can show that $\boldsymbol{s}$ satisfies the inequalities (4.4) and (4.5), therefore, the inequality (4.3) holds true. □

Next theorem is J-outlets version of Theorem A. We give an outline of the proof in Section 6.

THEOREM A_J. *Put*

$$(\mathrm{G_J}) \quad G_J(n) := \sup_{\boldsymbol{w}\in V^S(\Omega)} \frac{\sum_{j=1}^J((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_{j,n}^2\boldsymbol{W})_{\Omega_j}}{\|\nabla\boldsymbol{w}\|_\Omega^2}.$$

Then, it holds

$$\lim_{n\to\infty} G_J(n) = \max(\sigma_1, \sigma_2, \ldots, \sigma_J)$$

where σ_j should be replaced by σ_j^S if T_j is self-symmetric.

We continue the proof of Theorem 4.1. By Theorem A_J and the assumption

$$\nu - \max(\sigma_1, \ldots, \sigma_J) > 0,$$

we can choose $\varepsilon > 0$ and $m \in \mathbf{N}$ such that the following inequalities hold. C_0 is the constant in Lemma 4.4.

$$G_J(m) - \max(\sigma_1, \ldots, \sigma_J) \leqslant \frac{1}{4}\big\{\nu - \max(\sigma_1, \sigma_2, \ldots, \sigma_J)\big\}, \tag{4.7}$$

$$2\varepsilon + C_0 m^{-1} \leqslant \frac{1}{4}\big\{\nu - \max(\sigma_1, \sigma_2, \ldots, \sigma_J)\big\}. \tag{4.8}$$

Let Ω^n, $n = 1, 2, \ldots$, be a sequence of bounded symmetric domains with smooth boundary such that

$$\Omega^n \subset \Omega^{n+1} \subset \Omega, \quad \partial\Omega^n \cap \partial\Omega \subset \partial\Omega^{n+1} \cap \partial\Omega, \quad \Omega^n \to \Omega \quad (n \to \infty).$$

We suppose that the domain Ω^1 contains $\Omega \cap \{|x| < 2m\}$. We consider the stationary Navier–Stokes equations in Ω^n.

$$(\mathrm{NS})^n \quad \begin{cases} (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \nu\Delta\boldsymbol{u} - \nabla p + \boldsymbol{f} & \text{in } \Omega^n, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega^n, \end{cases}$$

with the boundary condition

$$(\mathrm{BC})^n \quad \begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on } \partial\Omega^n \cap \partial\Omega, \\ \boldsymbol{u} = \boldsymbol{W} & \text{on } \partial\Omega^n \setminus \partial\Omega, \end{cases}$$

where $\boldsymbol{W}$ is chosen as in Remark 4.3. A function $\boldsymbol{u}$ is called a weak solution to $(\mathrm{NS})^n$, $(\mathrm{BC})^n$, if $\boldsymbol{u} \in \mathbb{H}^1(\Omega^n)$, $\operatorname{div} \boldsymbol{u} = 0$,

$$\nu(\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) + \big((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{v}\big) = (\boldsymbol{f}, \boldsymbol{v}) \quad (\forall \boldsymbol{v} \in V(\Omega^n)),$$

and $\boldsymbol{u}$ satisfies the boundary condition $(\mathrm{BC})^n$ in the trace sense.

Let $\boldsymbol{u}_n$ be a symmetric weak solution to $(\mathrm{NS})^n$, $(\mathrm{BC})^n$, the existence of which was established by Fujita [7]. Put $\boldsymbol{u}_n = \boldsymbol{w}_n + \boldsymbol{W} + \boldsymbol{b}$, where $\boldsymbol{b}$ is the solenoidal symmetric extension (in Ω) of $\boldsymbol{\beta} - \boldsymbol{W}|_{\partial\Omega}$. Since $\boldsymbol{\beta} - \boldsymbol{W}|_{\partial\Omega}$ satisfies the conditions in Lemma 4.3, we can find its solenoidal extension $\boldsymbol{b}$ satisfying the inequality

$$(\mathrm{LF}) \quad \big|\big((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}\big)_\Omega\big| \leqslant \varepsilon\|\nabla\boldsymbol{v}\|_\Omega^2 \quad (\forall \boldsymbol{v} \in V^S(\Omega)).$$

It should be noted that the support of $\boldsymbol{b}$ is compact. Without loss of generality, we can assume that the support of $\boldsymbol{b}$ is contained in $\overline{\Omega}^1$. Since $\operatorname{div} \boldsymbol{w}_n = 0$ and $\boldsymbol{w}_n|_{\partial\Omega^n} = 0$, $\boldsymbol{w}_n$ belongs to $V^S(\Omega^n)$ and satisfies the following equation.

$$\begin{aligned} &\nu(\nabla\boldsymbol{w}_n, \nabla\boldsymbol{v}) + \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}\big) + \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{W} + (\boldsymbol{W} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}\big) \\ &\qquad + \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{v}\big) \\ &\quad = (\boldsymbol{F}, \boldsymbol{v}) - \nu\big(\nabla(\boldsymbol{b} + \boldsymbol{W}), \nabla\boldsymbol{v}\big) \quad (\forall \boldsymbol{v} \in V^S(\Omega^n)), \end{aligned} \tag{4.9}$$

where $\boldsymbol{F} = \boldsymbol{f} - \{(\boldsymbol{b} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{W} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{W} + (\boldsymbol{W} \cdot \nabla)\boldsymbol{W}\}$.

Substitute $\boldsymbol{v} = \boldsymbol{w}_n$ into (4.9), and we obtain

$$\begin{aligned} \nu\|\nabla\boldsymbol{w}_n\|^2 = &\big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{W}\big) + \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{b}\big) + (\boldsymbol{F}, \boldsymbol{w}_n) \\ &- \nu\big(\nabla(\boldsymbol{b} + \boldsymbol{W}), \nabla\boldsymbol{w}_n\big). \end{aligned} \tag{4.10}$$

Note that the flow $\boldsymbol{W}$ coincides with the Poiseuille flow $\boldsymbol{V}_j$ in $\Omega_j \setminus \widetilde{\Omega}_0$ for $1 \leqslant j \leqslant J$. P_j is the pressure corresponding to $\boldsymbol{V}_j$. Since $(\boldsymbol{V}_j \cdot \nabla)\boldsymbol{V}_j = 0$, we have

$$\big((\boldsymbol{W} \cdot \nabla)\boldsymbol{W}, \boldsymbol{v}\big) = \big((\boldsymbol{W} \cdot \nabla)\boldsymbol{W}, \boldsymbol{v}\big)_{\widetilde{\Omega}_0},$$

$$\nu(\nabla \boldsymbol{W}, \nabla \boldsymbol{v}) = \nu(\nabla \boldsymbol{W}, \nabla \boldsymbol{v})_{\widetilde{\Omega}_0} + \sum_{j=1}^{J} \left\{ \int_{\tilde{\gamma}_j} \nu \frac{\partial \boldsymbol{V}_j}{\partial n} \boldsymbol{v} \, \mathrm{d}\sigma - \int_{\tilde{\gamma}_j} P_j \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\sigma \right\},$$

where $\tilde{\gamma}_j = \partial \widetilde{\Omega}_0 \cap \Omega_j$. Using the trace theorem, we know the last term of the right-hand side is bounded by $C\|\nabla \boldsymbol{v}\|_{\widetilde{\Omega}_0}$ for some constant C. Choose $\boldsymbol{s}$ as in Lemma 4.4 for ε and m satisfying (4.7), (4.8). Then the support $\boldsymbol{s}$ is contained in $\{|x| < 2m\}$, and we obtain the following estimate.

$$\big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{W}\big) \leqslant \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\big) + \sum_{j=1}^{J} \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \theta_{j,m}^2 \boldsymbol{W}\big)_{\Omega_j} + \big(\varepsilon + C_0 m^{-1}\big)\|\nabla \boldsymbol{w}_n\|^2.$$

Therefore (4.10) is estimated as follows.

$$\nu\|\nabla \boldsymbol{w}_n\|^2 \leqslant \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\big) + \sum_{j=1}^{J} \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \theta_{j,m}^2 \boldsymbol{W}\big)_{\Omega_j} + \big(\varepsilon + C_0 m^{-1}\big)\|\nabla \boldsymbol{w}_n\|^2 + \varepsilon\|\nabla \boldsymbol{w}_n\|^2 + \|\boldsymbol{F}\|_{\widetilde{\Omega}_0}\|\boldsymbol{w}_n\| + C\|\nabla \boldsymbol{w}_n\|$$

for some constant C. The last two terms in the right-hand side are estimated by $k'''\|\nabla \boldsymbol{w}_n\|$ with some positive constant k''' which is independent of $\boldsymbol{w}_n$. Since $\boldsymbol{s} \in V^S(\Omega^1)$, we can substitute $\boldsymbol{s}$ for $\boldsymbol{v}$ in (4.9) and obtain

$$\big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\big) = -\nu(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{s}) - \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{W} + (\boldsymbol{W} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\big) - \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\big) + (\boldsymbol{F}, \boldsymbol{s}) - \nu\big(\nabla(\boldsymbol{b} + \boldsymbol{W}), \nabla \boldsymbol{s}\big).$$

Since the right-hand side is linear with respect to $\boldsymbol{w}_n$, it is estimated as follows with some positive constants k', k'' independent of $\boldsymbol{w}_n$.

$$\big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \boldsymbol{s}\big) \leqslant k' + k''\|\nabla \boldsymbol{w}_n\|.$$

Summing up, we obtain

$$\nu\|\nabla \boldsymbol{w}_n\|^2 \leqslant k' + k''\|\nabla \boldsymbol{w}_n\| + \sum_{j=1}^{J} \big((\boldsymbol{w}_n \cdot \nabla)\boldsymbol{w}_n, \theta_{j,m}^2 \boldsymbol{W}\big)_{\Omega_j} + \big(2\varepsilon + C_0 m^{-1}\big)\|\nabla \boldsymbol{w}_n\|^2 + k'''\|\nabla \boldsymbol{w}_n\|$$

$$\leqslant k' + k''\|\nabla \boldsymbol{w}_n\| + G_J(m)\|\nabla \boldsymbol{w}_n\|^2 + \big(2\varepsilon + C_0 m^{-1}\big)\|\nabla \boldsymbol{w}_n\|^2 + k'''\|\nabla \boldsymbol{w}_n\|,$$

where $G_J(m)$ is defined in Theorem A_J. According to (4.7) and (4.8), we have

$$\frac{1}{2}\big\{\nu - \max(\sigma_1, \sigma_2, \ldots, \sigma_N)\big\}\|\nabla \boldsymbol{w}_n\|^2 \leqslant k' + (k'' + k''')\|\nabla \boldsymbol{w}_n\|.$$

Therefore, there exists a positive constant M independent of n, for which the following inequality holds.

$$\|\nabla \boldsymbol{w}_n\| \leqslant M.$$

By the similar argument to the proof of Theorem 3.1, we can show the existence of $\boldsymbol{u} \in \mathcal{S}$ which is symmetric and satisfies (1.1), and Theorem 4.1 is proved. Cf. Morimoto and Fujita [13,14].

4.4. *Regularity and asymptotic behavior of the solution*

Let $\{\boldsymbol{u}, p\}$ be solution pair. It holds that

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) + ((\boldsymbol{u} \cdot \nabla)\boldsymbol{u}, \boldsymbol{\varphi}) - (p, \operatorname{div} \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}) \quad (\forall \boldsymbol{\varphi} \in \mathbb{C}_0^\infty(\Omega)).$$

Since $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{b} + \boldsymbol{W}$ and the support of $\boldsymbol{b}$ is compact,

$$\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{V}_j, \quad p = q + P_j \quad \text{in } \Omega_j$$

and $\{\boldsymbol{w}, q\}$ satisfies

$$\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) - (q, \operatorname{div} \boldsymbol{\varphi}) + \big((\boldsymbol{w} \cdot \nabla)\boldsymbol{w}, \boldsymbol{\varphi}\big) + \big((\boldsymbol{w} \cdot \nabla)\boldsymbol{V}_j + (\boldsymbol{V}_j \cdot \nabla)\boldsymbol{w}, \boldsymbol{\varphi}\big) = (\boldsymbol{f}, \boldsymbol{\varphi}) \quad (\forall \boldsymbol{\varphi} \in \mathbb{C}_0^\infty(\Omega_j)).$$

The regularity of $\{\boldsymbol{w}, \nabla q\}$ in Ω_j can be proved using the regularity results for the Stokes equations, and we obtain Theorem 4.3. Cf. the proof of Theorem 3.2.

Since $\{\boldsymbol{w}, \nabla q\}$ is smooth in Ω and the support of $\boldsymbol{b}$ is compact, it holds that

$$-\nu\Delta \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{V}_j + (\boldsymbol{V}_j \cdot \nabla)\boldsymbol{w} + \nabla q = \boldsymbol{f} \quad \text{in } \Omega_{j,R}$$

for a sufficiently large R. Applying the results for $J = 1$, we obtain Lemma 4.1 and Theorem 4.4.

5. Channel contained in T

In this section we investigate the behavior of the pressure near the infinity. Let $T = \mathbb{R}^1 \times (-1, 1)$. The domain Ω is contained in T, satisfy the conditions (Ω-1), (Ω-2) with $J = 2$, $N \geqslant 2$ and $\Gamma_0 = \partial T$.

Consider the problem

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \end{cases}$$

with the boundary condition

$$\begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on } \partial\Omega, \\ \boldsymbol{u} \to \boldsymbol{V}_{\pm} & \text{as } x_1 \to \pm\infty \text{ in } \Omega, \end{cases}$$

where $\boldsymbol{V}_{\pm} = (V_{\pm,1}, 0)$ is a Poiseuille flow in T. Let

$$\Phi_+ = \int_{-1}^{1} V_{+,1}\,\mathrm{d}x_2, \qquad \Phi_- = \int_{-1}^{1} V_{-,1}\,\mathrm{d}x_2.$$

If $\Phi_+ - \Phi_- \neq 0$ and $\int_{\partial\Omega} \boldsymbol{\beta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \Phi_1 - \Phi_2 = 0$, the condition (OC) holds but we do not know if a solution to this problem exists or not (Remark 4.1). So, we examine the case $\boldsymbol{V}_+ = \boldsymbol{V}_- = \boldsymbol{V}$, that is, the stationary Navier–Stokes flow in Ω,

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

with the boundary condition

$$\begin{cases} \boldsymbol{u} = \boldsymbol{\beta} & \text{on } \partial\Omega, \\ \boldsymbol{u} \to \boldsymbol{V} & \text{as } x_1 \to \pm\infty \text{ in } \Omega. \end{cases} \tag{5.2}$$

Here $\boldsymbol{u}$ is the velocity of the fluid, p the pressure, $\boldsymbol{f}$ the external force, $\boldsymbol{\beta}$ the given function defined on $\partial\Omega$ satisfying

$$\text{(GOC)} \qquad \int_{\partial\Omega} \boldsymbol{\beta}\cdot\boldsymbol{n}\,\mathrm{d}\sigma = 0,$$

where $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$, $\boldsymbol{V}$ the Poiseuille flow in T, P associated pressure.

$$\boldsymbol{V} = \frac{3\mu}{4}\big(1 - x_2^2, 0\big), \qquad P = \frac{3\nu\mu}{2}x_1.$$

This is a special case of the channel with 2 outlets($J = 2$). The results stated in this section are essentially the same as [12]. The difference is $\boldsymbol{\beta}$ might be none zero on $\partial\Omega$.

Let $\{\boldsymbol{u}, p\}$ be a smooth solution. Under some decay condition on $\boldsymbol{f}$, we can show $q = p - P$ tends to some constant $q_\pm$ as $x_1 \to \pm\infty$ in Ω. In general we cannot expect $q_+ = q_-$. Before stating our results, we show some examples. Consider the following problem.

$$\begin{cases} -\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } T, \\ \mathbf{div}\,\boldsymbol{u} = 0 & \text{in } T, \end{cases}$$

$$\begin{cases} \boldsymbol{u} = 0 & \text{on } \partial T, \\ \boldsymbol{u} \to \boldsymbol{V} & \text{as } x_1 \to \pm\infty \text{ in } T. \end{cases}$$

EXAMPLE 5.1. Let $\boldsymbol{f}(x) = a((\cosh x_1)^{-2}, 0)$ where a is non-zero constant. Let

$$\boldsymbol{w} = (0,0), \qquad q(x_1, x_2) = a\tanh x_1.$$

Then, $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{V}$ and $p = q + P$ is a solution. But,

$$\lim_{x_1\to-\infty} q = -a, \qquad \lim_{x_1\to\infty} q = a.$$

Therefore, $q_+ \neq q_-$.

EXAMPLE 5.2. Let $\boldsymbol{f} = (g(x_1), 0)$ where $g \in C_0^\infty(\mathbb{R})$, and $\int_{-\infty}^{\infty} g(t)\,\mathrm{d}t \neq 0$. Let

$$\boldsymbol{w} = (0,0), \qquad q(x_1, x_2) = \int_{-\infty}^{x_1} g(t)\,\mathrm{d}t.$$

Then, $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{V}$ and $p = q + P$ is a solution of the above equations. But,

$$\lim_{x_1\to-\infty} q = 0, \qquad \lim_{x_1\to\infty} q = \int_{-\infty}^{\infty} g(t)\,\mathrm{d}t \neq 0.$$

Therefore, $q_+ \neq q_-$.

Now we state our results. Suppose that every Γ_i $(1 \leqslant i \leqslant N)$ is symmetric with respect to the x_1-axis and intersects the x_1-axis, that $\boldsymbol{f} \in \mathbb{L}^2(\Omega) \cap \mathbb{C}^\infty(\overline{\Omega})$ is symmetric and satisfies

$$\|\boldsymbol{f}\|_{\Omega^{t,\infty}} \leqslant C\mathrm{e}^{-\lambda_0 t}, \quad \|\boldsymbol{f}\|_{\Omega^{-\infty,-t}} \leqslant C\mathrm{e}^{-\lambda_0 t} \quad \text{for } t > 0$$

for some positive constants C and λ_0, that $\boldsymbol{\beta} = (\beta_1, \beta_2)$ is symmetric, has a compact support, satisfies (GOC) and

$$\int_{x_2=1} \beta_2\,\mathrm{d}x_1 = \int_{x_2=-1} \beta_2\,\mathrm{d}x_1 = 0.$$

Let $\nu - \sigma^s > 0$. Then, Theorem 4.2 assure the existence of solution to (5.1) (5.2), and by Theorem 4.3 the solution is regular. Let $\{\boldsymbol{u}, p\}$ be the solution.

THEOREM 5.1. *There exist constants q_+ and q_- such that*

$$\begin{cases} \|p - P - q_+\|_{\Sigma(t)} \to 0 & (t \to \infty), \\ \|p - P - q_-\|_{\Sigma(t)} \to 0 & (t \to -\infty) \end{cases} \tag{5.3}$$

holds. Moreover the following relation holds.

$$\sum_{i=1}^{N} \int_{\Gamma_i} \left(-\nu \frac{\partial u_1}{\partial n} V_1 + p \boldsymbol{V} \cdot \boldsymbol{n} \right) + \left(\nu \frac{\partial V_1}{\partial n} \beta_1 - P \boldsymbol{\beta} \cdot \boldsymbol{n} \right) \mathrm{d}\sigma$$
$$+ \int_{\Omega} (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{V} \, \mathrm{d}x + (q_+ - q_-) \int_{-1}^{1} V_1 \, \mathrm{d}x_2 = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{V} \, \mathrm{d}x. \tag{5.4}$$

PROOF OF THEOREM 5.1. Firstly, we show the existence of q_+. Put

$$\bar{q}(s) = \frac{1}{2} \int_{-1}^{1} q(s, x_2) \, \mathrm{d}x_2.$$

Let $1 \ll t < s$. Choose an integer ℓ such that $t < s < t + 2\ell$ holds.

$$\begin{aligned} \left| \bar{q}(s) - \bar{q}(t) \right| &= \frac{1}{2} \left| \int_{-1}^{1} \int_{t}^{s} q_{x_1}(x_1, x_2) \, \mathrm{d}x_1 \, \mathrm{d}x_2 \right| \leqslant \frac{1}{2} \int_{-1}^{1} \int_{t}^{s} \left| q_{x_1}(x_1, x_2) \right| \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &\leqslant \frac{1}{2} \int_{-1}^{1} \int_{t}^{t+2\ell} \left| q_{x_1}(x_1, x_2) \right| \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= \sum_{j=1}^{\ell} \frac{1}{2} \int_{-1}^{1} \int_{t+2j-2}^{t+2j} \left| q_{x_1}(x_1, x_2) \right| \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &\leqslant \sum_{j=1}^{\ell} \| \nabla q \|_{\Omega^{t+2j-2, t+2j}}. \end{aligned}$$

Using Theorem 4.4, we have

$$\| \nabla q \|_{\Omega^{t+2j-2, t+2j}} \leqslant C_0 \mathrm{e}^{-\lambda_0 (t+2j-2)} = C_0 \mathrm{e}^{-\lambda_0 t} \mathrm{e}^{-2\lambda_0 (j-1)}$$

and

$$\left| \bar{q}(s) - \bar{q}(t) \right| \leqslant C_0 \mathrm{e}^{-\lambda_0 t} \sum_{j=1}^{\ell} \mathrm{e}^{-2\lambda_0 (j-1)} \leqslant C_0 \frac{1}{1 - \mathrm{e}^{-2\lambda_0}} \mathrm{e}^{-\lambda_0 t} \to 0 \quad (t \to \infty).$$

Therefore $\lim_{t\to\infty}\bar{q}(t)$ exists. We define $q_+ = \lim_{t\to\infty}\bar{q}(t)$. q_- is defined similarly.

Next, we derive the relation (5.4). We choose $t < s$ such that

$$\bigcup_{i=1}^{N}\Gamma_i \subset \big\{(x_1,x_2)\in T \mid t < x_1 < s\big\}.$$

Integrate $(-\nu\Delta\boldsymbol{u}+\nabla p)\cdot\boldsymbol{V}$ over $\Omega^{t,s}$. After integration by parts, we obtain

$$\begin{aligned}\int_{\Omega^{t,s}}(-\nu\Delta\boldsymbol{u}+\nabla p)\cdot\boldsymbol{V}\,\mathrm{d}x &= \int_{\Sigma(s)}\left(-\nu\frac{\partial u_1}{\partial x_1}V_1 + pV_1 - u_1P\right)\mathrm{d}x_2\\ &\quad -\int_{\Sigma(t)}\left(-\nu\frac{\partial u_1}{\partial x_1}V_1 + pV_1 - u_1P\right)\mathrm{d}x_2\\ &\quad +\sum_{i=1}^{N}\int_{\Gamma_i}\left(-\nu\frac{\partial\boldsymbol{u}}{\partial n}+p\boldsymbol{n}\right)\cdot\boldsymbol{V}\,\mathrm{d}\sigma\\ &\quad -\sum_{i=1}^{N}\int_{\Gamma_i}\left(-\nu\beta_1\frac{\partial V_1}{\partial n}+\boldsymbol{\beta}\cdot\boldsymbol{n}P\right)\mathrm{d}\sigma. \qquad (5.5)\end{aligned}$$

Since $\int_{\Sigma(s)} w_1(s,x_2)\,\mathrm{d}x_2 = 0$ for all s (Lemma 2.4), we have

$$\int_{\Sigma(s)} w_1(s,x_2)P(s,x_2)\,\mathrm{d}x_2 = \frac{3\mu\nu}{2}s\int_{-1}^{1} w_1(s,x_2)\,\mathrm{d}x_2 = 0.$$

For a sufficiently large s, $\boldsymbol{u} = \boldsymbol{w}+\boldsymbol{V}$ and $p = q+P$ on $\Sigma(s)$. In view of the fact that $q(s,x_2)-q_+\to 0$ as $s\to\infty$ in Ω, that V_1 is bounded in Ω, we have

$$\begin{aligned}\int_{\Sigma(s)}\left(-\nu\frac{\partial u_1}{\partial x_1}V_1 + pV_1 - u_1P\right)\mathrm{d}x_2 &= \int_{\Sigma(s)}\left(-\nu\frac{\partial w_1}{\partial x_1}V_1 + qV_1 - w_1P\right)\mathrm{d}x_2\\ &= \int_{\Sigma(s)}\left(-\nu\frac{\partial w_1}{\partial x_1}V_1 + qV_1\right)\mathrm{d}x_2 \to q_+\int_{-1}^{1}V_1\,\mathrm{d}x_2 \quad (s\to\infty \text{ in } \Omega).\end{aligned}$$

Similarly

$$\int_{\Sigma(t)}\left(-\nu\frac{\partial u_1}{\partial x_1}V_1 + pV_1 - u_1P\right)\mathrm{d}x_2 \to q_-\int_{-1}^{1}V_1\,\mathrm{d}x_2 \quad (t\to-\infty \text{ in } \Omega).$$

Letting $t\to-\infty$, $s\to\infty$ in (5.5), we obtain

$$\int_\Omega (-\nu\Delta \boldsymbol{u} + \nabla p)\cdot \boldsymbol{V}\,\mathrm{d}x$$
$$= \sum_{i=1}^{N}\int_{\Gamma_i}\left(-\nu\frac{\partial u_1}{\partial n}V_1 + p\boldsymbol{V}\cdot\boldsymbol{n}\right) + \left(\nu\frac{\partial V_1}{\partial n}\beta_1 - P\boldsymbol{\beta}\cdot\boldsymbol{n}\right)\mathrm{d}\sigma$$
$$+ (q_+ - q_-)\int_{-1}^{1} V_1\,\mathrm{d}x_2$$

and (5.4) follows. □

6. A characterization of Poiseuille flow

In this section, we give a proof of Theorem A_J announced in Section 3.3, that is, J-outlets version of Theorem A.

THEOREM A_J. *Put*

$$(\mathrm{G}_J)\quad G_J(n) := \sup_{\boldsymbol{w}\in V^S(\Omega)} \frac{\sum_{j=1}^{J}((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_{j,n}^2\boldsymbol{W})_{\Omega_j}}{\|\nabla\boldsymbol{w}\|_\Omega^2}.$$

Then, it holds

$$\lim_{n\to\infty} G_J(n) = \max(\sigma_1, \sigma_2, \ldots, \sigma_J),$$

where σ_j should be replaced by σ_j^S if T_j is self-symmetric.

OUTLINE OF THE PROOF. For $1 \leqslant j \leqslant J$, we put

$$G_J^j(n) := \sup_{\boldsymbol{w}\in V^S(\Omega)} \frac{((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta_{j,n}^2\boldsymbol{W})_{\Omega_j}}{\|\nabla\boldsymbol{w}\|_{\Omega_j}^2}.$$

We have only to show the following three inequalities.

$$\max(\sigma_1, \sigma_2, \ldots, \sigma_J) \leqslant G_J(n), \tag{6.1}$$
$$G_J(n) \leqslant \max\bigl(G_J^1(n), G_J^2(n), \ldots, G_J^J(n)\bigr), \tag{6.2}$$
$$G_J^j(n) \leqslant \sigma_j + \mathrm{O}\bigl(n^{-1}\bigr) \quad (1 \leqslant j \leqslant J). \tag{6.3}$$

Proof of (6.1) (i) The case where T_1 is self-symmetric.

Let $\boldsymbol{\varphi}$ be an arbitrary element in $\mathbb{C}_{0,\sigma}^{\infty,S}(T_1)$. By translation in the direction of the x_1-axis, the set $\mathbb{C}_{0,\sigma}^{\infty,S}(T_1)$ is invariant. Put $\boldsymbol{v}(x_1, x_2) = \boldsymbol{\varphi}(x_1 - k, x_2)$. Then $\boldsymbol{v} \in \mathbb{C}_{0,\sigma}^{\infty,S}(T_1)$. If

we choose k sufficiently large, we can suppose the support of $\boldsymbol{v}$ is contained in Ω_1 and $\theta_{1,n}(x) \equiv 1$ on the support of $\boldsymbol{v}$. Then,

$$\frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_1)_{T_1}}{\|\nabla\boldsymbol{\varphi}\|^2_{T_1}} = \frac{((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{V}_1)_{T_1}}{\|\nabla\boldsymbol{v}\|^2_{T_1}} = \frac{((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \theta^2_{1,n}\boldsymbol{W})_{\Omega_1}}{\|\nabla\boldsymbol{v}\|^2_{\Omega_1}}$$
$$= \frac{\sum_j((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \theta^2_{j,n}\boldsymbol{W})_{\Omega_j}}{\|\nabla\boldsymbol{v}\|^2_{\Omega}} \leqslant G_J(n).$$

Taking the supremum with respect to $\boldsymbol{\varphi} \in V^S(T_1)$, we obtain

$$\sigma_1^S \leqslant G_J(n).$$

(ii) The case where T_1 and T_2 are symmetric with respect to the x_1-axis.

We remark Ω_1 and Ω_2 are symmetric and $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are symmetric. It is clear that $\sigma_1 = \sigma_2$ holds true. Let $\forall\boldsymbol{\varphi} \in \mathbb{C}^\infty_{0,\sigma}(T_1)$. By the parallel translation in the direction of the axis of T_1, the set $\mathbb{C}^\infty_{0,\sigma}(T_1)$ is invariant. Let $\boldsymbol{v}$ be such translation of $\boldsymbol{\varphi}$. We can assume that $\boldsymbol{v}$ belongs to $\mathbb{C}^\infty_{0,\sigma}(\Omega_1)$, the support of $\boldsymbol{v}$ is contained in Ω_1 and $\theta_{1,n}(x) = 1$ on the support of $\boldsymbol{v}$. Put

$$\boldsymbol{w}(x_1, x_2) = \begin{cases} (v_1(x_1, x_2), v_2(x_1, x_2)) & \text{in } \Omega_1, \\ (v_1(x_1, -x_2), -v_2(x_1, -x_2)) & \text{in } \Omega_2, \\ 0 & \text{in } \Omega \setminus (\Omega_1 \cup \Omega_2). \end{cases}$$

Then it is easy to verify $\boldsymbol{w} \in \mathbb{C}^{\infty,S}_{0,\sigma}(\Omega)$. Straightforward calculation shows

$$\frac{((\boldsymbol{\varphi}\cdot\nabla)\boldsymbol{\varphi}, \boldsymbol{V}_1)_{T_1}}{\|\nabla\boldsymbol{\varphi}\|^2_{T_1}} = \frac{((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta^2_{1,n}\boldsymbol{W})_{\Omega_1}}{\|\nabla\boldsymbol{w}\|^2_{\Omega_1}}$$
$$= \frac{((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta^2_{1,n}\boldsymbol{W})_{\Omega_1} + ((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta^2_{2,n}\boldsymbol{W})_{\Omega_2} + 0}{\|\nabla\boldsymbol{w}\|^2_{\Omega_1} + \|\nabla\boldsymbol{w}\|^2_{\Omega_2} + 0}$$
$$= \frac{\sum_{j=1}^J((\boldsymbol{w}\cdot\nabla)\boldsymbol{w}, \theta^2_{j,n}\boldsymbol{W})_{\Omega_j}}{\|\nabla\boldsymbol{w}\|^2_{\Omega}} \leqslant G_J(n).$$

Taking the supremum with respect to $\boldsymbol{\varphi} \in V^S(T_1)$, we obtain

$$\sigma_1 \leqslant G_J(n).$$

Similarly we can show that $\sigma_j \leqslant G_J(n)$ holds for every $1 \leqslant j \leqslant J$. Therefore (6.1) is proved.

Proof of (6.2). We fix n. Since $\sigma_j > 0$, it is clear that $G_J(n)$ is also strictly positive, and there exists a sequence $\boldsymbol{w}_k \in V^S(\Omega)$ such that

$$0 < \frac{\sum_{j=1}^{J}((\boldsymbol{w}_k \cdot \nabla)\boldsymbol{w}_k, \theta_{j,n}^2 \boldsymbol{W})_{\Omega_j}}{\|\nabla \boldsymbol{w}_k\|_{\Omega}^2} \to G_J(n) \quad \text{as } k \to \infty.$$

Let $\sum_{j'}$ mean the sum with respect to j where $\|\nabla \boldsymbol{w}_k\|_{\Omega_j} \neq 0$. By straightforward calculation, we have

$$\begin{aligned}
\frac{\sum_{j'}((\boldsymbol{w}_k \cdot \nabla)\boldsymbol{w}_k, \theta_{j,n}^2 \boldsymbol{W})_{\Omega_j}}{\|\nabla \boldsymbol{w}_k\|_{\Omega}^2} &= \frac{\sum_{j'}((\boldsymbol{w}_k \cdot \nabla)\boldsymbol{w}_k, \theta_{j,n}^2 \boldsymbol{W})_{\Omega_j}}{\sum_{j'} \|\nabla \boldsymbol{w}_k\|_{\Omega_j}^2} \frac{\sum_{j'} \|\nabla \boldsymbol{w}_k\|_{\Omega_j}^2}{\|\nabla \boldsymbol{w}_k\|_{\Omega}^2} \\
&\leqslant \frac{\sum_{j'}((\boldsymbol{w}_k \cdot \nabla)\boldsymbol{w}_k, \theta_{j,n}^2 \boldsymbol{W})_{\Omega_j}}{\sum_{j'} \|\nabla \boldsymbol{w}_k\|_{\Omega_j}^2} \\
&\leqslant \max_{j'} \left\{ \frac{((\boldsymbol{w}_k \cdot \nabla)\boldsymbol{w}_k, \theta_{j,n}^2 \boldsymbol{W})_{\Omega_j}}{\|\nabla \boldsymbol{w}_k\|_{\Omega_j}^2} \right\} \\
&\leqslant \max\{G_J^1(n), G_J^2(n), \ldots, G_J^J(n)\}.
\end{aligned}$$

Letting $k \to \infty$, we obtain (6.2). Here we have used the following elementary inequality.

LEMMA 6.1. *Let $b_j > 0$ for $1 \leqslant j \leqslant J$. Then it holds*

$$\frac{\sum_{j=1}^{J} a_j}{\sum_{j=1}^{J} b_j} \leqslant \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_J}{b_J}\right).$$

Proof of (6.3). By rotation and translation, we can reduce the problem to the case $J = 1$, and show the inequality (6.3) similarly to [1].

Acknowledgement

The author wishes to express her sincere gratitude to Professor M. Chipot for his patience and encouragement.

References

[1] C.J. Amick, *Steady solutions of the Navier–Stokes equations for certain unbounded channels and pipes*, Ann. Scuola Norm. Pisa **4** (1977), 473–513.

[2] C.J. Amick, *Properties of steady Navier–Stokes solutions for certain unbounded channels and pipes*, Nonlinear Anal. TMA **2** (1978), 689–720.

[3] I. Babuška and A.K. Aziz, *Survey lectures on the mathematical foundations of the finite element method*, The Mathematical Foundation of the Finite Element Method with Applications to Partial Differential Equations, A.K. Aziz, ed., Academic Press (1972), 5–359.

[4] M.E. Bogovskii, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, Soviet Math. Dokl. **20** (1979), 1094–1098.

[5] L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Mat. Sem. Univ. Padova **31** (1961), 308–340.

[6] H. Fujita, *On the existence and regularity of the steady-state solutions of the Navier–Stokes equation*, J. Fac. Sci. Univ. Tokyo Sec. I **9** (1961), 59–102.

[7] H. Fujita, *On stationary solutions to Navier–Stokes equations in symmetric plane domains under general outflow condition*, Proceedings of International Conference on Navier–Stokes Equations, Theory and Numerical Methods, June 1997, Varenna, Italy, Pitman Research Notes in Mathematics, vol. 388, 16–30.

[8] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, Springer (1994).

[9] C.O. Horgan and L.T. Wheeler, *Spatial Decay Estimates for the Navier–Stokes Equations with Application to the Problem of Entry Flow*, SIAM J. Appl. Math. **35** (1978), 97–116.

[10] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York (1969).

[11] H. Morimoto, *A remark on the existence of 2-D steady Navier–Stokes flow in symmetric domain under general outflow condition*, MIMS report No. 042004; J. Math. Fluid Mech., to appear, published online 19 September 2006.

[12] H. Morimoto, *A remark on the pressure for the Navier–Stokes flows in 2-D straight channel with an obstacle*, Math. Methods Appl. Sci. **27** (2004), 891–906.

[13] H. Morimoto and H. Fujita, *A remark on the existence of steady Navier–Stokes flows in 2D semi-infinite channel involving the general outflow condition*, Math. Bohem.**126** (2001), 457–468.

[14] H. Morimoto and H. Fujita, *Stationary Navier–Stokes flow in 2-dimensional Y-shaped channel under general outflow condition*, Proceedings of the international Conference on the Navier–Stokes Equations: Theory and Numerical Methods, June 2000, Varenna, Italy, R. Salvi, ed., Lecture Note in Pure and Applied Mathematics, vol. 223, Marcel Decker (2002), 65–72.

[15] H. Morimoto and H. Fujita, *Stationary Navier–Stokes flow in 2-dimensional V-shaped infinite channel under general outflow condition*, Topics in Mathematical Fluid Dynamics, R. Russo, ed., Quaderni di Matematica, vol. 10 (2002), 125–135.

[16] H. Morimoto and H. Fujita, *A remark on the existence of steady Navier–Stokes flow in a certain two-dimensional infinite channel*, Tokyo J. Math. **25** (2) (2002), 307–321.

[17] P. Rabier, *Invertibility of the Poiseuille linearization for stationary 2-dim channel flow: symmetric case*, J. Math. Fluid Mech. **4** (2002), 327–350.

[18] P. Rabier, *Invertibility of the Poiseuille linearization for stationary 2-dim channel flow: nonsymmetric case*, J. Math. Fluid Mech. **4** (2002), 351–373.

[19] R. Temam, *Navier–Stokes Equations*, North-Holland, Amsterdam (1977).

This page intentionally left blank

CHAPTER 6

Maximum Principles for Elliptic Partial Differential Equations

Patrizia Pucci

Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, Perugia, Italy
E-mail: pucci@dipmat.unipg.it

James Serrin

Department of Mathematics, University of Minnesota, Minneapolis, USA
E-mail: serrin@math.umn.edu

Contents

Section 6.1. Introduction and Preliminaries 357
1.1. Introduction 357

Section 6.2. Tangency and Comparison Theorems for Elliptic Inequalities 362
2.1. The contributions of Eberhard Hopf 362
2.2. Tangency and comparison principles for quasilinear inequalities 368
2.3. Maximum and sweeping principles for quasilinear inequalities 371
2.4. Comparison theorems for divergence structure inequalities 376
2.5. Tangency theorems via Harnack's inequality 379
2.6. Uniqueness of the Dirichlet problem 382
2.7. The Boundary Point Lemma 383
2.8. Appendix: Proof of Eberhard Hopf's Maximum Principle 386
Notes 390

Section 6.3. Maximum Principles for Divergence Structure Elliptic Differential Inequalities 391
3.1. Distribution solutions 391
3.2. Maximum principles for homogeneous inequalities 393
3.3. A maximum principle for thin sets 397
3.4. A comparison theorem in $W^{1,p}(\Omega)$ 399
3.5. Comparison theorems for singular elliptic inequalities 401
3.6. Strongly degenerate operators 405
3.7. Maximum principles for non-homogeneous elliptic inequalities 409

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 4
Edited by M. Chipot
© 2007 Elsevier B.V. All rights reserved

3.8. Uniqueness of the singular Dirichlet problem 415
3.9. Maximum principles for structured inequalities 416
3.10. Appendix: Sobolev's inequality 419
Notes 420

Section 6.4. The Strong Maximum Principle and the Compact Support Principle 422
4.1. Introduction 422
4.2. A more general inequality 424
4.3. Existence theorems 425
4.4. The dead core lemma 426
4.5. Proof of the Strong Maximum Principle 429
4.6. Proof of the Compact Support Principle 431
4.7. Strong Maximum Principle: Generalized version 432
4.8. Further extensions of the Strong Maximum Principle 436
Notes 438

Section 6.5. Applications 440
5.1. Cauchy–Liouville theorems 440
5.2. Radial symmetry 444
5.3. Symmetry for overdetermined boundary value problems 451
5.4. The phenomenon of dead cores 458
5.5. The Harnack Inequality in $\mathbb{R}^2$ 470
5.6. The Strong Maximum Principle for Riemannian manifolds 478
Acknowledgement 480
References 480

Keywords: Quasilinear elliptic inequalities, Maximum principle, Comparison principle, Strong maximum principle, Compact support principle, Liouville theorem, Radial symmetry, Dead cores, Harnack inequality

MSC: primary 35J15; secondary 35J70

SECTION 6.1

Introduction and Preliminaries

1.1. Introduction

The maximum principles of Eberhard Hopf are classical and bedrock results of the theory of second order elliptic partial differential equations. They go back to the maximum principle for harmonic functions, already known to Gauss in 1839 on the basis of the mean value theorem. On the other hand, they carry forward to the maximum principles of Gilbarg and Trudinger and Serrin, and the maximum principles for singular quasilinear elliptic differential inequalities, a theory initiated particularly by Vázquez and Diaz in the 1980's, but with earlier intimations in the work of Benilan, Brezis and Crandall. The purpose of the present work is then to provide a clear explanation of the various maximum principles available for elliptic second order equations, from their beginnings in linear theory to recent work on nonlinear equations, operators and inequalities.

The first chapter concerns tangency and comparison theorems, based to begin with on the pioneering results of Eberhard Hopf. Section 2.1 includes in particular a discussion of Hopf's nonlinear contributions, which are in fact not nearly as well known as his classical linear principle. We continue with a treatment of quasilinear equations and inequalities, with linear equations of course being an important special case. We consider both non-singular and singular cases, that is, in the latter case, equations which lose ellipticity at special values of the gradient of solutions, particularly at critical points. The concern here with singular equations arises both from their growing importance in variational theory and applied mathematics, as well as their specific theoretical interest, e.g. the celebrated p-Laplace operator Δ_p.

The results of Hopf apply specifically to C^2 solutions of elliptic differential inequalities. In many cases, however, especially when the equations and inequalities in question are expressed in divergence form, as in the calculus of variations, one can expect solutions to be simply of class C^1 or even only weakly differentiable in some Sobolev space. In such cases, the solutions must naturally be taken in a distribution sense. Correspondingly, the study of maximum principles requires new techniques, alternative to Hopf's approach. These methods, necessarily integral in nature, originally arose from the work of a number of mathematicians, going back as far as Tonelli, Leray and Morrey in the years 1928–1935.

Sections 2.4 and 2.5 are devoted specifically to C^1 solutions of divergence structure inequalities, allowing both singular and non-singular operators. Theorem 2.4.1 and its attendant corollaries is of special interest for its simplicity and elegance; see also the corresponding uniqueness result for the singular Dirichlet problem (2.6.2). We note also the Tangency Theorem 2.5.2 obtained from the weak Harnack inequality.

Section 6.3 continues the study of divergence structure inequalities, but for more general operators for which the methods of Section 6.2 are inadequate. The principal results are

(i) the maximum principles of Section 3.2 for homogeneous inequalities; (ii) the "*thin set*" maximum principle in Section 3.3; (iii) the generalization of Theorem 2.4.1 given in Theorem 3.4.1 (applying to solutions in the Sobolev space $W^{1,p}$); (iv) Theorem 3.5.1 for weakly singular inequalities; and (v) the interesting Theorems 3.6.1 and 3.6.5 for strongly singular inequalities. We emphasize as well the Maximum Principles 3.7.2 and 3.7.4, and the series of uniqueness theorems in Section 3.8. These results, which extend well-known theorems of Gilbarg and Trudinger for the Dirichlet problem, see e.g. [40], Theorems 3.8.1 and 3.8.4, appear to be new in the generality given.

Further maximum principles for the complete quasilinear differential inequality

$$\operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du) \geqslant 0$$

can be obtained from appropriate test function arguments together with the well-known Moser iteration technique. We state the most important of these results in Section 3.9.

Chapter 6.4 is concerned with the Strong Maximum Principle and the Compact Support Principle for singular quasilinear differential inequalities

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + B(x, u, Du) \leqslant 0 \tag{1.1.1}$$

in a domain (connected open set) Ω in $\mathbb{R}^n$, under mild conditions of ellipticity, which allow both singular and degenerate behavior of the function A at $s = 0$, that is at critical points of u. The operator $\operatorname{div}\{A(|Du|)Du\}$ can be called the *A-Laplace operator*, to place it in the context of well-known elliptic theory. For the Laplace operator, that is when (1.1.1) takes the form of a Poisson inequality

$$\Delta u + B(x, u, Du) \leqslant 0,$$

we have $A(s) \equiv 1$. Similarly, for the degenerate p-Laplace operator $\operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, here denoted by Δ_p, we have $A(s) = s^{p-2}$, while for the mean curvature operator one has $A(s) = 1/\sqrt{1+s^2}$.

The final chapter includes recent applications of the maximum principle to Liouville theorems and dead core problems, and to differential inequalities on Riemannian manifolds. In Section 5.2 we also give various radial symmetry theorems for the semilinear Laplace–Poisson equation $\Delta u + f(u) = 0$ and the quasilinear divergence structure equation

$$\operatorname{div}\big\{A\big(u, |Du|\big)Du\big\} + B\big(u, |Du|\big) = 0$$

under mild Lipschitz continuity or monotonicity conditions on the functions f and B. The more delicate symmetry question for overdetermined boundary value problems is treated in Section 5.3. In Section 5.5 we use the maximum principle to obtain a Harnack inequality for the equation

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + B(x, u, Du) = 0$$

in $\mathbb{R}^2$. Although special in many respects, the result nevertheless covers the minimal surface equation and also is of interest for the unusual proof procedure which in essence depends on the Jordan curve theorem.

There are of course further applications of general interest, for example Phragmén–Lindelőf and other singularity theorems, and various uses of the maximum principle in the existence theory for nonlinear Dirichlet problems; the reader can be referred particularly to [35,40,66] and [98], and also to recent work of Marcus and Véron. The maximum principle can also be applied to obtain gradient bounds for solutions of elliptic equations, using "barrier methods" or the application of "P functions". For barrier methods, one can consult [40] Chapter 14, and for P-function approach the monograph of Sperb [90].

It is beyond the scope of this work to consider fully nonlinear equations in any detail. To do this would require the development and presentation of the techniques of Krylov and Safonov for Harnack inequalities for non-divergence second order linear equations, as well as the concept of viscosity solutions. This would altogether change our focus and require a lengthy treatment of its own to cover the literature which has grown up in this direction. The reader however can be referred to the survey works [46,47] and [16].

To conclude the introduction it is worth noting some further examples of second order elliptic equations of physical and geometric interest.

1. The equation of prescribed mean curvature:

$$\left(1+|Du|^2\right)\Delta u - \partial_{x_i} u\,\partial_{x_j} u\,\partial^2_{x_i x_j} u = n\mathcal{H}(x)\left(1+|Du|^2\right)^{3/2}, \tag{1.1.2}$$

or, equally, in divergence form,

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = n\mathcal{H}(x), \tag{1.1.3}$$

where $\mathcal{H}$ is the mean curvature of the non-parametric surface $x_{n+1} = u(x)$ in the $(n+1)$-dimensional (x, x_{n+1})-space. This equation arises naturally by considering the isoperimetric problem of least surface area bounding a given volume; it had already been derived by Lagrange in 1760. Of additional interest is the case when $\mathcal{H}$ is specified as a function of (x, u, Du). Some special examples of this type occur below.

2. The surface of a fluid under the combined action of gravity and surface tension (*capillary surface*)

$$\left(1+|Du|^2\right)\Delta u - \partial_{x_i} u\,\partial_{x_j} u\,\partial^2_{x_i x_j} u = \kappa u\left(1+|Du|^2\right)^{3/2},$$

where κ is an appropriate physical constant. In the physically central case of two dimensions this equation arises from balancing forces of tension (proportional to the mean curvature of the capillary surface) with the weight of the fluid supported. The constant κ is positive or negative depending on whether the surface in question is an upper or lower boundary of the fluid.

3. Central projection. Let S^n be the sphere of $\mathbb{R}^{n+1}$, which can be mapped conformally onto the Euclidean tangent space $\mathbb{R}^n$ at the South Pole by means of stereographic projection

from the North Pole. In this projection the volume element is $dm = dx/(1+|x|^2)^n$ and the gradient ∇ on S^n is expressed as $(1+|x|^2)D$, where D stands for the Euclidean gradient in $\mathbb{R}^n$ and x denotes a Euclidean coordinate centered at the South Pole.
As a particular example, the p-Dirichlet norm on S^n, $p > 1$, is then minimized by functions u on S^n which satisfy

$$\operatorname{div}_{S^n}\left(|\nabla u|^{p-2}\nabla u\right) = 0.$$

Reverting to the stereographic variables x, this takes the form

$$\rho^{-n}\operatorname{div}\left(\rho^{n-p}|Du|^{p-2}Du\right) = 0, \quad \rho(x) = 1/\left(1+|x|^2\right),$$

this being a particular example where the vector $\boldsymbol{A}$ depends on both x and Du. Of course, general variational integrals on S^n can be treated in the same way.

4. Subsonic gas dynamics. The velocity potential φ satisfies

$$\operatorname{div}(\varrho D\varphi) = 0,$$

where the velocity $D\varphi$ and the density ϱ are related through Bernoulli's law. For the important case of an ideal gas the relation is

$$\frac{1}{2}|D\varphi|^2 + \frac{c^2}{\gamma-1} = \text{Const.}, \quad c = \text{sound spead} \sim \varrho^{(\gamma-1)/2},$$

where $\gamma > 1$ is the ratio of specific heats in the gas.

5. The general equation of radiative cooling

$$\operatorname{div}\left(\kappa|Du|^{p-2}Du\right) = \sigma u^4, \quad p > 1,$$

where κ is the coefficient of heat conduction, depending on x and possibly also on u, while σ is the radiation, assumed to be constant. Replacing the right-hand side by various functions $f = f(x,u)$ yields further examples of physical interest.

6. The Euler–Lagrange equation. For the variational problem

$$\delta\int_\Omega \mathcal{G}(x,u,Du)\,dx = 0,$$

with $\mathcal{G} = \mathcal{G}(x,z,\boldsymbol{\xi})$ being of class C^3, this takes the form

$$\operatorname{div}\partial_{\boldsymbol{\xi}}\mathcal{G}(x,u,Du) = \partial_z\mathcal{G}(x,u,Du).$$

Ellipticity is equivalent to strong convexity of $\mathcal{G}$ with respect to $\boldsymbol{\xi}$, namely the figuratrix surface $x_{n+1} = \mathcal{G}(x,z,\boldsymbol{\xi})$ should have positive Gaussian curvature for fixed (x,z).

If $\mathcal{G}$ is jointly convex in z and $\boldsymbol{\xi}$ and satisfies mild regularity conditions then the solution of the Euler–Lagrange equation provides a minimizing function for the variational problem.

The case where $\mathcal{G}$ depends only on z and $|\boldsymbol{\xi}|$ is particularly to be noted since the corresponding problem is invariant under rotations of the underlying space.

7. The 2-dimensional Monge–Ampère equation

$$e\big(\partial^2_{xx}u\partial^2_{yy}u\big) + a\partial^2_{xx}u + 2b\partial^2_{xy}u + c\partial^2_{yy}u = d$$

is elliptic if and only if $ac - b^2 + ed > 0$. Here a, b, c, d, e depend on (x, y), or more generally on $(x, y, u, \boldsymbol{\xi})$, $\boldsymbol{\xi} = (\xi_1, \xi_2)$.

8. Calabi's equation

$$\det D^2u = f(x).$$

Ellipticity demands that the surface $x_{n+1} = u(x)$ be convex.

NOTATION. Throughout, we shall let $x = (x_1, \ldots, x_n)$ denote points of $\mathbb{R}^n$, $n \geqslant 1$, and will denote the solution variable by $u = u(x)$. We put as before $\partial u/\partial x_i = \partial_{x_i}u$, $\partial^2 u/\partial x_i \partial x_j = \partial^2_{x_i x_j}u$ when the solutions are assumed to be classical, that is of class C^2 in any domain of interest. We shall also write $Du = (\partial_{x_1}u, \ldots, \partial_{x_n}u)$ for the gradient vector of u, and $D^2u = [\partial^2_{x_i x_j}u]$ for the Hessian matrix of u. It is understood that repeated subscripts j, k etc. are summed over the appropriate range indicated by context.

A domain Ω in $\mathbb{R}^n$ is always understood to be a connected open set in $\mathbb{R}^n$. We denote the boundary of Ω by $\partial\Omega$, while by $\Omega' \Subset \Omega$ we mean that Ω' is a subdomain with compact closure in Ω. The notation $\langle \cdot, \cdot \rangle$ is always reserved for the inner product in the (vector) space $\mathbb{R}^n$.

We assume the reader to have a standard background in real analysis, but without need for linear operator theory. A useful assortment of classical results and techniques, including elementary Sobolev spaces, can be found in [40], Sections 7.1–7.7.

The authors are particularly grateful to Michel Chipot for a number of suggested improvements in this article.

SECTION 6.2

Tangency and Comparison Theorems for Elliptic Inequalities

2.1. The contributions of Eberhard Hopf

We begin with the classical maximum principle due to E. Hopf [42], together with an extended commentary and discussion of Hopf's original paper by J. Serrin [85].

The maximum principle for harmonic and subharmonic functions was known to Gauss on the basis of the mean value theorem (1839); an extension to elliptic inequalities however remained open until the twentieth century. Bernstein (1904), Picard (1905), Lichtenstein (1912, 1924) then obtained various results by difficult means, as well as use of regularity conditions for the coefficients of the highest order terms. It was Hopf's genius to see that a "*gänzlich elementares Begründen*" could be given. The comparison technique he invented for this purpose is essentially so transparent that it has generated an enormous number of important applications in many further directions.

Here is Hopf's theorem in its main form:

HOPF'S MAXIMUM PRINCIPLE. *Let $u = u(x)$, $x = (x_1, \dots, x_n)$, be a C^2 function which satisfies the differential inequality*

$$Lu \equiv \sum_{i,j} a_{ij}(x)\partial^2_{x_i x_j} u + \sum_i b_i(x)\partial_{x_i} u \geqslant 0$$

in a domain Ω. Suppose the (symmetric) matrix $[a_{ij}] = [a_{ij}(x)]$ is locally uniformly positive definite in Ω (that is, for any given compact subset Ω' of Ω, the quadratic form

$$\sum_{i,j} a_{ij}(x)\eta_i\eta_j$$

is positive and uniformly bounded from 0 for all x in Ω' and all vectors $\boldsymbol{\eta}$ in $\mathbb{R}^n$ with $|\boldsymbol{\eta}| = 1$), and the coefficients a_{ij}, $b_i = b_i(x)$ are locally bounded in Ω.

If u takes a maximum value M in Ω, then $u \equiv M$ in Ω.

Hopf's proof (Section I of [42]), now a classic of the subject, is reproduced in the monographs [66,40,35] and in many other texts as well, particularly the second volume of [21]. We give a proof in the Appendix of this chapter, Section 2.8.

The hypothesis that u is twice differentiable is essential for the theorem, though not always strictly noted in presentations of the result.

In Section II of [42] Hopf notices two important corollaries (Sätze 2, 3) dealing with the differential inequality $Lu + c(x)u \geqslant 0$. *First, for the case* $c = c(x) \leqslant 0$ *and a positive maximum, and second, when there is an extremum* $M = 0$ *irrespective of the sign of* c. The latter possibility is mentioned only in passing in [40], and not at all in Courant and Hilbert [21]. The formal statement of these corollaries is as follows.

THEOREM 2.1.1. *Let* u *be a* C^2 *function satisfying the differential inequality*

$$Lu + c(x)u \geqslant 0 \ (\leqslant 0) \tag{2.1.1}$$

in a domain Ω, *where the coefficients of* L *satisfy the previous conditions, and* $c = c(x) \leqslant 0$ *in* Ω. *If* u *takes a positive maximum* (*negative minimum*) *value* M *in* Ω, *then* $u \equiv M$.

The result is easy to prove. That is, near a positive maximum M of u we would have

$$Lu \geqslant -c(x)u \geqslant 0.$$

Hopf's main theorem then yields $u \equiv M$ near the maximum point; in turn $u \equiv M$ in all Ω (the set $\{x \in \Omega\colon\ u = M\}$ is non-empty and both open and closed in the connected set Ω).

Hopf's second result is

THEOREM 2.1.2. *Let the hypotheses of Theorem* 2.1.1 *hold, except that one now assumes alternatively that the function* c *is locally bounded below in* Ω. *If* u *takes on a vanishing maximum* (*minimum*) *value* $M = 0$ *in* Ω, *then* $u \equiv 0$.

PROOF (Hopf). Let $u \leqslant 0$ in Ω and define $v(x) = \mathrm{e}^{-\alpha x_1}u(x)$, $x \in \Omega$, $\alpha > 0$. Clearly $v \in C^2(\Omega)$, is non-positive and satisfies the differential inequality

$$Lv + \alpha \sum_i \tilde{b}_i \partial_{x_i} v \geqslant -\tilde{c}\mathrm{e}^{-\alpha x_1}u, \quad \tilde{c} = c + \alpha^2 a_{11} + \alpha b_1,$$

where $\tilde{b}_i(x) = 2a_{1i}$. In any domain Ω' with compact closure in Ω we have

$$c(x) \geqslant -\text{const.}, \quad \left|b_1(x)\right| \leqslant \text{const.}, \quad a_{11}(x) \geqslant \text{const.} > 0.$$

Therefore we can choose α sufficiently large so that $\tilde{c}(x)$ is positive in Ω'. In turn $Lv + \alpha \sum_i \tilde{b}_i \partial_{x_i} v \geqslant 0$ in Ω'. Let $y \in \Omega$ be such that $u(y) = 0$ and take Ω' containing y. Then $v(y) = 0$ and by Hopf's main theorem we get $u \equiv v \equiv 0$ in Ω', from which it follows at once that $u \equiv 0$ in the entire Ω. The case $u \geqslant 0$ in Ω is treated in the same way. □

It may be remarked that earlier statements of Theorems 2.1.1 and 2.1.2 have usually imposed stronger boundedness conditions on the function $c(x)$ than those required here.

As is customary, the term *strong maximum principle* will be used here to denote the main results of Hopf stated above, as well as related results, e.g., Theorem 2.1.1. On the other hand, the term *maximum principle* (in contrast to strong maximum principle) is reserved

to denote results in which a bound for a solution u of an elliptic equation, or inequality, is given in terms of an a priori bound for u on the boundary of its domain of definition. This terminology follows, e.g., Gilbarg and Trudinger [40] and Fraenkel [35].

To continue with the discussion of Hopf's work, in Section II of [42] Hopf observes that one can allow the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ to depend on the solution u itself, provided that when they are evaluated along the solution the resulting functions $\tilde{a}_{ij}(x)$, $\tilde{b}_i(x)$, $\tilde{c}(x)$ satisfy the conditions of the main theorems. This allows him to deal explicitly with nonlinear as well as linear equations.

The real depth of Hopf's nonlinear analysis shows up only in Section III, where he considered the *fully nonlinear equation of second order*

$$\mathcal{F}\big(x, u, Du, D^2u\big) = 0, \quad x \in \Omega.$$

The presentation is, however, seriously obscured by the restriction to exact equations, as well as to the case where one of the solutions in question is assumed to vanish identically ("*engere Voraussetzungen*" according to Hopf). Accordingly we shall restate the results in slightly greater generality and in more usual notation. Hopf's first result is a beautiful *tangency principle*, essentially Satz 3′ of [42].

THEOREM 2.1.3 (Tangency Principle). *Let u, v be $C^2(\Omega)$ solutions of the nonlinear differential inequality*

$$\mathcal{F}\big(x, u, Du, D^2u\big) \geqslant \mathcal{F}\big(x, v, Dv, D^2v\big),$$

where the function $\mathcal{F} = \mathcal{F}(x, z, \boldsymbol{\xi}, \boldsymbol{s})$ is continuously differentiable in the variables $z, \boldsymbol{\xi}, \boldsymbol{s}$, that is, the derivatives $\partial_z\mathcal{F}$, $\partial_{\boldsymbol{\xi}}\mathcal{F}$, $\partial_{\boldsymbol{s}}\mathcal{F}$ exist and are continuous functions of $(x, z, \boldsymbol{\xi}, \boldsymbol{s}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$. Suppose also that the matrix $Q = [Q_{ij}]$ given by

$$Q_{ij} \equiv \partial_{\boldsymbol{s}}\mathcal{F}\big(x, u, Du, \theta D^2u + (1-\theta)D^2v\big),$$

is positive definite in $x \in \Omega$ and all $\theta \in [0, 1]$.

If $u \leqslant v$ in Ω and $u = v$ at some point x_0 in Ω, then $u \equiv v$ in Ω.

The terms u, Du in Q can be replaced by v, Dv.

PROOF. Essentially following Hopf's proof of Satz 3′ of [42], we write

$$\begin{aligned}
0 &\geqslant \mathcal{F}\big(x, v, Dv, D^2v\big) - \mathcal{F}\big(x, u, Du, D^2u\big) \\
&= \mathcal{F}\big(x, u, Du, D^2v\big) - \mathcal{F}\big(x, u, Du, D^2u\big) + \mathcal{F}\big(x, u, Dv, D^2v\big) \\
&\quad - \mathcal{F}\big(x, u, Du, D^2v\big) + \mathcal{F}\big(x, v, Dv, D^2v\big) - \mathcal{F}\big(x, u, Dv, D^2v\big) \\
&= \sum a_{ij}\partial^2_{x_ix_j}(v-u) + \sum b_i\partial_{x_i}(v-u) + c(v-u) \\
&\equiv L(v-u) + c(v-u),
\end{aligned}$$

where, for some values $\theta, \theta_1, \theta_2 \in [0, 1]$, depending on x, we have by the mean value theorem

$$a_{ij} = \partial_s \mathcal{F}\big(x, u, Du, \theta D^2 v + (1-\theta) D^2 u\big) = Q_{ij}|_{\theta=\theta(x)}$$
$$b_i = \partial_{\xi_i} \mathcal{F}\big(x, u, \theta_1 Dv + (1-\theta_1) Du, D^2 v\big)\big|_{\theta_1=\theta_1(x)}$$
$$c = \partial_z \mathcal{F}\big(x, \theta_2 v + (1-\theta_2) u, Dv, D^2 v\big)\big|_{\theta_2=\theta_2(x)}.$$

Since Q_{ij} is continuous for $x \in \Omega$ and $\theta \in [0, 1]$, the principal condition on Q_{ij} shows that in fact it is *uniformly* positive definite for $x \in \Omega'$ and $\theta \in [0, 1]$, when Ω' is a compact subset of Ω. Consequently the coefficient matrix $[a_{ij}]$ is locally uniformly positive definite on Ω. By the same argument it is clear that also a_{ij}, b_i, c are locally bounded in Ω. Since by assumption $v - u \geqslant 0$ and $(v-u)(x_0) = 0$, it now follows from Theorem 2.1.2 that $v \equiv u$ in Ω.

To obtain the final conclusion of the theorem, one proceeds in the same way, though starting from the alternative decomposition

$$\begin{aligned} 0 &\geqslant \mathcal{F}\big(x, v, Dv, D^2 v\big) - \mathcal{F}\big(x, u, Du, D^2 u\big) \\ &= \mathcal{F}\big(x, v, Dv, D^2 v\big) - \mathcal{F}\big(x, v, Dv, D^2 u\big) + \mathcal{F}\big(x, v, Dv, D^2 u\big) \\ &\quad - \mathcal{F}\big(x, v, Du, D^2 u\big) + \mathcal{F}\big(x, v, Du, D^2 u\big) - \mathcal{F}\big(x, u, Du, D^2 u\big), \end{aligned}$$

but otherwise leaving the proof unchanged. □

Hopf's Theorems 2.1.1 and 2.1.2 are in fact tangency principles in which the second solution v is constant ($= M$).

The next result (essentially Satz 2′ of [42] in a more general context and formulation) is stated here as a *comparison result*, rather than a maximum principle, this being the underlying content of Hopf's theorem. By $u \leqslant v$ on $\partial\Omega$ we mean explicitly that for every $\delta > 0$ there is a neighborhood of $\partial\Omega$ in which $u \leqslant v + \delta$.

THEOREM 2.1.4 (Comparison Principle). *Let u, v be $C^2(\Omega)$ solutions of the nonlinear differential inequality given in Theorem* 2.1.3. *Suppose that the matrix $Q = [Q_{ij}]$ is positive definite in Ω and that for every fixed $x \in \Omega$ the function*

$$\mathcal{F}\big(x, \cdot, Dv(x), D^2 v(x)\big) : \mathbb{R} \to \mathbb{R} \tag{2.1.2}$$

is non-increasing on the semiline $[v(x), \infty)$ – but not necessarily differentiable. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.[1]

The terms u, Du in Q can be replaced by v, Dv if at the same time the terms Dv, $D^2 v$ in (2.1.2) *are replaced by Du, $D^2 u$ and the semiline $[v(x), \infty)$ is replaced by $(-\infty, u(x)]$.*

[1] In fact by Theorem 2.1.3 if $\partial_z \mathcal{F}$ is also continuously differentiable then either $u \equiv v$ in Ω or $u < v$ in Ω.

PROOF. Suppose for contradiction that the conclusion $v - u \geqslant 0$ in Ω fails.

Then there will be a subdomain Ω' of Ω in which $v - u < 0$ but is not identically constant, and in which also $v - u$ takes on a negative minimum M at a point y. As in the proof of Theorem 2.1.3, one obtains with the help of (2.1.2) that $L(v - u) \leqslant 0$ in Ω', where L has the obvious meaning. Hence by Hopf's main theorem we get $v - u \equiv M$ in Ω', a contradiction.

The final conclusion is obtained from the alternative decomposition in the proof of Theorem 2.1.3. □

Using other decompositions, one can obtain various related results, see e.g. Theorem 31 of Chapter 2 of [66].

A direct consequence of Theorem 2.1.4 is a uniqueness theorem for the Dirichlet problem for the nonlinear equation $\mathcal{F}(x, u, Du, D^2u) = 0$, a fact mentioned by Hopf in the final paragraph of [42], though not explicitly formulated by him. Since the result is important, and a precise formulation is in fact not immediate from Hopf's analysis, it is worth stating the definite result here.

THEOREM 2.1.5. *Let u and v be $C^2(\Omega)$ solutions of the nonlinear equation*

$$\mathcal{F}\big(x, u, Du, D^2u\big) = 0 \tag{2.1.3}$$

in a domain Ω, with $u = v$ on $\partial\Omega$. Suppose Q is positive definite in Ω for all $\theta \in [0, 1]$, and that $\mathcal{F}(x, \cdot, Dv(x), D^2v(x))$ is non-increasing on the entire line $\mathbb{R}$; see (2.1.2). *Then $u \equiv v$.*

This is an immediate corollary of Theorem 2.1.4, the main result being used to establish that $u \leqslant v$, and the final part used to get $v \leqslant u$. Here it is crucial that (2.1.2) holds on the *entire line* $\mathbb{R}$.

It is surprising that the matrix Q in the hypothesis of Theorem 2.1.5 is, insofar as its second and third arguments are concerned, to be evaluated solely on the functions u and Du, *without any symmetric reference to v and Dv*.

The maximum principle, simple enough in essence, nevertheless lends itself to a quite remarkable number of uses when combined appropriately with other notions. We discuss several here, reserving more subtle applications until the final chapter of the book.

A general quasilinear equation of second order, for example, has the form

$$\boldsymbol{a}(x, u, Du)D^2u + B(x, u, Du) = 0, \quad x \in \Omega, \tag{2.1.4}$$

where $\boldsymbol{a} = \boldsymbol{a}(x, z, \boldsymbol{\xi})$ and $B = B(x, z, \boldsymbol{\xi})$ are respectively a given $n \times n$ matrix $[a_{ij}]$ and a given scalar function of the variables $(x, z, \boldsymbol{\xi}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. The notation $\boldsymbol{a}D^2u$ denotes the natural contraction $a_{ij}\partial^2_{x_i x_j}u$.

A classical solution $u \in C^2(\Omega)$ of (2.1.4) is calles *elliptic* if the matrix $\boldsymbol{a}(x, u, Du)$ is positive definite when evaluated at $u = u(x)$, $x \in \Omega$. The equation itself is called elliptic in Ω, or simply elliptic, if $\boldsymbol{a}(x, z, \boldsymbol{\xi})$ is positive definite for all $(x, z, \boldsymbol{\xi}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

In view of Theorem 2.1.5, a sufficient condition for uniqueness of the corresponding Dirichlet problem for (2.1.4), with $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and u given on $\partial\Omega$, is that there exists at least one (!), elliptic solution u, while the matrix $\boldsymbol{a}$ is independent of z and the scalar $B(x, z, \boldsymbol{\xi})$ is a non-increasing function of z for arbitrary arguments x, $\boldsymbol{\xi}$. This conclusion is essentially due to Hopf, though not explicitly mentioned or stated by him, and seems to have appeared first in [40], first edition, Chapter 8.

The result applies at once to the quasilinear operator

$$\left(1 + |Du|^2\right)\Delta u - \sum \partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u$$

(mean curvature) for which the corresponding matrix

$$Q_{ij} = a_{ij} = \left(1 + |Du|^2\right)\delta_{ij} - \partial_{x_i} u \partial_{x_j} u$$

is positive definite for all values of its arguments (that is, the mean curvature operator is elliptic). Here of course there is no need to use the full strength of Theorem 2.1.5. On the other hand, if we consider the Dirichlet problem

$$\left(1 + |Du|^2\right)\Delta u - 2\sum \partial_{x_i} u \partial_{x_j} u \partial^2_{x_i x_j} u = 0$$

in Ω, with $u = 0$ on $\partial\Omega$, then the matrix is *not* positive definite for arbitrary arguments D^2u. Nevertheless $Q = \mathbb{I}$ for the function $u \equiv 0$, whence it follows that 0 is the *unique* solution of the Dirichlet problem.

A second and more subtle example is the elementary Monge–Ampère equation in $\mathbb{R}^2$

$$\partial^2_{x^2} u \partial^2_{y^2} u - \left(\partial^2_{xy} u\right)^2 = g(x, y).$$

Here one checks that

$$Q_{ij}\xi_i\xi_j = \partial^2_{y^2} u \xi_1^2 - 2\partial^2_{xy} u \xi_1\xi_2 + \partial^2_{x^2} u \xi_2^2.$$

The discriminant of Q is then

$$\det Q = \det \mathbb{H}u = \partial^2_{x^2} u \partial^2_{y^2} u - \left(\partial^2_{xy} u\right)^2,$$

which is precisely $g = g(x, y)$ when evaluated at a solution u.

Suppose in particular that $g > 0$. It is easy to see then, that *any solution u is either everywhere strictly convex or everywhere strictly concave*.

From this, one can check without difficulty that *if u and v are two convex solutions then Q is positive definite for the arguments $\partial^2_{x_i x_j}(\theta u + (1 - \theta)v)$*.

Hence the Dirichlet problem for the elementary Monge–Ampère equation above has at most one convex solution. On the other hand, if u and v are concave solutions, then $-u$ and $-v$ are convex solutions and so, similarly, the Dirichlet problem can have at most one

concave solution; altogether then *the problem can have at most two solutions*. This result is a special case of a theorem of Rellich [78]; see [21, page 324].

Other related maximum and comparison principles are discussed in the Notes to Chapter 2 of [66], to which the reader is strongly referred; see also the references cited on page 314 of [98]. Several recent maximum principles for singular fully nonlinear equations are given in [7,14], based on the "*viscosity*" method.

Hopf's proof technique, as noted above, leads to other results of fundamental interest, particularly the celebrated Boundary Point Lemma and a Harnack principle for nonlinear elliptic equations in two variables, see Section 5.5 and [40], Chapter 3.

2.2. Tangency and comparison principles for quasilinear inequalities

We consider the pair of differential inequalities

$$a_{ij}(x, u, Du)\partial^2_{x_i x_j} u + B(x, u, Du) \geqslant 0, \tag{2.2.1}$$

$$a_{ij}(x, v, Dv)\partial^2_{x_i x_j} v + B(x, v, Dv) \leqslant 0, \tag{2.2.2}$$

where the standard summation convention is assumed to be in effect. Let $\boldsymbol{P}$ be an open subset of $\mathbb{R}^n$ and let the matrix of coefficients

$$[a_{ij}] = \big[a_{ij}(x, z, \boldsymbol{\xi})\big] : K \to \mathbb{R}^{n^2}, \quad K = \Omega \times \mathbb{R} \times \boldsymbol{P},$$

be continuous, and also continuously differentiable with respect to z and $\boldsymbol{\xi}$, in the set K. Similarly, let $B = B(x, z, \boldsymbol{\xi}) : K \to \mathbb{R}$ be continuously differentiable with respect to $\boldsymbol{\xi}$ in K.

The set $\boldsymbol{P}$ is called the *regular set*, while $\boldsymbol{Q} = \mathbb{R}^n \setminus \boldsymbol{P}$ is the *singular set* for (2.2.1) and (2.2.2). It is not necessary that the inequalities (2.2.1) and (2.2.2) even have meaning for points x in Ω for which $Du(x)$ or $Dv(x)$ are in the singular set. These conditions apply in particular to the p-Laplace operator Δ_p, where

$$[a_{ij}] = \big[a_{ij}(\boldsymbol{\xi})\big] = |\boldsymbol{\xi}|^{p-2}\left[\mathbb{I} + (p-2)\frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right], \quad \boldsymbol{\xi} \neq \mathbf{0};$$

this is singular when $p \neq 2$, with the singular set $\boldsymbol{Q} = \{\mathbf{0}\}$. (The matrix $[a_{ij}]$ is undefined at $\boldsymbol{\xi} = \mathbf{0}$ when $p < 2$.)

The inequalities (2.2.1), (2.2.2) are called *elliptic* if $\boldsymbol{a}(x, z, \boldsymbol{\xi})$ is positive definite for $(x, z, \boldsymbol{\xi}) \in \Omega \times \mathbb{R} \times \boldsymbol{P}$. Similarly, a solution v of (2.2.2) is called *elliptic* if the matrix $\boldsymbol{a}(x, v, Dv)$ is positive definite when evaluated at $v = v(x)$, $x \in \Omega$. The corresponding terminology applies of course to solutions of (2.2.1).

THEOREM 2.2.1 (Tangency Principle). *Let u and v be of class $C^2(\Omega)$ with $Dv(x) \in \boldsymbol{P}$ for all $x \in \Omega$. Suppose that u is a solution of* (2.2.1) *in the open set $U = \{x \in \Omega\colon\ Du(x) \in \boldsymbol{P}\}$, and that v is an elliptic solution of* (2.2.2) *in Ω.*

Assume moreover that $B(x, z, \boldsymbol{\xi})$ is locally lower Lipschitz continuous with respect to the variable z in K.[2]

If $u \leqslant v$ in Ω and $u = v$ at some point $x_0 \in \Omega$, then $u \equiv v$ in Ω.

The conclusion can be informally restated as saying that if u and v are one-sidedly tangent at a point in Ω, then they coincide. It is interesting to note that *no condition of ellipticity* is required of (2.2.1) itself. The same remark applies also to the next two theorems.

When the regular set is all of $\mathbb{R}^n$ (that is, $\boldsymbol{Q} = \emptyset$) Theorem 2.2.1 as well as later theorems has a simpler formulation.

THEOREM 2.2.2 (Tangency Principle). *Let $\boldsymbol{P} = \mathbb{R}^n$. Suppose that $u, v \in C^2(\Omega)$ are respectively solutions of* (2.2.1) *and* (2.2.2) *in Ω, with v being elliptic in Ω.*

Assume also that $B(x, z, \boldsymbol{\xi})$ is locally lower Lipschitz continuous with respect to z in K. If $u \leqslant v$ in Ω and $u = v$ at some point in Ω, then $u \equiv v$ in Ω.

PROOF. It is enough to prove Theorem 2.2.1. Let $E = \{x \in \Omega\colon\ u(x) = v(x)\}$. By assumption $E \neq \emptyset$, while of course E is closed. Fix $y \in E$. Since $u - v \leqslant 0$ in Ω and $(u - v)(y) = 0$, we have $D(u - v)(y) = \mathbf{0}$.

Since $Dv(y) \in \boldsymbol{P}$ there is a suitably small $\sigma > 0$ such that $Du(x), Dv(x) \in \boldsymbol{P}$ for all $x \in B_\sigma$, where $B_\sigma = B_\sigma(y)$ is the closed ball with center y and radius σ in Ω. Obviously $B_\sigma \subset U$. As in the proof of Theorem 2.1.3, but now with $\mathcal{F}(x, u, Du, D^2u) = a_{ij}(x, u, Du)\partial^2_{x_i x_j} u + B(x, u, Du)$, we obtain the inequality

$$a_{ij}(x, v, Dv)\partial^2_{x_i x_j}(u - v) + b_i \partial_{x_i}(u - v) + c(u - v) \geqslant 0 \quad \text{in } B_\sigma \subset U, \tag{2.2.3}$$

where

$$b_k = \big[\partial_{\xi_k}\big(a_{ij}\partial^2_{x_i x_j} u + B\big)\big(x, v, \theta_1 Du + (1 - \theta_1)Dv\big)\big],$$
$$c = \big(\partial_z a_{ij}\partial^2_{x_i x_j} u\big)\big(x, \theta_2 u + (1 - \theta_2)v, Du\big) - L,$$

for some values $\theta_1, \theta_2 \in [0, 1]$. Clearly a_{ij}, b_i, are bounded, and c is bounded below, in B_σ, and equally by continuity the coefficient matrix $[a_{ij}(x, v, Dv)]$ is uniformly positive definite in B_σ. Because $u - v$ has a zero maximum in B_σ, it now follows from Theorem 2.1.2 applied to the linear inequality (2.2.3) that $u \equiv v$ in B_σ, that is $B_\sigma \subset E$. Hence E is also an open set. By the connectedness of Ω it follows that $E = \Omega$, as required. □

[2] That is, for every compact subset of K there is a number $L > 0$ such that if $\bar{z} > z$ then

$$B(x, \bar{z}, \boldsymbol{\xi}) - B(x, z, \boldsymbol{\xi}) \geqslant -L(\bar{z} - z)$$

in the subset.

In the formulation of Theorem 2.2.1 the inequalities (2.2.1) and (2.2.2) could be taken in the form $Lu - Lv \geqslant 0$. The present formulation is equivalent and perhaps easier to visualize.

THEOREM 2.2.3 (Comparison Principle). *As in Theorem* 2.2.1, *let u and v be of class $C^2(\Omega)$ with $Dv(x) \in \boldsymbol{P}$ for all $x \in \Omega$. Suppose that u is a solution of* (2.2.1) *in the open set $U = \{x \in \Omega\colon\ Du(x) \in \boldsymbol{P}\}$, while v is an elliptic solution of* (2.2.2) *in Ω.*

Assume that $[a_{ij}]$ is independent of z and that B is non-increasing with respect to z in K. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.

REMARK. The reader should note the rather different hypotheses in Theorems 2.2.1 and 2.2.3. It can be shown by example that the specific monotonicity stated for B in these results cannot be reversed.

In view of conclusion $u \leqslant v$ of Theorem 2.2.3, solutions of the inequalities (2.2.1) and (2.2.2) are frequently called, respectively, *subsolutions* and *supersolutions* of the equation

$$a_{ij}(x, u, Du)\partial^2_{x_i x_j} u + B(x, u, Du) = 0.$$

PROOF OF THEOREM 2.2.3. The proof is by contradiction, essentially the same as for Theorem 2.1.4. Let Ω' be a subdomain of Ω in which $v - u < 0$ but is not identically constant, and in which also $v - u$ takes on a negative minimum M at a point y. Obviously $D(v - u) = \mathbf{0}$ at y.

Hence, as in the proof of Theorem 2.2.1, there exists a closed ball $B_\sigma \subset \Omega'$ centered at y such that $Du(x), Dv(x) \in \boldsymbol{P}$ for all $x \in B_\sigma$. Clearly $B_\sigma \subset U$. Moreover, as in the proof of Theorem 2.2.1, but using the fact that $[a_{ij}]$ is independent of z and also the monotonicity of B in z, we get (see (2.2.3))

$$a_{ij}(x, Dv)\partial^2_{x_i x_j}(u - v) + b_k \partial_{x_k}(u - v) \geqslant 0 \quad \text{in } B_\sigma.$$

It now follows from Hopf's main theorem that $v - u \equiv M < 0$ in B_σ. The subset of Ω' where $v - u \equiv M$ is thus both open and relatively closed, so $v - u \equiv M$ in Ω', a contradiction. □

As in the case of Theorem 2.2.1, when the regular set is all of $\mathbb{R}^n$ the proof can be considerably simplified.

Norman Meyers [53] has shown that the comparison Theorem 2.2.3 fails if the coefficient matrix $[a_{ij}]$ depends on the z variable. At the same time, by considering the function v in Theorem 2.2.3 as a "*comparison function*", the conclusion can be interpreted as a maximum principle. We take up this idea in the next section.

The next result applies to semilinear rather than quasilinear inequalities, for example

$$\Delta u + f(x, u) \geqslant 0, \qquad \Delta v + f(x, v) \leqslant 0.$$

THEOREM 2.2.4 (Comparison Principle). *Let L be the linear differential operator given in Hopf's main theorem* (*Section* 2.1), *and let $u, v \in C^2(\Omega)$ be solutions of the differential inequalities*

$$Lu + f(x, u) \geqslant 0, \qquad Lv + f(x, v) \leqslant 0$$

in Ω with $v > 0$ in $\overline{\Omega}$.

Suppose that $z \mapsto f(x, z)/z$, $z > 0$, is a non-increasing function for each fixed $x \in \Omega$. Then if $u \leqslant v$ on $\partial\Omega$ we have $u \leqslant v$ in Ω.

The condition on f here is more general than simple monotonicity, as one sees from the example $f(z) = z^q$, which is non-increasing when $q \leqslant 0$, while $f(z)/z = z^{q-1}$ is non-increasing when $q \leqslant 1$.

PROOF. Put $w = w(x) = u(x)/v(x)$ in Ω, so

$$Lw + \frac{2}{v} a_{ij} \partial_{x_j} v \partial_{x_i} w = \frac{1}{v} Lu - \frac{u}{v^2} Lv \geqslant \left\{ \frac{f(x, v)}{v} - \frac{f(x, u)}{u} \right\} w. \tag{2.2.4}$$

Since $u \leqslant v$ and $v > 0$ on $\partial\Omega$ it follows that also $w \leqslant 1$ on $\partial\Omega$. If the conclusion $w \leqslant 1$ fails at some point in Ω, there would be a point x_0 in Ω where w takes a maximum value $M > 1$. In the neighborhood of x_0 the right side of (2.2.4) would then be non-negative according to hypothesis, so by Hopf's main theorem, with b_i replaced by $b_i + (2/v)a_{ij}\partial_{x_j} v$, we would have $w \equiv M > 1$ in this neighborhood, and then $w \equiv M$ in Ω, which is impossible. □

EXAMPLE. As a consequence of this theorem, Protter and Weinberger [66] have observed that when $L = \Delta$ and $f(x, u) = 2u$ then there can be no positive solutions of $\Delta v + 2v \leqslant 0$ in the 2-dimensional square $\Omega = \{|x| \leqslant \pi/2,\ |y| \leqslant \pi/2\}$. Indeed, if this were the case, then any solution of $\Delta u + 2u = 0$ in Ω with $u = 0$ on $\partial\Omega$ would be bounded above by v. But obviously $u(x, y) = c \sin x \sin y$ is a solution, which can be made as large as one wishes by taking the constant $c > 0$ suitably large.

2.3. Maximum and sweeping principles for quasilinear inequalities

As a main consequence of the comparison Theorem 2.2.3 of the previous section we have the following

THEOREM 2.3.1 (Maximum Principle). *Let $v \in C^2(\Omega)$ be a comparison function for* (2.2.1), *in sense that there exists M such that*

(i) *$v(x) \geqslant M$ and $Dv(x) \in \boldsymbol{P}$ for all $x \in \Omega$;*

(ii) *v is an elliptic solution of the inequality*

$$a_{ij}\big(x, z, Dv(x)\big) \partial^2_{x_i x_j} v + B\big(x, z, Dv(x)\big) \leqslant 0 \tag{2.3.1}$$

for all fixed values $z > M$.

If $u \in C^2(\Omega)$ is a solution of (2.2.1) *in $U = \{x \in \Omega\colon\ Du(x) \in \boldsymbol{P}\}$ and $u \leqslant v$ on $\partial\Omega$, then either $u(x) \equiv v(x)$ or $u(x) < v(x)$ in Ω.*

PROOF. Define

$$\tilde{a}_{ij}(x,\xi)=a_{ij}\big(x,u(x),\xi\big),\qquad \widetilde{B}(x,\xi)=B\big(x,u(x),\xi\big)$$

and

$$\mathcal{L}[v]=\tilde{a}_{ij}(x,Dv)\partial^2_{x_ix_j}v+\widetilde{B}(x,Dv).$$

By (ii) the operator $\mathcal{L}$ is elliptic and $\mathcal{L}[v]\leqslant 0$ when $u(x)>M$. Moreover, obviously, $\mathcal{L}[u]\geqslant 0$ in U.

Let

$$\Omega'=\big\{x\in\Omega\colon\ u(x)>M\big\}\quad\text{and}\quad U'=\big\{x\in\Omega'\colon\ Du(x)\in\boldsymbol{P}\big\}\subset U.$$

Since $u=M$ on $\partial\Omega'\cap\Omega$ and $u\leqslant v$ on $\partial\Omega$ it follows that $u\leqslant v$ on $\partial\Omega'$. Then by Theorem 2.2.3 applied to any component $\mathcal{C}'$ of Ω' we have $u\leqslant v$ in $\mathcal{C}'$, and so $u\leqslant v$ in Ω'. Hence $u\leqslant v$ in Ω.

The required conclusion now follows at once with the help of Theorem 2.2.1. □

Theorem 2.3.1 is somewhat abstract, in that it depends on the *existence* of the comparison function v. When $[a_{ij}]$ and B are more specialized we can avoid this difficulty. In particular, consider the case where $\boldsymbol{Q}\subset\boldsymbol{B}_\varrho$ for some $\varrho\geqslant 0$ (the possibility $\boldsymbol{P}=\mathbb{R}^n$ is included when $\varrho=0$). Assume that

$$\begin{cases}[a_{ij}(x,z,\xi)]\quad\text{is positive definite,}\\ B(x,z,\xi)\leqslant\alpha|\xi|E(x,z,\xi)+\gamma,\quad E(x,z,\xi)=a_{ij}(x,z,\xi)\xi_i\xi_j/|\xi|^2,\end{cases}\tag{2.3.2}$$

in $\Omega\times\mathbb{R}^+\times\boldsymbol{P}$, where α and γ are non-negative constants.

THEOREM 2.3.2 (Maximum Principle). *Let $\boldsymbol{A}$ and B satisfy* (2.3.2), *and suppose that*

$$|\xi|E(x,z,\xi)\geqslant\Psi\big(|\xi|\big)\quad\textit{in }\Omega\times\mathbb{R}^+\times\boldsymbol{P},\quad \boldsymbol{P}=\mathbb{R}^n\setminus\boldsymbol{Q},\tag{2.3.3}$$

where $\Psi=\Psi(t)$ is a strictly increasing function on (ϱ,∞), $\varrho\geqslant 0$.

Let $u\in C^2(\Omega)$ be a solution of the boundary value problem

$$\begin{aligned}&a_{ij}(x,u,Du)\partial^2_{x_ix_j}u+B(x,u,Du)\geqslant 0 && \textit{in }\Omega,\\ &u\leqslant 0 && \textit{on }\partial\Omega,\end{aligned}\tag{2.3.4}$$

where $\Omega\subset\{x\in\mathbb{R}^n\colon\ 0<x_1<R\}$. Then there holds

$$u(x)\leqslant R\max\{\rho,\,C\}\big(\mathrm{e}^k-1\big),\tag{2.3.5}$$

where[3]

$$\begin{aligned} &C = \Psi^{-1}(R\gamma), \quad k = 1 + \alpha R, && \text{when } \lim_{t\to\infty} \Psi(t) > 2\gamma R, \\ &C = \Psi^{-1}(\ell), \quad k = 1 + \left(\alpha + \frac{\gamma}{\ell}\right) R, && \text{when } \lim_{t\to\infty} \Psi(t) = 2\ell \leqslant 2\gamma R. \end{aligned} \tag{2.3.6}$$

For the important subcase of the p-Laplace operator one has $E(t) = (p-1)t^{p-2}$, $\Psi(t) = (p-1)t^{p-1}$ and $R\Psi^{-1}(Rs) = [s/(p-1)]^{1/(p-1)} R^{p'}$.

PROOF. It is enough to construct a comparison function $v = v(x)$ such that $v(x) > 0$ in Ω and (2.3.1) holds. Accordingly, we choose

$$v(x) = K\left(\mathrm{e}^{mR} - \mathrm{e}^{mx_1}\right), \quad x \in \Omega,$$

where $m = k/R$, $K > R\max\{\varrho, C\}$. Then

$$\partial_{x_1} v(x) = -Km\mathrm{e}^{mx_1}$$

so $|Dv| \geqslant mK$ and $Dv \in \boldsymbol{P}$, since $m > 1/R$. Also

$$\partial_{x_1}^2 v(x) = -Km^2\mathrm{e}^{mx_1} = -m|Dv|.$$

With the help of (2.3.2), a calculation shows that (2.3.1) is valid provided

$$m|Dv|a_{11}(x, z, Dv) \geqslant \alpha|Dv|E(x, z, Dv) + \gamma \tag{2.3.7}$$

for all $x \in \Omega$ and $z > 0$. But $E(x, z, Dv) = a_{11}(x, z, Dv)$, so (2.3.7) becomes

$$m|Dv|E(x, z, Dv) \geqslant \alpha|Dv|E(x, z, Dv) + \gamma. \tag{2.3.8}$$

Obviously (2.3.8) is satisfied if $(m-\alpha)|Dv|\,E(x, z, Dv) \geqslant \gamma$ for all $z > 0$. At the same time

$$|Dv|E(x, z, Dv) \geqslant \Psi\left(|Dv|\right) \geqslant \Psi(mK) \geqslant \Psi(C) \geqslant \min\{\gamma R, \ell\},$$

[3]If $\Psi(\varrho) = \lim_{t\to\varrho+} \Psi(t) = \ell' > 0$ then we define $\Psi^{-1}(s) = \varrho$ when $s \leqslant \ell'$. Note that the case $\lim_{t\to\infty} \Psi(t) < \infty$ is possible. That is, take for $\boldsymbol{\xi} \neq \boldsymbol{0}$

$$a_{ij}(\boldsymbol{\xi}) = \frac{2\ell}{|\boldsymbol{\xi}| + 1} \cdot \frac{\xi_i \xi_j}{|\boldsymbol{\xi}|^2};$$

an easy computation yields

$$E(\boldsymbol{\xi}) = \frac{2\ell}{|\boldsymbol{\xi}| + 1}, \quad \Psi(t) = \frac{2\ell t}{t+1}, \quad \Psi^{-1}(s) = \frac{s}{2\ell - s},$$

so $\Psi(t) \to 2\ell$ as $t \to \infty$. (In fact in this case $\Psi^{-1}(\ell) = 1$.)

since $mK > (k/R)R\max\{\varrho, C\} \geqslant C$. Therefore (2.3.8) holds when k and C are given as in (2.3.6), and in turn (2.3.1) holds, as required.

We now apply Theorem 2.3.1, giving

$$u(x) \leqslant v(x) \leqslant K\left(\mathrm{e}^k - 1\right) \quad \text{in } \Omega.$$

Letting $K \to R\max\{\varrho, C\}$ completes the proof. □

REMARKS.

1. The condition $u \leqslant 0$ on the boundary can obviously be replaced by $u \leqslant M$, by adding M to the right side of (2.3.5).
2. The condition $\Omega \subset \{x \in \mathbb{R}^n\colon 0 < x_1 < R\}$ can (by appropriate translation and rotation of coordinates) always be satisfied by any domain whose minimum diameter is R.
3. Finally, the theorem simplifies considerably when either $\boldsymbol{Q} = \emptyset$ or $\{\mathbf{0}\}$ and range $\Psi = \mathbb{R}^+$. Then $\varrho = 0$ and $u(x) \leqslant R\Psi^{-1}(\gamma R)[\exp(1 + \alpha R) - 1]$.
4. The possibility that $\boldsymbol{Q} \supsetneq \{\mathbf{0}\}$, say $\boldsymbol{Q} = \boldsymbol{B}_\varrho$, $\varrho > 0$, is discussed later in Section 3.7.

The next result shows that when B is homogeneous the global condition (2.3.2) need be assumed only for $|\boldsymbol{\xi}|$ small, clearly of importance in applications.

THEOREM 2.3.3. *Assume* $\boldsymbol{P} = \mathbb{R}^n$ *or* $\boldsymbol{P} = \mathbb{R}^n \setminus \{\mathbf{0}\}$. *Let the hypotheses of Theorem* 2.3.2 *hold, with the exceptions that* $\gamma = 0$, *and* (2.3.2) *and* (2.3.3) *are assumed to be valid only in* $\Omega \times \mathbb{R}^+ \times \boldsymbol{R}_1$, $\boldsymbol{R}_1 = \{\boldsymbol{\xi} \in \mathbb{R}^n\colon 0 < |\boldsymbol{\xi}| < 1\}$. *Let* $u \in C^2(\Omega)$ *be a solution of the boundary value problem* (2.3.4) *where* Ω *is now an arbitrary bounded domain in* $\mathbb{R}^n$. *Then* $u \leqslant 0$ *in* Ω.

In the generality of the present hypotheses, this seems to be a new result.

PROOF. Since $\gamma = 0$ only the first case of (2.3.6) applies and so $C = \Psi^{-1}(0) = \varrho = 0$. In this case the constant $K > 0$ in the proof of Theorem 2.3.2 can be chosen arbitrarily small, and in particular so small that $|Dv(x)| \leqslant Kme^{mR} \leqslant 1$ in Ω. The rest of the proof of Theorem 2.3.2 then applies without change, giving $u \leqslant 0$ whatever the value of R. □

Theorem 2.3.3 is false if one weakens condition (2.3.2), as follows from the example

$$\Delta_4 u + |Du|^2 = 0, \quad n = 2. \tag{2.3.9}$$

Indeed, this equation has the solution $u(x) = \frac{1}{8}(R^2 - |x|^2)$ in B_R, which vanishes on the boundary, and at the same time is positive in the interior.

THEOREM 2.3.4. *Let the hypotheses of Theorem* 2.3.2 *be satisfied, with the exception that* (2.3.2) *is replaced by the condition that*

$$B(x, z, \boldsymbol{\xi}) \leqslant \left(\alpha|\boldsymbol{\xi}| + \beta|\boldsymbol{\xi}|^q\right)E(x, z, \boldsymbol{\xi}) + \gamma, \quad 0 < q < 1,$$

in $\Omega \times \mathbb{R}^+ \times \boldsymbol{P}$, *where* α, β, γ *are non-negative constants.*

Then (2.3.5) *holds with the previous constant C replaced by* $C + \beta^{1/(1-q)}$ *and the previous constant k replaced by* $k + 1$.

The proof is essentially the same as before. The additional term $\beta|\boldsymbol{\xi}|^q$ (in the case $q = 0$) was first introduced by Gilbarg and Trudinger ([40], Theorem 10.3).

The idea of Theorem 2.3.1 can be extended in the form of a "*field version*" of the result.

THEOREM 2.3.5 (Sweeping Principle). *For* $\lambda \in [0, 1]$, *let* $\lambda \mapsto v_\lambda = v(x, \lambda)$ *be a family of* $C^2(\Omega) \cap C(\overline{\Omega})$ *functions which are strictly increasing in* λ *for each* $x \in \overline{\Omega}$, *and are such that* v *is of class* $C(\overline{\Omega} \times [0, 1])$. *Define*

$$\mathcal{L}[u](x) = a_{ij}(x, u, Du)\partial^2_{x_i x_i} u + B(x, u, Du), \quad x \in \Omega. \tag{2.3.10}$$

Assume that

$$\mathcal{L}[v_\lambda] \leqslant 0 \quad \text{in } \Omega,\ 0 \leqslant \lambda \leqslant 1,$$

that $\mathcal{L}[u] \geqslant 0$ *and that* $\mathcal{L}$ *is elliptic either for* u *or for the family* $\{v_\lambda\}_\lambda$.

If $u \leqslant v_1$ *in* Ω *and* $u \leqslant v_0$ *on* $\partial\Omega$, *then either* $u \equiv v_0$ *or* $u < v_0$ *in* Ω.

The proof is an immediate consequence of Theorem 2.2.1, the idea being illustrated in the accompanying Figure 1, where the (solid line) function u is shown satisfying the conditions $u \leqslant v_1$ in Ω and $u \leqslant v_0$ on $\partial\Omega$, but at the same time contradicting the conclusion

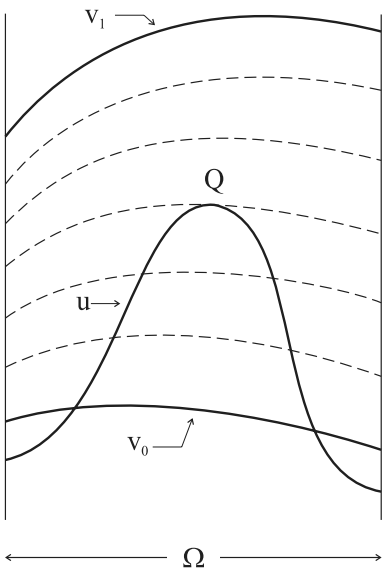

Fig. 1. Proof of the Sweeping Principle: a contradiction with the Tangency Principle occurs at Q.

of the theorem. Note that in Theorem 2.3.5 no statement need be made concerning ellipticity of $\mathcal{L}$ along u or the monotonicity of $B(x, z, \boldsymbol{\xi})$ in z. The setting of Theorem 2.3.5 can be compared with the usual notion of a field used for sufficiency proofs in the calculus of variations.

The maximum principle Theorem 2.3.1 can be considered as essentially the special case $v_\lambda = v + C\lambda$ of the sweeping principle, C being chosen so that $u \leqslant v + C$ in Ω. A version of the sweeping principle in which the operator has a singular set $\boldsymbol{Q}$ can be left to the reader.

2.4. Comparison theorems for divergence structure inequalities

We consider the pair of differential inequalities

$$\operatorname{div} \boldsymbol{A}(x, Du) + B(x, u) \geqslant 0, \tag{2.4.1}$$

$$\operatorname{div} \boldsymbol{A}(x, Dv) + B(x, v) \leqslant 0, \tag{2.4.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. Let $\boldsymbol{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be in $L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R}^n)$, and $B : \Omega \times \mathbb{R} \to \mathbb{R}$ be in $L^\infty_{\mathrm{loc}}(\Omega \times \mathbb{R})$.

For the purpose of this section, by a *solution of* (2.4.1) *or* (2.4.2) *in* Ω we mean a (classical) distribution solution of class $C^1(\Omega)$, with the test function space consisting of all non-negative functions $\varphi \in C^1(\Omega)$ such that $\varphi \equiv 0$ near $\partial\Omega$. As is well known (see [40], Section 7.3) the test function space can without loss of generality be enlarged to include Lipschitz continuous functions which vanish near the boundary.

We shall treat here the simplest comparison theorems for divergence structure inequalities. More general results are given in Sections 3.4–3.6. Strong comparison theorems, under alternative hypotheses, have been obtained by Tolksdorf [94] and by Cuesta and Takáč [22].

THEOREM 2.4.1 (Comparison Principle). *Let u and v be respective solutions of* (2.4.1) *and* (2.4.2) *in* Ω. *Suppose that* $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ *is independent of z and monotone in* $\boldsymbol{\xi}$, *i.e.*

$$\big\langle \boldsymbol{A}(x, \boldsymbol{\xi}) - \boldsymbol{A}(x, \boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta} \big\rangle > 0, \quad \text{when } \boldsymbol{\xi} \neq \boldsymbol{\eta}; \tag{2.4.3}$$

while $B = B(x, z)$ *is independent of* $\boldsymbol{\xi}$ *and non-increasing in z.*

If $u \leqslant v$ *on* $\partial\Omega$, *then* $u \leqslant v$ *in* Ω.

PROOF. Assume for contradiction that there exists $x_0 \in \Omega$ such that $u(x_0) > v(x_0)$. Let Γ be the open set $\{x \in \Omega\colon\ u(x) - v(x) > \varepsilon\}$, non-empty for $\varepsilon > 0$ sufficiently small. The function $\varphi = (u - v - \varepsilon)^+$ is uniformly Lipschitz continuous, has compact support in Ω, and $D\varphi = \boldsymbol{0}$ a.e. in $\Omega \setminus \Gamma$.

Subtracting (2.4.2) from (2.4.1) and using φ as test function yields

$$\int_\Gamma \big\langle \boldsymbol{A}(x, Du) - \boldsymbol{A}(x, Dv), Du - Dv \big\rangle \leqslant \int_\Gamma \big[B(x, u) - B(x, v) \big](u - v - \varepsilon).$$

By (2.4.3) the left-hand side is positive unless $Du \equiv Dv$ in Γ, while the right-hand side is $\leqslant 0$ since B is non-increasing in z. Let $\mathcal{C}$ be any component of Γ, so that $u - v \equiv$ const. $= c$ in $\mathcal{C}$. If $\partial\mathcal{C} \cap \Omega \neq \emptyset$, then $c = \varepsilon$ which contradicts the fact that $u - v > \varepsilon$ in $\mathcal{C}$. Otherwise $\mathcal{C} = \Omega$ and c must be positive since $u(x_0) > v(x_0)$. This violates the fact that $u \leqslant v$ on $\partial\Omega$. □

REMARK. If B is non-increasing only for values $z \in (-\infty, \delta)$, then the conclusion of Theorem 2.4.1 continues to hold provided $u < \delta$ in Ω.

Because the condition (2.4.3) is somewhat abstract, it is of interest to give explicit vector functions $\boldsymbol{A}$ for which it is satisfied. One of the simplest examples is

$$\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\boldsymbol{\xi}, \quad \boldsymbol{\xi} \neq \mathbf{0}; \qquad \boldsymbol{A}(\mathbf{0}) = \mathbf{0}, \tag{2.4.4}$$

where $s \mapsto A(s)$, $s > 0$, is positive and $\Phi(s) = sA(s)$ is strictly increasing on $\mathbb{R}^+$. We state this as

PROPOSITION 2.4.2. *Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be vectors in $\mathbb{R}^n$. Then for the function* (2.4.4) *we have*

$$\big\langle \boldsymbol{A}(\boldsymbol{\xi}) - \boldsymbol{A}(\boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta} \big\rangle > 0$$

whenever $\boldsymbol{\xi} \neq \boldsymbol{\eta}$.

PROOF. If one of the vectors is $\mathbf{0}$ the assertion is trivial. Otherwise, $\boldsymbol{\xi}, \boldsymbol{\eta} \neq \mathbf{0}$ and $\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle \leqslant |\boldsymbol{\xi}| \cdot |\boldsymbol{\eta}|$, so that

$$\begin{aligned}
\big\langle \boldsymbol{A}(\boldsymbol{\xi}) - \boldsymbol{A}(\boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta} \big\rangle &= A\big(|\boldsymbol{\xi}|\big)|\boldsymbol{\xi}|^2 + A\big(|\boldsymbol{\eta}|\big)|\boldsymbol{\eta}|^2 - A\big(|\boldsymbol{\xi}|\big)\langle \boldsymbol{\xi}, \boldsymbol{\eta} \rangle - A\big(|\boldsymbol{\eta}|\big)\langle \boldsymbol{\eta}, \boldsymbol{\xi} \rangle \\
&\geqslant \Phi\big(|\boldsymbol{\xi}|\big)|\boldsymbol{\xi}| + \Phi\big(|\boldsymbol{\eta}|\big)|\boldsymbol{\eta}| - \Phi\big(|\boldsymbol{\xi}|\big)|\boldsymbol{\eta}| - \Phi\big(|\boldsymbol{\eta}|\big)|\boldsymbol{\xi}| \\
&= \big\{\Phi\big(|\boldsymbol{\xi}|\big) - \Phi\big(|\boldsymbol{\eta}|\big)\big\}\big(|\boldsymbol{\xi}| - |\boldsymbol{\eta}|\big)
\end{aligned}$$

and the conclusion now comes from the strict monotonicity of Φ. □

Proposition 2.4.2 obviously covers the p-Laplace operator $A(s) = s^{p-2}$, $p > 1$, as a special case. A second example of interest is the following

PROPOSITION 2.4.3. *Suppose that $\boldsymbol{A}(x, \boldsymbol{\xi})$ is continuous in $\Omega \times \mathbb{R}^n$ and continuously differentiable with respect to $\boldsymbol{\xi}$ in the set $\Omega \times \mathbb{R}^n \setminus \{\mathbf{0}\}$, with the Jacobian matrix $[\partial_{\xi_j} A_i(x, \boldsymbol{\xi})]$ being positive definite.*

Then (2.4.3) *is valid.*

PROOF. First we observe that if $\boldsymbol{\xi} \neq \boldsymbol{\eta}$ and the line segment $[\boldsymbol{\xi}, \boldsymbol{\eta}]$ does not include the point $\mathbf{0}$, then by the mean value theorem, for some point $\boldsymbol{\zeta}$ in the segment,

$$\big\langle \boldsymbol{A}(x, \boldsymbol{\xi}) - \boldsymbol{A}(x, \boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta} \big\rangle = \big\langle \partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, \boldsymbol{\zeta})(\boldsymbol{\xi} - \boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta} \big\rangle > 0,$$

since $[\partial_{\boldsymbol{\xi}} A(x, \boldsymbol{\xi})]$ is positive definite in $\Omega \times \mathbb{R}^n \setminus \{\mathbf{0}\}$.

When $\mathbf{0} \in [\boldsymbol{\xi}, \boldsymbol{\eta}]$, we apply the mean value theorem in each segment $[\boldsymbol{\xi}, \mathbf{0}]$, $[\mathbf{0}, \boldsymbol{\eta}]$, using the continuity of $\boldsymbol{A}$ with respect to $\boldsymbol{\xi}$ in $\Omega \times \mathbb{R}^n$. □

A delicate application of this proposition occurs when

$$\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\boldsymbol{a}(x)\boldsymbol{\xi}, \qquad \boldsymbol{A}(x, \mathbf{0}) = \mathbf{0},$$

where $\boldsymbol{a} = \boldsymbol{a}(x) = [a_{ij}(x)]$, $i, j = 1, \dots, n$, is a continuous real symmetric matrix defined in Ω, uniformly positive definite and satisfying

$$\lambda|\boldsymbol{\zeta}|^2 \leqslant a_{ij}(x)\zeta_i\zeta_j \leqslant \Lambda|\boldsymbol{\zeta}|^2, \quad \lambda > 0, \tag{2.4.5}$$

for all $x \in \Omega$ and all $\boldsymbol{\zeta} \in \mathbb{R}^n$; we assume A has the properties noted before Proposition 2.4.2 and is continuously differentiable in $\mathbb{R}^+$.

PROPOSITION 2.4.4. *Let* $0 < \tau \leqslant \infty$ *and assume*

$$\inf_{0<s<\tau} \frac{sA'(s)}{A(s)} = c_1 > -1, \qquad \sup_{0<s<\tau} \frac{sA'(s)}{A(s)} = c_2 < \infty, \tag{2.4.6}$$

and

$$\sqrt{\frac{\Lambda}{\lambda}} < \min\big\{\phi(c_1), \phi(c_2)\big\}, \quad \phi(c) = \frac{2 + c + 2\sqrt{1+c}}{|c|}. \tag{2.4.7}$$

Define $\boldsymbol{P} = \{\boldsymbol{\xi} \in \mathbb{R}^n\colon\ 0 < |\boldsymbol{\xi}| < \tau\}$. *Then the operator* $\boldsymbol{A}$ *is elliptic in* $\Omega \times \boldsymbol{P}$, *in the sense that the Jacobian matrix* $[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, \boldsymbol{\xi})]$ *is positive definite in* $\Omega \times \boldsymbol{P}$. *Moreover, for* $x \in \Omega$

$$\big\langle A\big(|\boldsymbol{\xi}|\big)\boldsymbol{a}(x)\boldsymbol{\xi} - A\big(|\boldsymbol{\eta}|\big)\boldsymbol{a}(x)\boldsymbol{\eta}, \boldsymbol{\xi} - \boldsymbol{\eta}\big\rangle > 0$$

whenever $\boldsymbol{\xi}, \boldsymbol{\eta} \in \overline{\boldsymbol{P}}$ *and* $\boldsymbol{\xi} \neq \boldsymbol{\eta}$.

The terms $\phi(c_1)$ or $\phi(c_2)$ respectively should be omitted from (2.4.7) if c_1 or $c_2 = 0$. If $c_1 = c_2 = 0$ then (2.4.7) itself should be omitted. Also, since $\phi(c) > 1$ for all $c > -1$ it is evident that (2.4.7) is automatically satisfied whenever $\boldsymbol{a}$ is a multiple of the identity.

PROOF. We have $\partial_{\xi_j} A_i(x, \boldsymbol{\xi}) = A(|\boldsymbol{\xi}|)a_{ik}(x)b_{kj}(\boldsymbol{\xi})$ in $\Omega \times \boldsymbol{P}$, where

$$\boldsymbol{b} = \big[b_{ij}(\boldsymbol{\xi})\big] = \mathbb{I} + c\frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}, \quad c = c\big(|\boldsymbol{\xi}|\big) = \frac{|\boldsymbol{\xi}|A'(|\boldsymbol{\xi}|)}{A(|\boldsymbol{\xi}|)}.$$

The eigenvalues of $\boldsymbol{b}$ are 1, with multiplicity $n - 1$, and $1 + c$. Then from the Nicholson–Strang theorem, see [19, Theorem 2.1], it follows that $\boldsymbol{ab}$, the product of the real matrices

$\boldsymbol{a}$ and $\boldsymbol{b}$, will be positive definite provided $c = 0$ or

$$\left(\sqrt{\frac{\Lambda}{\lambda}} - 1\right)\left(\sqrt{1+c} - 1\right) < 2 \quad \text{if } c > 0,$$
$$\left(\sqrt{\frac{\Lambda}{\lambda}} - 1\right)\left(\sqrt{\frac{1}{1+c}} - 1\right) < 2 \quad \text{if } c < 0.$$

This however reduces to

$$\sqrt{\frac{\Lambda}{\lambda}} < \frac{2 + c + 2\sqrt{1+c}}{|c|} = \phi(c). \tag{2.4.8}$$

By (2.4.6) we have $c_1 \leqslant c \leqslant c_2$ so by the monotonicity properties of ϕ there holds $\phi(c) \geqslant \min\{\phi(c_1), \phi(c_2)\}$. Therefore in view of (2.4.7) the condition (2.4.8) holds for all $\boldsymbol{\xi} \in \boldsymbol{P}$. Thus $\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, \boldsymbol{\xi})$ is positive definite in $\Omega \times \boldsymbol{P}$.

Application of Proposition 2.4.3 then completes the proof (replacing $\mathbb{R}^n \setminus \{\mathbf{0}\}$ in the proposition by the more general set $\boldsymbol{P}$ causes no difficulty). □

With the help of the abstract comparison Theorem 2.4.1, the preceding Propositions 2.4.2 and 2.4.4 give explicit comparison principles for operators of the type

$$\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\boldsymbol{\xi}, \quad \boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\boldsymbol{a}(x)\boldsymbol{\xi} \quad (\boldsymbol{A} = \mathbf{0} \text{ at } \boldsymbol{\xi} = \mathbf{0}).$$

For the special p-Laplace case $A(s) = s^{p-2}$, $p > 1$, that is $\boldsymbol{A}(x, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{a}(x)\boldsymbol{\xi}$, we have $c_1 = c_2 = p - 2 > -1$ in (2.4.6), whence (2.4.8) takes the form (for $p \neq 2$)

$$\sqrt{\frac{\Lambda}{\lambda}} < \frac{p + 2\sqrt{p-1}}{|p-2|}.$$

2.5. Tangency theorems via Harnack's inequality

Tangency theorems for non-divergence inequalities also have counterparts in the divergence structure case. We begin by considering the singular differential inequality

$$\operatorname{div} \tilde{\boldsymbol{A}}(x, u, Du) + \widetilde{B}(x, u, Du) \leqslant 0 \quad \text{in } \Omega, \qquad u \geqslant 0, \tag{2.5.1}$$

where $\tilde{\boldsymbol{A}}$ and $\widetilde{B}$ are in $L^{\infty}_{\mathrm{loc}}(\Omega)$ and have the following homogeneity and ellipticity properties for all $x \in \Omega$, $z \in \mathbb{R}^+$ and $\boldsymbol{\xi} \in \mathbb{R}^n$

$$\begin{aligned} &\tilde{\boldsymbol{A}}(x, z, \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geqslant a_1|\boldsymbol{\xi}|^p - a_2 z^p, \qquad \big|\tilde{\boldsymbol{A}}(x, z, \boldsymbol{\xi})\big| \leqslant a_3|\boldsymbol{\xi}|^{p-1} + a_4 z^{p-1}, \\ &-\widetilde{B}(x, z, \boldsymbol{\xi}) \leqslant b_1|\boldsymbol{\xi}|^{p-1} + b_2 z^{p-1}, \end{aligned} \tag{2.5.2}$$

with $p > 1$; $a_1, a_3 > 0$; $a_2, a_4, b_1, b_2 \geqslant 0$ being constants (see [81], where these conditions apparently appear first). The p-Laplace operator $\tilde{A}(\xi) = |\xi|^{p-2}\xi$, $p > 1$, clearly obeys the first line of (2.5.2) with $a_1 = a_3 = 1$ and $a_2 = a_4 = 0$.

Trudinger [95, Theorem 1.2], closely using the ideas of [81], has observed that under these conditions the following beautiful Harnack inequality is valid for non-negative solutions $u \in C^1(\Omega)$ of (2.5.1).

For any ball B_R, such that $0 < R \leqslant 1$ and $B_{2R} \subset \Omega$, there holds

$$\|u\|_{q,B_R} \leqslant C|R|^{n/q} \inf_{B_R} u(x), \tag{2.5.3}$$

where C depends only on $p, n, q; a_1, a_2, a_3, a_4, b_1, b_2$, while $q \in (0, (p-1)n/(n-p))$ (or $q \in \mathbb{R}^+$ if $p \geqslant n$).

This theorem holds equally for non-negative solutions of (2.5.1) in $W^{1,p}_{\mathrm{loc}}(\Omega) \cap C(\Omega)$. The case when (2.5.1) is a linear inequality ($p = 2$) is of course included in the result.

The Harnack inequality immediately implies the following Strong Maximum Principle.[4]

THEOREM 2.5.1 (Strong Maximum Principle). *Assume that the conditions* (2.5.2) *are valid only for $x \in \Omega$, $0 < z \leqslant 1$ and $|\xi| \leqslant 1$. Let $u \in C^1(\Omega)$ be a* (*non-negative*) *distribution solution of* (2.5.1) *in Ω.*

Then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

PROOF. We first modify $\tilde{A}$ and $\tilde{B}$ for values $u \geqslant 1$ and $|\xi| > 1$, so that the modified functions remain in $L^\infty_{\mathrm{loc}}(\Omega)$ but now also satisfy (2.5.2) for the complete set of variables. Then, corresponding to any classical (non-negative) solution of (2.5.1) for which $u(y) = 0$, there is some neighborhood $\mathcal{N}$ of y where $u \leqslant 1$ and $|Du| \leqslant 1$.

Let B_{2R} be a ball centered at y, with R so small that B_{2R} is in $\mathcal{N}$. Then $\min_{B_R} u(x) = 0$. In turn $\|u\|_{q,B_R} = 0$ by (2.5.3). That is, $u = 0$ in B_R. The conclusion $u \equiv 0$ in Ω now follows from connectedness, see the argument in the proof of Theorem 2.1.1. □

Corresponding to Theorem 2.2.1, it is natural to seek a tangency principle which applies to C^1 solutions of divergence structure inequalities. To this end, we consider the partial differential inequalities

$$\begin{aligned} &\operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du) \geqslant 0 \quad \text{in } \Omega, \\ &\operatorname{div} \boldsymbol{A}(x, v, Dv) + B(x, v, Dv) \leqslant 0 \quad \text{in } \Omega, \end{aligned} \tag{2.5.4}$$

where $\boldsymbol{A}$ and B are, respectively, a given vector and a given scalar function. Specifically, we assume that

$$\boldsymbol{A}(x, z, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

is continuous, and continuously differentiable in the variables z and ξ; while

$$B(x, z, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \tag{2.5.5}$$

[4] The special case $a_2 = a_4 = 0$ and $B = 0$ was noted by Granlund [41].

is locally Lipschitz continuous in $\boldsymbol{\xi}$ and locally lower Lipschitz continuous in z, that is,

$$B(x, v, \boldsymbol{\eta}) - B(x, u, \boldsymbol{\xi}) \geqslant -b_1|\boldsymbol{\eta} - \boldsymbol{\xi}| - L(v - u), \quad \text{when } v > u,$$

for any compact set of arguments. The principal result of [82] is now the following

THEOREM 2.5.2 (Tangency Principle). *Let $u = u(x)$ and $v = v(x)$ be functions of class $C^1(\Omega)$, satisfying the respective differential inequalities* (2.5.4). *Suppose that $u \leqslant v$ in Ω, and that at least one of the matrices*

$$\left[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, u, Du)\right] \quad \textit{or} \quad \left[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, v, Dv)\right] \tag{2.5.6}$$

is positive definite in Ω. Then either $u \equiv v$ or else $u < v$ throughout Ω.

In [3] Almgren has obtained a related result for variational problems under somewhat less smoothness of the integrand than noted above. This generalization is paid for, however, by a weaker conclusion, namely, that either $u \equiv v$ in Ω or else the set of equality is at most *of capacity zero*. Moreover, his theorem applies only to extremals and not to differential inequalities as is the case here.

At the end of the section, we also discuss the corresponding case when the solutions u and v are strongly differentiable rather than of class C^1.

PROOF OF THEOREM 2.5.2. Let Ω' denote the subset of Ω where $u = v$. Obviously Ω' is relatively closed with respect to Ω. To complete the proof of the theorem it is therefore enough to show that Ω' is open, for then it must either be empty or coincide with Ω, since Ω is a connected set.

Thus assume that Ω' is not empty, and let y be an arbitrary point in Ω'. Obviously $u = v$ and $Du = Dv$ at y. Let B_R denote the closed ball of radius R centered at y, with $R \in (0, 1]$ so small that B_{3R} is contained in Ω. By subtracting and using the definition of weak solution we obtain, for x in B_{3R},

$$\operatorname{div}\left\{\boldsymbol{A}(x, v, Dv) - \boldsymbol{A}(x, u, Du)\right\} + B(x, v, Dv) - B(x, u, Du) \leqslant 0. \tag{2.5.7}$$

By assumption, we have in B_{3R}

$$\begin{aligned} &\left|\boldsymbol{A}(x, v, Dv) - \boldsymbol{A}(x, u, Du)\right| \leqslant a|Dw| + bw, \\ &-\left\{B(x, v, Dv) - B(x, u, Du)\right\} \leqslant b_1|Dw| + Lw, \end{aligned} \tag{2.5.8}$$

where $w = v - u \geqslant 0$ and a, b are suitable constants depending only on the structure of $\boldsymbol{A}$ and B and on bounds for u, v, Du, Dv in B_{3R}. Let us assume that the matrix $[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, u, Du)]$ in (2.5.6) is positive definite in Ω (the other case is treated similarly).

By continuity, the least eigenvalue of $[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, u, Du)]$ in B_{3R} is then positive, say equal to λ. In turn, for some vector $\boldsymbol{\zeta}$ in the line segment joining Du and Dv we have

$$\begin{aligned}\big\langle \boldsymbol{A}(x,u,Dv)-\boldsymbol{A}(x,u,Du),Dw\big\rangle &= \big\langle \partial_{\boldsymbol{\xi}}\boldsymbol{A}(x,u,\boldsymbol{\zeta})Dw,Dw\big\rangle\\ &= \big\langle \partial_{\boldsymbol{\xi}}\boldsymbol{A}(x,u,Du)Dw,\ Dw\big\rangle + \mathrm{o}\big(|Dw|^2\big)\\ &\geqslant \frac{1}{2}\lambda|Dw|^2,\end{aligned}$$

if R is taken even smaller if necessary (since $Dw=\mathbf{0}$ at y).

Again using the differentiability properties of $\boldsymbol{A}$, we have next, for $x\in B_{3R}$,

$$\begin{aligned}\big\langle \boldsymbol{A}(x,v,Dv)-\boldsymbol{A}(x,u,Du),Dw\big\rangle &= \big\langle \boldsymbol{A}(x,u,Dv)-\boldsymbol{A}(x,u,Du),Dw\big\rangle\\ &\quad+\big\langle \boldsymbol{A}(x,v,Dv)-\boldsymbol{A}(x,u,Dv),Dw\big\rangle\\ &\geqslant \frac{\lambda}{2}|Dw|^2-bw|Dw|\\ &\geqslant \frac{\lambda}{4}|Dw|^2-\frac{b^2}{\lambda}w^2 \qquad (2.5.9)\end{aligned}$$

by the Cauchy inequality.

We are now in position to apply the Harnack inequality (2.5.3). In particular, let the non-negative function $w=v-u$ be considered as a solution of the differential inequality (2.5.7), which we can write in the form (2.5.1) with w replacing u. Then in view of (2.5.8) and (2.5.9) the hypotheses (2.5.2) are satisfied with $p=2$. Consequently, since $w=0$ at y, we obtain the inequality (take $q=1$)

$$\int_{B_{2R}} w\leqslant 0.$$

Hence $w=0$ in B_{2R}. Therefore Ω' is an open set, completing the proof. □

If the continuity and differentiability hypotheses on $\boldsymbol{A}$ and B in Theorem 2.5.2 are strengthened to hold *uniformly* in their variables, one can obtain a result applying not only to C^1 solutions of (2.5.4) but even to solutions in $W^{1,2}_{\mathrm{loc}}(\Omega)\cap C(\Omega)$. Supposing also that at least one of the matrices (2.5.6) is *uniformly* positive definite in any compact subset of Ω, we have the following conclusion.

THEOREM 2.5.3. *Let $u=u(x)$ and $v=v(x)$ be solutions of* (2.5.4) *in the class $W^{1,2}_{\mathrm{loc}}(\Omega)\cap C(\Omega)$. Suppose that $u\leqslant v$ in Ω. Then either $u\equiv v$ or else $u<v$ throughout Ω.*

2.6. Uniqueness of the Dirichlet problem

A first case concerns semilinear equations in $\mathbb{R}^n$:

$$\begin{aligned}&Lu+f(x,u)=h(x) && \text{in } \Omega,\\ &u=g(x) && \text{on } \partial\Omega,\end{aligned} \qquad (2.6.1)$$

where L is the elliptic operator in Section 2.1.

THEOREM 2.6.1. *Suppose* $z \mapsto f(x,z)/z$, $z > 0$, *is a non-increasing function for each fixed* $x \in \Omega$, *and assume* $h(x) \leqslant 0$, $g(x) > 0$. *Then the Dirichlet problem* (2.6.1) *can have at most one positive solution.*

PROOF. Since $h(x) \leqslant 0$, one sees that the function $[f(x,z) - h(x)]/z$ is non-increasing in z. Let u, v be two positive solutions of (2.6.1). Since $u \leqslant v$ on $\partial\Omega$, it follows from Theorem 2.2.4 that $u \leqslant v$ in Ω. Similarly $v \leqslant u$ in Ω, and the proof is done. □

As observed earlier, the problem (2.6.1) may possibly have no positive solutions at all; one can only state that if there does exist a positive solution it is unique.

The structure built up in Section 2.4 has as a consequence several uniqueness theorems for C^1 solutions of the singular Dirichlet problem

$$\begin{aligned} &\operatorname{div} \boldsymbol{A}(x, Du) + B(x,u) = 0 \quad &&\text{in } \Omega, \\ &u = u_0 &&\text{on } \partial\Omega, \end{aligned} \tag{2.6.2}$$

where $u_0 \in C(\partial\Omega)$, Ω is a bounded domain of $\mathbb{R}^n$, and $\boldsymbol{A}$ and B are as in Section 2.4.

THEOREM 2.6.2. *Let condition* (2.4.3) *hold and assume that* B *is non-increasing in* z. *Then problem* (2.6.2) *can have at most one* $C^1(\Omega)$ *solution.*

This is an immediate consequence of Theorem 2.4.1. The special cases

$$\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\boldsymbol{\xi}, \quad \boldsymbol{A} = \boldsymbol{A}(x,\boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\boldsymbol{a}(x)\boldsymbol{\xi} \quad (\boldsymbol{A} = \boldsymbol{0} \text{ at } \boldsymbol{\xi} = \boldsymbol{0}),$$

given in Section 2.4 are of particular interest. For example, for the p-Laplace operator, $p > 1$, one has the following conclusion.

COROLLARY 2.6.3. *Let* $B = B(x,z)$ *be non-increasing in* z. *Then the Dirichlet problem*

$$\begin{aligned} &\Delta_p u + B(x,u) = 0 \quad &&\textit{in } \Omega, \\ &u = u_0 &&\textit{on } \partial\Omega, \end{aligned} \tag{2.6.3}$$

where $u_0 \in C(\partial\Omega)$, *has at most one* $C^1(\Omega)$ *solution.*

Of equal interest is a corresponding uniqueness theorem for C^1 solutions of the (non-singular) mean curvature equation (1.1.2); the formal statement can be omitted. For the restricted class of C^2 solutions this result was already noted in Section 2.1.

2.7. The Boundary Point Lemma

Hopf's tangency Theorem 2.2.1 does not apply when $u - v$ attains a maximum at a boundary point of Ω. The following boundary point theorem treats this case.

THEOREM 2.7.1. *Let $u = u(x)$ and $v = v(x)$ be solutions of the inequalities* (2.2.1) *and* (2.2.2) *in Ω, of class $C^2(\Omega) \cap C^1(\overline{\Omega})$. Assume $\boldsymbol{P} = \mathbb{R}^n$, so that in particular the coefficient matrix $[a_{ij}]$ is continuous, and continuously differentiable with respect to z and $\boldsymbol{\xi}$ in the set $K = \Omega \times \mathbb{R} \times \mathbb{R}^n$, while similarly the scalar term B is continuously differentiable with respect to $\boldsymbol{\xi}$ in K.*

Assume that at least one of the solutions u and v is elliptic, and that B is locally lower Lipschitz continuous in the variable z, as in Theorem 2.2.1.

If $u < v$ in Ω, and $v = u$ at some point y on the boundary of Ω admitting an internally tangent sphere, then

$$\partial_{\boldsymbol{\nu}} u > \partial_{\boldsymbol{\nu}} v \quad \text{at } y.$$

McNabb [52] has treated the fully nonlinear version of Theorem 2.7.1, though his assumptions, when reduced to the quasilinear case, are stronger than required here. The theorem as stated follows directly from the boundary point Theorem 2.8.4, after applying the differencing procedure of Theorem 2.2.1 to obtain an appropriate linear inequality for the function $u - v$.

When we turn to $C^1(\overline{\Omega})$ solutions u and v of (2.5.4), it is a surprising fact that the analog of Theorem 2.7.1 is no longer true. This is shown by the following example due to Gilbarg ([39], p. 169). Consider the function

$$u = u(x, y) = x\mathrm{e}^{-\sqrt{|\log 4/r|}}, \quad r^2 = x^2 + y^2,$$

where $n = 2$ and Ω is the domain $(x - 1)^2 + y^2 = 1$ in the (x, y)-plane. This function is of class C^1 in the closure of Ω and satisfies there the linear elliptic equation $\operatorname{div} \boldsymbol{A}(x, y, Du) = 0$, where

$$\boldsymbol{A}(x, y, Du) = (a\partial_x u + b\partial_y u, b\partial_x u + c\partial_y u)$$

with continuous coefficients

$$a = \frac{1}{\mu} + \frac{\mu^2 - 1}{r^2\mu} y^2, \quad b = \frac{1 - \mu^2}{r^2\mu} xy, \quad c = \frac{1}{\mu} + \frac{\mu^2 - 1}{r^2\mu} x^2,$$

and $\mu = 1 + (2\sqrt{|\log 4/r|})^{-1}$. Clearly $u > 0$ in Ω, but u and Du are zero at the origin, contradicting the conclusion of Theorem 2.7.1.

In spite of this negative result, there are nevertheless two related results, analogous to Theorem 2.7.1 but applying to C^1 solutions of the divergence structure inequalities (2.5.4).

In the first case, the boundary point lemma holds for $C^1(\overline{\Omega})$ solutions of (2.5.4) *when $\boldsymbol{A}(x, \boldsymbol{\xi})$ is linear in $\boldsymbol{\xi}$ and continuously differentiable in x, and B satisfies condition* (2.5.5).[5] This is a consequence of Hopf's construction (Lemma 2.8.2), together with the comparison principle Theorem 2.4.1. The proof can be left to the reader. Whether the condition of linearity can be avoided is an open question.

[5] A particular case of interest is the model Poisson equation $\Delta u + f(u) = 0$ when $f(u)$ is a locally Lipschitz continuous function.

For convenience in stating the second result, we shall say that a boundary point y of Ω admits *an internal cone condition* provided there exists a right circular cone V with height h and vertex y which is contained in Ω.

THEOREM 2.7.2. *Let $u = u(x)$ and $v = v(x)$ be functions of class $C^1(\overline{\Omega})$, satisfying the respective inequalities* (2.5.4). *Suppose that $u < v$ in Ω, and that at least one of the solutions is elliptic in Ω.*

Assume finally that $u = v$ at some point y on the boundary of Ω admitting an internal cone condition. Then the zero of $u - v$ at y is of finite order.

PROOF. Assume for contradiction that $u - v$ has a zero of infinite order at y. Then $Du = Dv$ at y and the estimates (2.5.8) and (2.5.9) hold in the associated cone V (we may, of course, suppose that $h > 0$ is suitably small).

We can therefore apply the Harnack inequality to the positive function $w = v - u$ in any ball contained in V. This being the case, let us consider in particular a sequence of balls $B(y, \varrho)$, each of which is internally tangent to V and whose successive centers $y = y_{\varrho_k}$ and radii $\varrho = \varrho_k$ are such that

$$B\left(y_k, \frac{\varrho_k}{3}\right) \subset B\left(y_{k+1}, \frac{2\varrho_{k+1}}{3}\right), \qquad k = 0, 1, 2, \ldots.$$

If ϑ is the half-angular opening of V, it is easy to see that the successive radii and centers can be chosen to satisfy the relation

$$\frac{\varrho_{k+1}}{\varrho_k} = \frac{|y_{k+1}|}{|y_k|} = \frac{1 + (1/3)\sin\vartheta}{1 + (2/3)\sin\vartheta} = \kappa < 1,$$

so that the sequence $B(y_k, \varrho_k)$ converges to y (for convenience we assume that y is the origin).

By Theorem 1.2 of [95], see Section 2.5, there exists a constant C such that

$$\varrho^{-n} \int_{B(y,2\varrho/3)} w \leqslant C \min_{B(y,\varrho/3)} w(x)$$

for any ball $B(y, \varrho)$ in the sequence. On the other hand, for the ball $B(y', \varrho')$ preceding $B(y, \varrho)$ in the sequence we have (since $w > 0$)

$$\min_{B(y',\varrho'/3)} w(x) \leqslant \frac{3^n}{\omega_n \varrho'^n} \int_{B(y',\varrho'/3)} w \leqslant \frac{3^n}{\omega_n \varrho^n} \int_{B(y,2\varrho/3)} w,$$

where ω_n denotes the volume of the unit ball in n dimensions. Combining the last two inequalities now yields

$$\min_{B(y,\varrho/3)} w(x) \geqslant L \min_{B(y',\varrho'/3)} w(x),$$

where $L = \omega_n/3^n C$. If this relation is iterated backward to successively larger radii ϱ, we find easily that

$$\min_{B(y_k,\varrho_k/3)} w(x) \geqslant L^k \min_{B(y_0,\varrho_0/3)} w(x),$$

whence $w(y_k) \geqslant \text{const.}\, L^k$ for some positive constant and all positive integers k.

Now, by assumption, w has a zero of infinite order at y. Hence for any integer m there exists a constant $c(m)$ such that

$$w(y_k) \leqslant c(m)|y_k|^m = c(m)|y_0|^m \kappa^{mk}.$$

By combining the preceding two inequalities we obtain

$$\text{const.} L^k \leqslant c(m)|y_0|^m \kappa^{mk}.$$

Letting k tend to infinity there results finally

$$\kappa^m \geqslant L,$$

which is impossible for sufficiently large m, since $\kappa < 1$. This completes the proof. □

It is evident from the proof that one could determine an upper bound for the order of the zero at y depending on the structure of the coefficients $\boldsymbol{A}$ and B near the solution $u(x)$, namely, $m < \log L/|\log \kappa|$. We also note that an alternate proof of Theorem 2.7.2, in the case when equality holds in both relations (2.5.4), can be given on the basis of a result of Widman [99], though the proof as a whole would then be considerably more involved.

If the hypotheses on $\boldsymbol{A}$ and B are strengthened as in the last part of Section 2.6, then we can drop the condition that u and v are of class C^1. Specifically, in this case the following result holds.

THEOREM 2.7.3. *Let $u = u(x)$ and $v = v(x)$ be continuous functions in the closure of $\overline{\Omega}$, possessing strong derivatives of class $L^2_{\text{loc}}(\Omega)$. Suppose that $u \leqslant v$ in Ω and that* (2.5.4) *holds. Assume finally that $u = v$ at some point y on the boundary of Ω, admitting an internal cone condition. Then either $u \equiv v$ or else $u < v$ in Ω and the zero of $u - v$ at y is of finite order.*

PROOF. Since (2.5.8) and (2.5.9) are valid in the present circumstances (see the demonstration of Theorem 2.5.3), the result follows exactly as in the proof of Theorem 2.7.2. □

2.8. Appendix: Proof of Eberhard Hopf's Maximum Principle

We begin with a simple but striking consequence of elementary calculus.

THEOREM 2.8.1 (Weak Maximum Principle). *Let $u = u(x)$ be a C^2 function which satisfies the differential inequality*

$$Lu = \sum_{i,j} a_{ij}(x)\partial^2_{x_i x_j} u + \sum_i b_i(x)\partial_{x_i} u > 0$$

in a domain Ω, where the (symmetric) matrix $[a_{ij}]$ is positive semi-definite in Ω, but otherwise the coefficients a_{ij}, b_i are merely defined and finite at each point of Ω.

Then u cannot achieve an (interior) maximum in Ω. In particular, if $u \leqslant M$ on $\partial\Omega$, then $u \leqslant M$ in Ω.

PROOF. If u reached the maximum value M at a point $y \in \Omega$, then since Ω is open we would have $Du(y) = \mathbf{0}$, while by elementary calculus the Hessian matrix $[\partial^2_{x_i x_j} u(y)]$ would be negative semi-definite, so that $\sum_{i,j} a_{ij}(y)\partial^2_{x_i x_j} u(y) \leqslant 0$, i.e. $Lu(y) \leqslant 0$, a contradiction. □

LEMMA 2.8.2. *Let B_R be an arbitrary open ball of radius R in the domain Ω. Suppose that the (symmetric) matrix $[a_{ij}] = [a_{ij}(x)]$ is uniformly positive definite in B_R and the coefficients $a_{ij}, b_i = b_i(x)$ are uniformly bounded in B_R. Then for every constant $m > 0$ there exists a function $v \in C^2(\overline{B_R})$ such that*

(i) $v = 0$ *on* ∂B_R;
(ii) $v = m$ *on* $\partial B_{R/2}$;
(iii) $\partial_{\boldsymbol{\nu}} v < 0$ *on* ∂B_R, *where* $\boldsymbol{\nu}$ *is the exterior unit normal to* B_R;
(iv) $Lv > 0$ *in* $B_R \setminus \overline{B_{R/2}}$.

PROOF. For a constant exponent $\alpha > 0$ still to be determined, we define

$$\tilde{v}(x) = \mathrm{e}^{-\alpha r^2} - \mathrm{e}^{-\alpha R^2}, \quad x \in B_R, \tag{2.8.1}$$

where r denotes the distance from x to the center of B_R. Then

$$L\tilde{v}(x) = \mathrm{e}^{-\alpha r^2}\left\{4\alpha^2 \sum_{i,j} a_{ij}(x)x_i x_j - 2\alpha \sum_i \big[a_{ii}(x) + b_i(x)x_i\big]\right\},$$

where for simplicity we have taken the center of B_R as the origin 0 and $r = |x|$. Since by hypothesis $\sum_{i,j} a_{ij}(x)x_i x_j \geqslant \lambda r^2$, the constant α can be chosen so large that $L\tilde{v}(x) > 0$ for all x with $r = |x| \geqslant R/2$. Thus conditions (i), (iii) and (iv) hold for $\tilde{v}$. Define $v(x) = m\tilde{v}(x)/\tilde{v}(R/2)$, $x \in B_R$. Then v satisfies (ii) and of course continues to verify (i), (iii) and (iv). □

THEOREM 2.8.3 (Hopf's Boundary Point Lemma). *Suppose that the (symmetric) matrix $[a_{ij}] = [a_{ij}(x)]$ is uniformly positive definite in the domain Ω and that the coefficients a_{ij}, $b_i = b_i(x)$ are uniformly bounded in Ω. Let $u \in C^2(\Omega)$ satisfy the differential inequality $Lu \geqslant 0$ in Ω and let $x_0 \in \partial\Omega$ be such that*

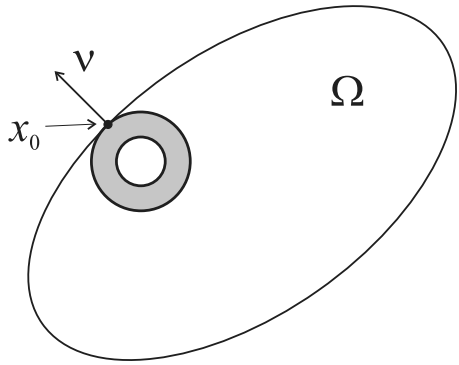

Fig. 2. Proof of the Boundary Point Theorem; the annular region $B_R \setminus \overline{B_{R/2}}$ is shaded.

(i) *u is continuous at x_0 and $\partial_{\boldsymbol{\nu}} u$ exists at x_0, where $\boldsymbol{\nu}$ is the outer normal vector to Ω at x_0;*
(ii) *$u(x) < u(x_0)$ for all $x \in \Omega$;*
(iii) *there exists a ball $B_R \subset \Omega$, with $x_0 \in \partial B_R$ (interior sphere condition).*

Then $\partial_{\boldsymbol{\nu}} u(x_0) > 0$.

PROOF. Let $u(x_0) = M$ and $\ell = \sup_{|x|=R/2} u(x) < M$. The function $w = u + v - M$ then satisfies $Lw > 0$ in $B_R \setminus \overline{B_{R/2}}$, while also $w \leqslant 0$ on ∂B_R and $\partial B_{R/2}$, provided $m = M - \ell$.

Consequently $w \leqslant 0$ in $B_R \setminus \overline{B_{R/2}}$ by Theorem 2.8.1, so that $\partial_{\boldsymbol{\nu}} w(x_0) \geqslant 0$. In turn $\partial_{\boldsymbol{\nu}} u(x_0) \geqslant -\partial_{\boldsymbol{\nu}} v(x_0) > 0$. □

PROOF OF HOPF'S MAXIMUM PRINCIPLE. Suppose u takes a maximum value M in Ω. The subset Ω_0 of Ω where $u = M$ is then non-empty and relatively closed in Ω. We must show that $\Omega_0 = \Omega$.

Thus suppose for contradiction that $\Omega_0 \neq \Omega$. By the connectedness of Ω it follows that the set $\partial\Omega_0 \cap \Omega$ must be non-empty (otherwise Ω_0 would be open as well as closed, and thus identical to Ω).

Fix $x_1 \in \partial\Omega_0 \cap \Omega$, and in turn let $\mathbf{0}$ be a point of Ω, as near to x_1 as we like, such that $u(\mathbf{0}) < M$. Taking $\mathbf{0}$ nearer to x_1 than to $\partial\Omega$, it follows that there is a largest open ball B_R in $\mathbb{R}^n$, with center at $\mathbf{0}$, which does not intersect Ω_0. Obviously $\overline{B}_R \subset \Omega$, so that in particular $u < M$ in B_R and $u = M$ at some point x_0 on the boundary of both B_R and Ω_0.

But then $\partial_{\boldsymbol{\nu}} u(x_0) > 0$ by the boundary point Theorem 2.8.3. At the same time, x_0 is an *interior* maximum point of u; hence $Du(x_0) = \mathbf{0}$, an immediate contradiction. Thus $\Omega_0 = \Omega$, completing the proof. □

The function $\tilde{v}$ in (2.8.1) was introduced by Hopf in [42]. An elegant alternative to $\tilde{v}$ is

$$\hat{v}(x) = r^{-\alpha} - R^{-\alpha}, \quad \alpha > 0.$$

In fact

$$L\hat{v}(x) = \alpha r^{-\alpha}\left\{(\alpha+2)\sum_{i,j} a_{ij}(x)x_i x_j - \sum_i [a_{ii}(x) + b_i(x)x_i]r^2\right\},$$

which is clearly positive in $B_R \setminus \overline{B_{R/2}}$ for suitably large α, as required.

The techniques used for the proof of the boundary point lemma yield another result of interest.

THEOREM 2.8.4. *Let the hypotheses of Theorem* 2.8.3 *hold, with the exception that* (a) *the inequality* $Lu \geqslant 0$ *is replaced by* $[L + c(x)]u \geqslant 0$, *where* c *is bounded below in a neighborhood of* x_0, *and* (b) *either* $u(x_0) = 0$ *or* $u(x_0) > 0$ *and* $c(x) \leqslant 0$. *Then*

$$\partial_\nu u(x_0) > 0.$$

PROOF. Consider first the case when $u(x_0) = 0$. Let d be a positive constant. From the proof of Lemma 2.8.2 it is easy to see that if the constant α is chosen even larger if necessary, then the function v given in Lemma 2.8.2 can equally be supposed to satisfy

(iv′) $(L - d)v > 0$ in $E_R \equiv B_R \setminus \overline{B_{R/2}}$.

In turn $L(u + v) > -cu + dv$ in E_R.

As in the proof of Theorem 2.8.3, put $\ell = \sup_{|x|=R/2} u(x) < 0$. We claim that $u + v \leqslant 0$ in E_R. In fact, obviously $u + v \leqslant 0$ on $\partial B_R \cup \partial B_{R/2} = \partial E_R$, provided that $m = -\ell$. If the claim were false, there would be a point $y \in E_R$ at which $u + v$ would attain a positive maximum. Then we would have

$$L(u + v) > -(c + d)u > 0 \quad \text{at } y, \tag{2.8.2}$$

provided d is chosen so that $\inf_{x \in E_R} c(x) + d > 0$ (recall that c is bounded below in a neighborhood of x_0 and $u < 0$ in Ω). On the other hand, as in the proof of the weak maximum principle Theorem 2.8.1, we have necessarily $L(u + v) \leqslant 0$ at y, a contradiction with (2.8.2).

Thus $u + v \leqslant 0$ in E_R and in turn, since $u + v = 0$ at x_0, we obtain $\partial_\nu u(x_0) \geqslant -\partial_\nu v(x_0) > 0$, as required.

When $u(x_0) = M > 0$ we define $w = u - M$. Then $w(x_0) = 0$ and $[L + c(x)]w \geqslant -Mc(x) \geqslant 0$. The previous argument therefore yields $\partial_\nu u(x_0) = \partial_\nu w(x_0) > 0$. □

COROLLARY 2.8.5. *Let the hypotheses of Theorem* 2.8.4 *hold, with the exception that in condition* (ii) *of Theorem* 2.8.3 *one assumes only that* $u(x) \leqslant u(x_0)$ *for* $x \in \Omega$. *Then either*

$$u \equiv u(x_0) \quad \text{in } \Omega, \quad \text{or} \quad \partial_\nu u(x_0) > 0.$$

PROOF. By Theorems 2.1.1 and 2.1.2, if $u \leqslant u(x_0)$ in Ω then either $u \equiv u(x_0)$ or $u < u(x_0)$ in Ω. The conclusion then follows from Theorem 2.8.4. □

Notes

The results in Section 2.1 are due to Eberhard Hopf. They are stated, however, in greater generality and in more usual notation. Theorems 2.2.1 and 2.2.3 are variants of Hopf's results; they are, however, new in the form given. The maximum principle Theorem 2.3.2 corresponds to Theorem 10.3 of [40], though again formulated for the case of singular inequalities.

For maximum principles when u is not of class C^2, and even possibly only measurable, see e.g. Littman [50] and Chapter 2 of Fraenkel [35]. For the case of distribution solutions, see Sections 2.4, 2.5 and Section 3.1.

The results of Section 2.4 are for the most part new, especially Proposition 2.4.4. The tangency principle Theorem 2.5.2 is due to Serrin [82]. The uniqueness Theorem 2.6.1 seems to be new. The proofs in Section 2.7 also follow those of [82].

The proof of the Hopf maximum principle in Section 2.8 is a streamlined version of that in [40]. The boundary point lemma, Theorem 2.8.3, appears first in [43]; see also Oleinik [60].

When the matrix $[a_{ij}]$ is semidefinite rather than positive definite, many of Hopf's results remain valid in appropriately modified and weakened forms, see [55].

SECTION 6.3

Maximum Principles for Divergence Structure Elliptic Differential Inequalities

3.1. Distribution solutions

For a large number of divergence structure equations, including equations which involve the important p-Laplacian operator Δ_p, there is a further series of Maximum Principles. In particular, in this chapter we study the differential inequality

$$\operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du) \geqslant 0 \quad \text{in } \Omega, \tag{3.1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ (unless otherwise stated explicitly), and

$$\boldsymbol{A}(x, z, \boldsymbol{\xi}) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \qquad B(x, z, \boldsymbol{\xi}) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}.$$

Throughout the chapter, by a *solution u of* (3.1.1) *in Ω* we mean specifically a distribution or weak solution, in the sense that $u \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable in Ω (that is, all its weak derivatives of first order exist); $\boldsymbol{A}(\cdot, u, Du)$, $B(\cdot, u, Du) \in L^1_{\mathrm{loc}}(\Omega)$; and

$$\int_\Omega \big\langle \boldsymbol{A}(x, u, Du), D\varphi \big\rangle \leqslant \int_\Omega B(x, u, Du)\varphi \tag{3.1.2}$$

for all $\varphi \in C^1(\Omega)$ such that $\varphi \geqslant 0$ in Ω and $\varphi \equiv 0$ near $\partial\Omega$.

In order to treat solutions in the natural Sobolev space $W^{1,p}(\Omega)$ we shall require several preliminary results. We say that u is a *p-regular solution*, $p \geqslant 1$, if also[6]

$$\boldsymbol{A}(\cdot, u, Du) \in L^{p'}_{\mathrm{loc}}(\Omega), \quad p' = p/(p-1). \tag{3.1.4}$$

Furthermore by $u \leqslant M$ on $\partial\Omega$ for some $M \in \mathbb{R}$ we mean explicitly that for every $\delta > 0$ there is a neighborhood of $\partial\Omega$ in which $u \leqslant M + \delta$.

For simplicity in printing, we shall write $\|\cdot\|_{\nu,\Gamma}$ for $\|\cdot\|_{L^\nu(\Gamma)}$ when Γ is a measurable subset of Ω, and $\|\cdot\|_\nu$ for $\|\cdot\|_{L^\nu(\Omega)}$.

[6]Condition (3.1.4) is obviously satisfied when $u \in W^{1,p}(\Omega)$ under the "*natural*" additional condition that, for all $(x, z, \boldsymbol{\xi})$ in $\Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$\big|\boldsymbol{A}(x, z, \boldsymbol{\xi})\big| \leqslant a_3|\boldsymbol{\xi}|^{p-1} + a_4|z|^{p-1} + a_5, \tag{3.1.3}$$

where a_3, a_4 are constants and $a_5 \in L^{p'}_{\mathrm{loc}}(\Omega)$. The condition of p-regularity was noted in [81].

The principal requirement that $\boldsymbol{A}(\cdot, u, Du)$, $B(\cdot, u, Du) \in L^1_{\mathrm{loc}}(\Omega)$ can be met if for example one assumes, in addition to (3.1.3), a corresponding condition on $B(x, z, \boldsymbol{\xi})$ and that both $\boldsymbol{A}(\cdot, u, Du)$, $B(\cdot, u, Du)$ are measurable.

LEMMA 3.1.1. *Let f_h be the regularization (mollification) of a function $f \in L^p(\Omega)$, $p \geqslant 1$, with mollification radius h.*

Then $\|f_h - f\|_p \to 0$ as $h \to 0$; also a subsequence of $(f_h)_h$, which by agreement we identify as $(f_h)_h$, converges a.e. in Ω. Moreover if $f \in L^1(\Omega)$ and f is weakly differentiable in Ω, then in any domain $\Omega' \Subset \Omega$ we have $Df_h = [Df]_h$ for h sufficiently small.

For the proof of this lemma we refer to Lemmas 7.2 and 7.3 of [40].

LEMMA 3.1.2. *Let $\psi : \mathbb{R} \to \mathbb{R}_0^+$ be a non-decreasing continuous function such that $\psi(t) = 0$ for $t \in (-\infty, \ell]$ and $\psi \in C^1$ for $t \in [\ell, \infty)$, with a possible corner at $t = \ell$ and with ψ' uniformly bounded. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a p-regular solution of* (3.1.1), *and suppose that $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is such that $f \leqslant \ell' < \ell$ on $\partial\Omega$.*

Then (3.1.2) *is valid for $\varphi = \psi(f)$, in the sense that*

$$\int_\Omega \big\langle \boldsymbol{A}(x, u, Du), D\varphi \big\rangle \leqslant \int_\Omega \big[B(x, u, Du)\big]^+ \varphi, \tag{3.1.5}$$

where $D\varphi = \psi'(f)Df$ when $f \neq \ell$, and $D\varphi = \mathbf{0}$ a.e. when $f = \ell$.

PROOF. The last line is a consequence of [40, Lemma 7.8].

Let $\varphi_N = \psi_N(f)$ be the truncation of $\psi(f)$ at the level $N > \ell$, that is, equal to $\psi(f)$ when $f < N$ and to $\psi(N)$ when $f \geqslant N$. By the properties of ψ and the fact that $f < \ell$ on $\partial\Omega$ it is clear that $\varphi_N \in W^{1,p}(\Omega)$ with $\varphi_N \equiv 0$ near $\partial\Omega$.

The regularization $\varphi_{N,h}$ of φ_N is in $C^1(\Omega)$ and vanishes near $\partial\Omega$ for h sufficiently small, and of course also $\varphi_{N,h} \geqslant 0$. Thus $\varphi_{N,h}$ can serve as a test function for (3.1.2), that is, by (3.1.2),

$$\int_\Omega \big\langle \boldsymbol{A}(x, u, Du), D\varphi_{N,h} \big\rangle \leqslant \int_\Omega \big[B(x, u, Du)\big]^+ \varphi_{N,h}. \tag{3.1.6}$$

By Lemma 3.1.1 we have $D\varphi_{N,h} = [D\varphi_N]_h$ for h sufficiently small; therefore

$$\|D\varphi_{N,h} - D\varphi_N\|_p \to 0, \quad \varphi_{N,h} \to \varphi_N \quad \text{a.e. in } \Omega \tag{3.1.7}$$

as $h \to 0$.

Clearly $A(\cdot, u, Du) \in L^{p'}_{\mathrm{loc}}(\Omega)$ and $[B(x, u, Du)]^+ \varphi_{N,h} \leqslant N[B(x, u, Du)]^+$ a.e. in Ω. Thus we can apply (3.1.7) to the left side of (3.1.6) and the dominated convergence theorem to the right side, since $[B(\cdot, u, Du)]^+ \in L^1_{\mathrm{loc}}(\Omega)$. Hence for $h \to 0$ one gets

$$\int_\Omega \big\langle \boldsymbol{A}(x, u, Du), D\varphi_N \big\rangle \leqslant \int_\Omega \big[B(x, u, Du)\big]^+ \varphi_N. \tag{3.1.8}$$

Finally

$$\|D\varphi_N - D\varphi\|_p = \|D\varphi\|_{p,\{f \geqslant N\}} \to 0$$

as $N \to \infty$. Using the monotone convergence theorem (since $\varphi_N \nearrow \varphi$) proves the lemma. □

The integral $\int B^+\varphi$ in (3.1.5) can at the moment possibly be infinite, though in our applications in the sequel it will in fact prove to be finite.

LEMMA 3.1.3. *Lemma* 3.1.2 *applies to solutions* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *with* p*-regularity no longer being required.*

The proof is essentially the same, with the exception that (3.1.7) is replaced by $D\varphi_{N,h} \to D\varphi_N$ a.e. in Ω as $h \to 0$, while by the definition of weak solution we have $A(\cdot, u, Du) \in L^1_{\mathrm{loc}}(\Omega)$.

APPENDIX. The condition of p-regularity is necessary for the demonstration of Lemma 3.1.2. The delicacy of the structure can be emphasized by observing first that Gilbarg and Trudinger define weak solutions exactly as we do here (see equation (8.30) in [40]), while in their following Theorem 8.15 (for the case of linear equations) they consider solutions in $W^{1,2}(\Omega)$, these being 2-regular by linearity and so legitimate in forming test functions.

On the other hand, for Lemma 10.8 [40, page 273] their solution is assumed to be in $C^1(\Omega)$, so one then must have $\boldsymbol{A}(\cdot, u, Du) \in L^1_{\mathrm{loc}}(\Omega)$ in order to use the theory of weak solutions. While not explicitly indicated in Lemma 10.8, this condition can be obtained from their earlier remark (page 260) that $\boldsymbol{A}$ is a differentiable function. But, once this is assumed, their structure condition (10.23) no longer applies, except when the exponent $p = 2$! There seems no way to avoid this dilemma other than giving up the differentiability of $\boldsymbol{A}$ and setting conditions so that $\boldsymbol{A}(\cdot, u, Du) \in L^1_{\mathrm{loc}}(\Omega)$, say that $\boldsymbol{A}$ is continuous in all its variables.

Even here, however, one must also deal with their later statement that solutions can be allowed in the space $W^{1,p}(\Omega)$, see [40, page 277]. This in turn requires the p-regularity condition $\boldsymbol{A}(\cdot, u, Du) \in L^{p'}_{\mathrm{loc}}(\Omega)$, a condition which is not indicated in [40].

Of course, this begs the question, under what conditions can one in fact obtain $\boldsymbol{A}(\cdot, u, Du) \in L^{p'}_{\mathrm{loc}}(\Omega)$ when $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$? The simplest (though not the only) answer is found in the footnote above.

3.2. Maximum principles for homogeneous inequalities

Let the functions $\boldsymbol{A}$ and B in (3.1.1) be defined in the set $\Omega \times \mathbb{R}^+ \times \mathbb{R}^n$, and satisfy an alternative version of the natural p-homogeneous structure condition (2.5.2); that is, there are constants $a_1 > 0$ and $a_2, b_1, b_2 \geqslant 0$ such that for all $(x, z, \boldsymbol{\xi}) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n$ there holds

$$\big\langle \boldsymbol{A}(x, z, \boldsymbol{\xi}), \boldsymbol{\xi} \big\rangle \geqslant a_1 |\boldsymbol{\xi}|^p - a_2 z^p, \qquad B(x, z, \boldsymbol{\xi}) \leqslant b_1 |\boldsymbol{\xi}|^{p-1} + b_2 z^{p-1}, \tag{3.2.1}$$

where $p \in [1, \infty)$ describes the level of homogeneity of $\boldsymbol{A}$ and B. In particualr, the case $p = 2$ covers linear elliptic inequalities.

THEOREM 3.2.1 (Maximum principle). *Assume* $\boldsymbol{A}$ *and* B *satisfy* (3.2.1), *with* $a_2 = b_2 = 0$. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > 1$, *be a* p*-regular solution of* (3.1.1). *If* $u \leqslant M$ *on* $\partial\Omega$ *for some* $M \geqslant 0$, *then* $u \leqslant M$ *a.e. in* Ω.

PROOF. Since $a_2 = b_2 = 0$ it is enough to consider the case $M = 0$. Thus assume for contradiction that $\operatorname{essup}_\Omega u > 0$.

Let $\operatorname{essup}_\Omega u = V$. For $V < \infty$ fix $\ell \in (V/2, V)$ and for $V = \infty$ take $\ell > 1$. Define $\psi(t) = (t - \ell)^+$ and, as in Lemma 3.1.2, take $\varphi = \psi(u)$ as a (non-negative) test function for the inequality (3.1.1). Let

$$\Gamma' = \big\{x \in \Omega\colon\ \ell < u(x)\big\}.$$

Then since $\varphi = 0$, $D\varphi = \mathbf{0}$ a.e. in $\Omega \setminus \Gamma'$ and $D\varphi = Du$ in Γ', we see from (3.1.5) that

$$\int_{\Gamma'} \big\langle \boldsymbol{A}(x, u, Du), Du\big\rangle \leqslant \int_{\Gamma'} \big[B(x, u, Du)\big]^+ \varphi.$$

Observing that $u > 0$ at all points where $\varphi > 0$, we can apply (3.2.1) with $a_2 = b_2 = 0$ to get

$$a_1 \int_{\Gamma'} |Du|^p \leqslant b_1 \int_{\Gamma'} |Du|^{p-1} \cdot \varphi. \tag{3.2.2}$$

Introduce the further set

$$\Gamma = \big\{x \in \Omega\colon\ \ell < u(x) < V\big\}. \tag{3.2.3}$$

We assert that (3.2.2) holds equally with the integration set Γ' replaced by Γ. If $V = \infty$ this is trivial. On the other hand if $V < \infty$ then

$$\int_{\Gamma'} |Du|^{p-1} \cdot \varphi = \left(\int_{\Gamma} + \int_{\{u=V\}} + \int_{\{u>V\}}\right) |Du|^{p-1} \cdot \varphi = \int_{\Gamma} |Du|^{p-1} \cdot \varphi$$

since $Du = 0$ a.e. where $u = V$ ([40], Lemma 7.7) while the set where $u > V$ has measure zero. Of course, in the same way $\|Du\|_{p,\Gamma'} = \|Du\|_{p,\Gamma}$, so the assertion is proved.

Put $s = p^* = pn/(n-p)$ if $1 \leqslant p < n$ and $s = 2p$ if $p \geqslant n$. Then replacing Γ' by Γ in (3.2.2) and applying Hőlder's inequality to the right side yields

$$a_1 \|Du\|^p_{p,\Gamma} \leqslant b_1 |\Gamma|^{1/p - 1/s} \|\varphi\|_{s,\Gamma} \|Du\|^{p-1}_{p,\Gamma}. \tag{3.2.4}$$

We claim next that $\|Du\|_{p,\Gamma} > 0$. Indeed by Poincaré's inequality (Theorem 3.10.4)

$$\|\varphi\|_{p,\Omega} \leqslant Q \|D\varphi\|_{p,\Omega} = Q \|Du\|_{p,\Gamma'} = Q \|Du\|_{p,\Gamma}.$$

But $\|\varphi\|_{p,\Omega} = \|u\|_{p,\Gamma'} > 0$, since $\ell < \operatorname{essup}_\Omega u$, proving the claim. Now dividing (3.2.4) by $\|Du\|_{p,\Gamma}^{p-1}$ gives

$$a_1 \|Du\|_{p,\Gamma} \leqslant b_1 |\Gamma|^{1/p-1/s} \|\varphi\|_{s,\Gamma}. \tag{3.2.5}$$

Because φ vanishes near $\partial\Omega$, we have by Sobolev's inequality (see Theorem 3.10.3)

$$\|\varphi\|_s \leqslant C\|D\varphi\|_p = C\|Du\|_{p,\Gamma} \leqslant (b_1/a_1)C|\Gamma|^{1/p-1/s}\|\varphi\|_s,$$

where $C = S(n, p; s)|\Omega|^{1/n-1/p+1/s}$. Dividing by $\|\varphi\|_s (> 0)$ gives finally

$$a_1 \leqslant C|\Gamma|^{1/p-1/s} b_1; \tag{3.2.6}$$

here $1/p - 1/s = 1/n$ if $p < n$ and $1/p - 1/s = 1/2p$ if $p \geqslant n$. But $\Gamma \to \emptyset$ as $\ell \to V$ by (3.2.3). This contradicts (3.2.6) and completes the proof. □

THEOREM 3.2.2 (Maximum principle). *Assume that* $\boldsymbol{A}$ *and* B *satisfy* (3.2.1) *with* $b_1 = b_2 = 0$. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > 1$, *be a* p*-regular solution of the inequality* (3.1.1) *in* Ω. *If* $u \leqslant 0$ *on* $\partial\Omega$, *then* $u \leqslant 0$ *a.e. in* Ω.

PROOF. Assume for contradiction that $V = \operatorname{essup}_\Omega u > 0$, possibly infinite. For $\varepsilon > 0$ define $\psi(t) = 0$ when $t \leqslant \varepsilon$ and

$$\psi(t) = 1 - \left(\frac{\varepsilon}{t}\right)^{p-1}$$

for $t \geqslant \varepsilon$. Lemma 3.1.2 applies with $\ell = \varepsilon$, so that $\varphi = \psi(u)$ can be used as a (non-negative) test function for (3.1.1). That is, by (3.1.5) with $b_1 = b_2 = 0$,

$$\int_\Gamma \big\langle \boldsymbol{A}(x, u, Du), D\varphi \big\rangle \leqslant 0, \tag{3.2.7}$$

where $\Gamma = \{x \in \Omega\colon\ u(x) > \varepsilon\}$. Using the relations

$$D\varphi = \psi'(u)Du, \quad \psi'(u) = (p-1)\varepsilon^{p-1}u^{-p} \quad \text{a.e. in } \Gamma,$$

we obtain from (3.2.7) and (3.2.1), after dividing by $(p-1)\varepsilon^{p-1}$,

$$0 \geqslant \int_\Gamma \frac{\langle \boldsymbol{A}(x, u, Du), Du\rangle}{u^p} \geqslant \int_\Gamma \frac{a_1|Du|^p - a_2 u^p}{u^p}, \tag{3.2.8}$$

that is

$$a_1 \int_\Gamma |D \log u|^p \leqslant a_2 |\Gamma|. \tag{3.2.9}$$

Define $\varphi_1(x) = \log(u(x)/\varepsilon)$ if $u(x) > \varepsilon$ and $\varphi_1(x) = 0$ if $u(x) \leqslant \varepsilon$. As in the proof of Theorem 3.1.1 it is clear that φ_1 is in $W^{1,p}(\Omega)$. Moreover, since $\varphi_1 = 0$ in $\Omega \setminus \Gamma$, it then follows from Sobolev's inequality (Theorem 3.10.3) that

$$\|\varphi_1\|_{s,\Omega} \leqslant C\|D\varphi_1\|_{p,\Omega} = C\left\|D\log\frac{u}{\varepsilon}\right\|_{p,\Gamma}, \tag{3.2.10}$$

where, as before, $s = p^*$ if $p < n$ and $s \in (p, \infty)$ if $p \geqslant n$.

Now take $\varepsilon \leqslant \min\{1, V/2\}$, and define

$$\Sigma = \begin{cases} \{x \in \Omega\colon\ V/2 \leqslant u(x) \leqslant V\}, & \text{when } V < \infty, \\ \{x \in \Omega\colon\ u(x) \geqslant 1\}, & \text{when } V = \infty. \end{cases}$$

In the first case, since $\varphi_1 \geqslant \log(V/2\varepsilon)$ in Σ, we find from (3.2.9) and (3.2.10) that

$$|\Sigma|^{1/s}\log\frac{V}{2\varepsilon} \leqslant C\left(\frac{a_2}{a_1}|\Gamma|\right)^{1/p},$$

which gives a contradiction as $\varepsilon \to 0$ (since Σ is independent of ε and $|\Gamma| \leqslant |\Omega|$). In the second, similarly, since $\varphi_1 \geqslant \log(1/\varepsilon)$ in Σ,

$$|\Sigma|^{1/s}\log\frac{1}{\varepsilon} \leqslant C\left(\frac{a_2}{a_1}|\Gamma|\right)^{1/p},$$

and again there is a contradiction as $\varepsilon \to 0$. □

REMARKS.

1. An alternative formulation of the boundary condition requires that $(u - M)^+ \in W_0^{1,p}(\Omega)$. In this case, (3.1.4) must be strengthened to

   $$\boldsymbol{A}(\cdot, u, Du), B(\cdot u, Du) \in L^{p'}(\Omega),$$

   and corresponding changes are needed for the following proofs.
2. It is obvious that condition (3.2.1) in the previous theorems needs to be valid only for the range of values $u(x)$, $Du(x)$, $x \in \Omega$. We shall take advantage of this remark in later sections where it is assumed that $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ rather than $u \in W^{1,p}_{\text{loc}}(\Omega)$.
3. If Ω is unbounded and the boundary condition is understood to include the limit relation

   $$\limsup_{|x|\to\infty,\, x\in\Omega} u(x) \leqslant M, \tag{3.2.11}$$

   then the conclusions of Theorems 3.2.1 and 3.2.2 continue to hold.

THEOREM 3.2.3. *In Theorem* 3.2.1 *the coefficient* b_1 *can be taken in an appropriate Lebesgue space, that is*

$$b_1 \in \begin{cases} L^{n/(1-\theta)}_{\mathrm{loc}}(\Omega), & \text{when } 1 < p \leqslant n, \\ L^{p}_{\mathrm{loc}}(\Omega), & \text{when } p > n, \end{cases} \tag{3.2.12}$$

for some $\theta \in (0, 1]$.

The same result holds for Theorem 3.2.2 *provided that* $a_2 \in L^1(\Omega)$ *(for all* $p > 1$*).*

PROOF. When $1 < p \leqslant n$ the proof of Theorem 3.2.1 is valid exactly as before, with (3.2.4) replaced by

$$a_1 \|Du\|^p_{p,\Gamma} \leqslant |\Gamma|^{\theta/n} \|b_1\|_{n/(1-\theta),\Gamma} \|u\|_{p^*,\Gamma} \|Du\|^{p-1}_{p,\Gamma}.$$

For the case $p \geqslant n$, see Theorems 3.9.4 and 3.9.5 below.

The second result is obvious from the proof as given. □

THEOREM 3.2.4. *The conclusions of Theorems* 3.2.1 *and* 3.2.2 *remain valid when the right side of* (3.2.1) *is replaced by*

$$\big\langle \boldsymbol{A}(x, z, \boldsymbol{\xi}), \boldsymbol{\xi} \big\rangle \geqslant a_1 |\boldsymbol{\xi}|^p - a_2 z^p, \qquad B(x, z, \boldsymbol{\xi}) \leqslant b_1 \big(|\boldsymbol{\xi}|^{p-1} + |\boldsymbol{\xi}|^{q-1} \big), \tag{3.2.13}$$

with $1 < q < p$.

The proofs are essentially the same as before, except that (3.2.2), for example, now becomes

$$a_1 \int_\Gamma |Du|^p \leqslant b_1 \bigg(\int_\Gamma \big\{ |Du|^{p-1} + |Du|^{q-1} \big\} \cdot |w| \bigg).$$

One then applies Hőlder's inequality to the separate terms on the right side, as before. The details may be left to the reader.

3.3. A maximum principle for thin sets

When the coefficients a_2, b_1, b_2 in (3.2.1) do not vanish, the maximum principles Theorems 3.2.1 and 3.2.2 are no longer valid, as one can see from obvious examples, e.g., the equation $\Delta u + u = 0$ in a ball, as in elementary eigenvalue theory.

Nevertheless, if the domain in question has sufficiently small measure, that is, is sufficiently "*thin*", then the maximum principle remains correct even when a_2, b_1, b_2 are non-zero.

THEOREM 3.3.1 (Maximum principle). *Assume* $\boldsymbol{A}$ *and* B *satisfy* (3.2.1), *and let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p \geqslant 1$, *be a* p*-regular solution of* (3.1.1). *Suppose also that the measure of* Ω

is so small that

$$\left[\left(\frac{b_1}{a_1}\right)^p + p\frac{a_2+b_2}{a_1}\right]^{n/p} |\Omega| < \omega_n, \tag{3.3.1}$$

where ω_n is the measure of the unit ball in $\mathbb{R}^n$.

If $u \leqslant 0$ on $\partial\Omega$ then $u \leqslant 0$ a.e. in Ω.

PROOF. Define $\varphi = (u-\varepsilon)^+$ for $\varepsilon > 0$. Then, as in the proof of Theorem 3.2.1, see (3.2.2),

$$\int_\Gamma \left(a_1|Du|^p - a_2 u^p\right) \leqslant \int_\Gamma \left(b_1 u|Du|^{p-1} + b_2 u^p\right),$$

where $\Gamma = \{x \in \Omega\colon\ u(x) > \varepsilon\}$. In turn, using the Hőlder and Young inequalities, with $c = a_2 + b_2$,

$$\begin{aligned}\int_\Gamma |Du|^p &\leqslant \frac{b_1}{a_1}\|u\|_{p,\Gamma}\|Du\|_{p,\Gamma}^{p-1} + \frac{c}{a_1}\int_\Gamma u^p \\ &\leqslant \frac{1}{p'}\|Du\|_{p,\Gamma}^p + \frac{1}{p}\left(\frac{b_1}{a_1}\|u\|_{p,\Gamma}\right)^p + \frac{c}{a_1}\int_\Gamma u^p\end{aligned}$$

(note $1/p' = 0$ when $p = 1$). Hence

$$\|Du\|_{p,\Gamma} \leqslant \left[\left(\frac{b_1}{a_1}\right)^p + p\frac{c}{a_1}\right]^{1/p}\|u\|_{p,\Gamma}.$$

Next, by Poincaré's inequality (Theorem 3.10.4),

$$\begin{aligned}\|u\|_{p,\Gamma} &\leqslant \|u-\varepsilon\|_{p,\Gamma} + \|\varepsilon\|_{p,\Gamma} \\ &= \left\|(u-\varepsilon)^+\right\|_{p,\Omega} + \varepsilon|\Gamma|^{1/p} \\ &\leqslant \left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\|Du\|_{p,\Gamma} + \varepsilon|\Omega|^{1/p},\end{aligned}$$

since $D[(u-\varepsilon)]^+ = \mathbf{0}$ a.e. in $\Omega \setminus \Gamma$ and $D[(u-\varepsilon)]^+ = Du$ in Γ. Combining the previous two lines gives

$$\begin{aligned}\|u\|_{p,\Gamma} &\leqslant \left(\frac{|\Omega|}{\omega_n}\right)^{1/n}\left[\left(\frac{b_1}{a_1}\right)^p + p\frac{c}{a_1}\right]^{1/p}\|u\|_{p,\Gamma} + \varepsilon|\Omega|^{1/p} \\ &\leqslant (1-\theta)\|u\|_{p,\Gamma} + \varepsilon|\Omega|^{1/p}\end{aligned}$$

for some $\theta \in (0,1)$. Hence $\|u\|_{p,\Gamma} \leqslant (\varepsilon/\theta)|\Omega|^{1/p}$. Letting $\varepsilon \to 0$ and using the monotone convergence theorem then gives $\|u^+\|_p = 0$. Consequently $u \leqslant 0$ a.e. in Ω. □

Condition (3.2.1) includes the p-Laplace operator Δ_p. For this case the coefficient in (3.3.1) becomes

$$\left(b_1^p + pb_2\right)^{n/p};$$

in particular, $p = 2$ in the Laplace case.

A real number λ such that the Dirichlet problem

$$\begin{aligned} &\operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du) + \lambda |u|^{p-2}u = 0 \quad \text{in } \Omega,\ p > 1, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.3.2}$$

has a non-trivial solution is called an eigenvalue for (3.3.2). With the help of the thin set Theorem 3.3.1 one can give a lower estimate for any possible eigenvalue of (3.3.2). We state this as

COROLLARY 3.3.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > 1$, be a non-trivial p-regular solution of* (3.3.2). *Assume $\boldsymbol{A}$ and B satisfy* (3.2.1) *in the stronger form*

$$\left\langle \boldsymbol{A}(x, z, \boldsymbol{\xi}), \boldsymbol{\xi} \right\rangle \geqslant a_1 |\boldsymbol{\xi}|^p - a_2 |z|^p, \qquad \left| B(x, z, \boldsymbol{\xi}) \right| \leqslant b_1 |\boldsymbol{\xi}|^{p-1} + b_2 |z|^{p-1}.$$

Then

$$\lambda + a_2 + b_2 \geqslant \frac{a_1}{p}\left[\kappa^p - \left(\frac{b_1}{a_1}\right)^p\right], \quad \kappa = \left(\frac{\omega_n}{|\Omega|}\right)^{1/n}.$$

The proof is left to the reader. In the canonical case $B = 0$ the corollary yields the estimate $\lambda + a_2 \geqslant a_1 \kappa^p / p$ for the eigenvalues of the pure operator $\operatorname{div} \boldsymbol{A}(x, u, Du)$ with homogeneous Dirichlet data.

3.4. A comparison theorem in $W^{1,p}(\Omega)$

As in Section 2.4, consider the pair of differential inequalities (2.4.1) and (2.4.2), with $\boldsymbol{A}$ and B no longer required to be in L^∞_{loc}.

THEOREM 3.4.1. *Let u and v be respectively p-regular solutions of* (2.4.1) *and* (2.4.2) *of class $W^{1,p}_{\mathrm{loc}}(\Omega)$. Suppose that $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ is independent of z and monotone in $\boldsymbol{\xi}$, i.e.* (2.4.3) *holds, while $B = B(x, z)$ is independent of $\boldsymbol{\xi}$ and non-increasing in z.*

If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ a.e. in Ω.

PROOF. By definition of distribution solution we get by subtraction

$$\int_\Omega \left\langle \boldsymbol{A}(x, Du) - \boldsymbol{A}(x, Dv), D\varphi \right\rangle \leqslant \int_\Omega \left[B(x, u) - B(x, v)\right]\varphi.$$

Taking $\varphi = (u - v - \varepsilon)^+$, $\varepsilon > 0$, as test function, we find from Lemma 3.1.2 and (2.4.3) that

$$\begin{aligned} 0 &\leqslant \int_\Gamma \big\langle \boldsymbol{A}(x, Du) - \boldsymbol{A}(x, Dv), Du - Dv \big\rangle \\ &\leqslant \int_\Omega \big[B(x, u) - B(x, v)\big]^+ (u - v - \varepsilon)^+, \end{aligned}$$

where $\Gamma = \{x \in \Omega\colon u - v - \varepsilon > 0\}$. Since B is non-increasing in the variable z the right-hand side is zero. Hence $Du = Dv$ a.e. in Γ. Consequently, in view of [40, Lemma 7.6(a)], we have

$$\begin{aligned} D\varphi &= \begin{cases} Du - Dv & \text{in } \Gamma \\ \mathbf{0} & \text{in } \Omega \setminus \Gamma \end{cases} \\ &= \mathbf{0} \quad \text{a.e. in } \Omega. \end{aligned}$$

That is, the function φ, considered as an element of $W^{1,p}(\Omega)$, has weak derivative zero and vanishes near $\partial\Omega$. Hence by the Poincaré inequality, Theorem 3.10.4, there holds

$$\|\varphi\|_p \leqslant C \|D\varphi\|_p = 0.$$

Therefore $\varphi = (u - v - \varepsilon)^+ = 0$ a.e. in Ω, that is, $u \leqslant v + \varepsilon$ a.e. in Ω. Letting $\varepsilon \to 0$ completes the proof. □

REMARK. The special case where $\boldsymbol{A}$ satisfies (2.4.4), noted in Section 2.4, is of particular interest, since it includes the p-Laplace operator $A(s) = s^{p-2}$, $p > 1$. We have the following

COROLLARY 3.4.2. *Let $\boldsymbol{A}$ have the form* (2.4.4), *and suppose that $B = B(x, z)$ is independent of $\boldsymbol{\xi}$ and non-increasing in z. Let u and v be respectively p-regular solutions of* (2.4.1) *and* (2.4.2) *of class $W^{1,p}_{\mathrm{loc}}(\Omega)$. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.*

PROOF. This is a direct consequence of Theorem 3.4.1 and Proposition 2.4.2. □

For the case of the p-Laplace operator, it is clear that the p-regularity of any solution is automatic. Corollary 3.4.2 applies also to the mean curvature operator $A(s) = 1/\sqrt{1 + s^2}$. Here $|\boldsymbol{A}(\boldsymbol{\xi})| < 1$, so 1-regularity of a solution $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is again automatic.

Proposition 2.4.3 can also be applied in the present case, though we can omit the details. Finally, the uniqueness theorems given in Section 2.6 obviously carry over to the present case. In particular, if $B = B(x, z)$ is non-increasing in z, the Dirichlet problem (2.6.3) when $u_0 \in C(\partial\Omega)$ has at most one solution in $W^{1,p}_{\mathrm{loc}}(\Omega)$. Similarly the mean curvature Dirichlet problem

$$\begin{aligned} &\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) + B(x, u) = 0 \quad \text{in } \Omega, \\ &u = u_0 \qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{aligned}$$

has at most one solution in $W^{1,1}_{\mathrm{loc}}(\Omega)$. This last result seems to be new.

3.5. Comparison theorems for singular elliptic inequalities

When the relatively simple assumptions of the previous section do not apply, in particular when the function B depends explicitly on the variable $\boldsymbol{\xi}$, or the operator $\boldsymbol{A}$ is singular at more than isolated points, one can nevertheless reach useful conclusions by applying the maximum principles of Section 3.2. These include the well-known results of Chapter 10 of [40], and in turn lead to the mostly new uniqueness theorems in the later Section 3.8.

Consider the pair of differential inequalities

$$\operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du) \geqslant 0 \quad \text{in } \Omega, \tag{3.5.1}$$

$$\operatorname{div} \boldsymbol{A}(x, v, Dv) + B(x, v, Dv) \leqslant 0 \quad \text{in } \Omega, \tag{3.5.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$, and

$$\boldsymbol{A} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \qquad B : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}.$$

As in Section 3.1, by a *solution of* (3.5.2) *in* Ω we mean a distribution or weak solution, in the sense that $v \in L^1_{\mathrm{loc}}(\Omega)$ is weakly differentiable in Ω, $\boldsymbol{A}(\cdot, v, Dv)$, $B(\cdot, v, Dv) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\int_\Omega \big\langle \boldsymbol{A}(x, v, Dv), D\varphi \big\rangle \geqslant \int_\Omega B(x, v, Dv)\varphi \tag{3.5.3}$$

for all non-negative functions $\varphi \in C^1(\Omega)$ such that $\varphi \equiv 0$ near $\partial\Omega$.

Among other topics, we shall deal with singular or degenerate inequalities in which ellipticity disappears as $\boldsymbol{\xi} \to \mathbf{0}$, as for the p-Laplace operator Δ_p, $p \neq 2$, where $\boldsymbol{A} = \boldsymbol{A}_p(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$. In fact, in many cases the arguments by which a singular point $\mathbf{0}$ is treated can be generalized to allow for larger singular sets. A structure in which such behavior can be studied is described in the following principal conditions, which we assume throughout this and the next two sections.

(i) $\boldsymbol{A}$ *is continuous with respect to* $\boldsymbol{\xi}$ *in* $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

(ii) *There exists a non-empty open subset* $\boldsymbol{P}$ *of* $\mathbb{R}^n$ (*possibly* $\boldsymbol{P} = \mathbb{R}^n$) *such that* $\boldsymbol{A}$ *is continuously differentiable with respect to* $\boldsymbol{\xi}$ *in* $\Omega \times \mathbb{R} \times \boldsymbol{P}$.

$\boldsymbol{P}$ is called the *regular* set for the inequalities (3.5.1) and (3.5.2), while

$$\boldsymbol{Q} = \mathbb{R}^n \setminus \boldsymbol{P}$$

is the *singular* set. If $\boldsymbol{Q} = \emptyset$ the problem is called *regular*, while otherwise it is *singular*. We say that the operator $\boldsymbol{A}$ is (*strictly*) *elliptic* in a set $K \subset \Omega \times \mathbb{R} \times \boldsymbol{P}$ if the Jacobian matrix $[\partial_{\boldsymbol{\xi}} \boldsymbol{A}]$ is (*uniformly*) *positive definite* in K.[7]

In stating our next results, it is convenient to define

$$\boldsymbol{B}_r = \big\{\boldsymbol{\xi} \in \mathbb{R}^n \colon |\boldsymbol{\xi}| \leqslant r\big\}, \qquad \boldsymbol{R}_r = \boldsymbol{B}_r \setminus \{\mathbf{0}\}.$$

[7] The concept of strict ellipticity can be illustrated with the example of the p-Laplace operator, where $\boldsymbol{A}(\boldsymbol{\xi}) = \boldsymbol{A}_{\boldsymbol{p}}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$. This is elliptic for $\boldsymbol{\xi} \neq \mathbf{0}$ when $p > 1$, but strictly elliptic only when $1 < p \leqslant 2$.

The following comparison principle then holds, both for regular operators as well as singular operators for which the singular set is the single point $\boldsymbol{Q} = \{\mathbf{0}\}$.

THEOREM 3.5.1 (Comparison Principle). *Let $\boldsymbol{Q} = \emptyset$ or $\boldsymbol{Q} = \{\mathbf{0}\}$. Suppose that $\boldsymbol{A} = \boldsymbol{A}(x, \xi)$ is independent of z and strictly elliptic in $\Omega \times \boldsymbol{R}_r$ for all $r > 0$. Assume additionally that $B(x, z, \xi)$ is locally Lipschitz continuous with respect to ξ in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and moreover is non-increasing in z.*

Let u and v be solutions of (3.5.1) *and* (3.5.2) *of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ in Ω. If $u \leqslant v + M$ on $\partial\Omega$, where M is constant, then $u \leqslant v + M$ in Ω.*

PROOF. We treat only the case $\boldsymbol{Q} = \{\mathbf{0}\}$. When $\boldsymbol{Q}$ is empty the proof is slightly simpler, and can be omitted. moreover, since $\boldsymbol{A}$ is independent of z it is enough to consider only the case $M = 0$.

Step 1. Suppose that $(x, \xi'), (x, \eta') \in K' = \Omega \times \boldsymbol{R}_W$ for some $W > 0$, $[\xi', \eta']$, given by

$$\zeta(t) = t\xi' + (1 - t)\eta', \quad t \in [0, 1], \tag{3.5.4}$$

does *not* include $\mathbf{0}$, then by the mean value theorem

$$\big\langle \boldsymbol{A}(x, \xi') - \boldsymbol{A}(x, \eta'), \xi' - \eta' \big\rangle = \big\langle \partial_\xi \boldsymbol{A}(x, \zeta)(\xi' - \eta'), \xi' - \eta' \big\rangle$$

for some $\zeta \in [\xi', \eta']$. Since by hypothesis the matrix $[\partial_\xi \boldsymbol{A}(x, \xi)]$ is uniformly positive definite in $\Omega \times \boldsymbol{R}_W$, it follows that

$$\big\langle \boldsymbol{A}(x, \xi') - \boldsymbol{A}(x, \eta'), \xi' - \eta' \big\rangle \geqslant a_1 |\xi' - \eta'|^2, \tag{3.5.5}$$

where

$$a_1 = \inf_{x \in \Omega, \xi \in \boldsymbol{R}_W} \big\{ \text{min eigenvalue of } \big[\partial_\xi \boldsymbol{A}(x, \xi)\big] \big\} > 0.$$

We claim that (3.5.5) holds also when $\mathbf{0} \in [\xi', \eta']$. First, if $\mathbf{0}$ is an end point of $[\xi', \eta']$, say $\eta' = \mathbf{0}$, it is enough to let $\eta' \to \mathbf{0}$ in (3.5.5), since $\boldsymbol{A}$ is continuous at $\mathbf{0}$ and a_1 remains unchanged. The remaining possibility, when $\mathbf{0}$ is in the interior of $[\xi', \eta']$ is now obvious.

Next, if $(x, u, \xi'), (x, v, \eta') \in K''$, where K'' is a compact subset of $\Omega \times \mathbb{R} \times \mathbb{R}^n$, then by local Lipschitz continuity of B we have

$$B(x, u, \xi') - B(x, v, \eta') \leqslant b_1 |\xi' - \eta'| + B(x, u, \eta') - B(x, v, \eta'),$$

where b_1 is the Lipschitz constant of B in the set K''. In particular, since B is non-increasing in z,

$$B(x, u, \xi') - B(x, v, \eta') \leqslant b_1 |\xi' - \eta'| \quad \text{when } u > v. \tag{3.5.6}$$

Step 2. By subtracting (3.5.1) and (3.5.2) we get

$$\operatorname{div}\big\{ \boldsymbol{A}(x, Du) - \boldsymbol{A}(x, Dv) \big\} + B(x, u, Du) - B(x, v, Dv) \geqslant 0 \tag{3.5.7}$$

in Ω. Let $w = u - v$ and define

$$\tilde{\boldsymbol{A}}(x, \boldsymbol{\xi}) = \boldsymbol{A}\big(x, \boldsymbol{\xi} + Dv(x)\big) - \boldsymbol{A}\big(x, Dv(x)\big).$$

Clearly

$$\tilde{\boldsymbol{A}}(x, Dw) = \boldsymbol{A}(x, Du) - \boldsymbol{A}(x, Dv),$$

so that in view of (3.5.7) the function w can be considered as a solution of the differential inequality

$$\operatorname{div} \tilde{\boldsymbol{A}}(x, Dw) + \widetilde{B}(x, w, Dw) \geqslant 0 \tag{3.5.8}$$

where $\widetilde{B}(x, z, \boldsymbol{\xi}) = B(x, z + v(x), \boldsymbol{\xi} + Dv(x)) - B(x, v(x), Dv(x))$ is defined analogously to $\tilde{\boldsymbol{A}}$. Of course, also $w = u - v \leqslant 0$ on $\partial\Omega$.

Since u, $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ it follows that in any compact subset Ω' of Ω we have Du, $Dv \in \boldsymbol{R}_W$ for some $W > 0$. Thus (3.5.5) and (3.5.6) hold in Ω' with the identifications $\boldsymbol{\xi}' = Du$ and $\boldsymbol{\eta}' = Dv$ (so $\boldsymbol{\xi}' - \boldsymbol{\eta}' = Dw$); that is we have

$$\big\langle \tilde{\boldsymbol{A}}(x, Dw), Dw \big\rangle \geqslant a_1 |Dw|^2$$

and

$$\widetilde{B}(x, w, Dw) \leqslant b_1 |Dw| \quad \text{when } w > \ell$$

with $\ell \geqslant \text{Const.} > 0$. Stated in other terms, the functions $\tilde{\boldsymbol{A}}$ and $\widetilde{B}$ in (3.5.8) obey the structural conditions (3.2.1) *along the solution* w, that is, with $\boldsymbol{\xi} = Dw$ and with also $a_2 = b_2 = 0$, $p = 2$.

Since $w \leqslant 0$ on $\partial\Omega$ we can therefore apply Theorem 3.2.1 to obtain $w \leqslant 0$ in Ω, that is $u \leqslant v$. □

REMARKS. This is essentially Theorem 10.7(i) of [40] with the important exceptions that $\boldsymbol{A}$ and B are allowed to be singular at $\boldsymbol{\xi} = \boldsymbol{0}$, and that the class $C^1(\Omega)$ is weakened to $W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Compare also Theorem 10.3 of [71].

If Ω is unbounded and the boundary condition is understood to include the limit relation

$$\limsup_{|x| \to \infty,\, x \in \Omega} \big\{u(x) - v(x)\big\} \leqslant M,$$

then the conclusion of Theorem 3.5.1 continues to hold. The same conclusion is valid for the later results of the section.

In the important case of the p-Laplace operator (where $\boldsymbol{Q} = \{\boldsymbol{0}\}$) we have the following corollary of Theorem 3.5.1.

COROLLARY 3.5.2. *Let u and v be solutions in $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of the inequalities*

$$\Delta_p u + B(x, u, Du) \geqslant 0, \quad \Delta_p v + B(x, v, Dv) \leqslant 0 \quad \text{in } \Omega,$$

where $1 < p \leqslant 2$. Assume also that $B = B(x, z, \boldsymbol{\xi})$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and is non-increasing in the variable z. If $u \leqslant v + M$ on $\partial\Omega$, where M is constant, then $u \leqslant v + M$ in Ω.

PROOF. Here $\boldsymbol{A}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi}$ (and $\boldsymbol{A}(\mathbf{0}) = \mathbf{0}$), so by direct calculation

$$\partial_{\boldsymbol{\xi}} \boldsymbol{A}(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{p-2}\left[\mathbb{I}_n + (p-2)\frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|^2}\right], \quad \boldsymbol{\xi} \neq \mathbf{0}.$$

Therefore the minimum eigenvalue of the Jacobian matrix $[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(\boldsymbol{\xi})]$ when $\boldsymbol{\xi} \neq \mathbf{0}$ is $(p-1)|\boldsymbol{\xi}|^{p-2}$ and so $a_1 = (p-1)W^{p-2}$ in (3.5.5) since $p \leqslant 2$. That is, $\boldsymbol{A}$ is strictly elliptic in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. □

Corollary 3.5.2 can be compared with the results of Section 2.4. In particular, by the final remarks there *the restriction $1 < p \leqslant 2$ is unnecessary when $B = B(x, z)$ is independent of the variable $\boldsymbol{\xi}$*. See also Corollary 3.6.3 below.

THEOREM 3.5.3 (Comparison Principle). *Suppose that $\boldsymbol{A}$ is strictly elliptic in $\Omega \times B_r \times \boldsymbol{R}_r$ for every $r > 0$, and $\partial_z \boldsymbol{A}$ is locally bounded in $\Omega \times \mathbb{R} \times \mathbb{R}^n$. Assume additionally that $B = B(x, z)$ does not depend on $\boldsymbol{\xi}$ and is non-increasing in the variable z.*

Let u and v be solutions of (3.5.1) *and* (3.5.2) *of class $W^{1,\infty}(\Omega)$. If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.*

PROOF. The proof is essentially the same as that for Theorem 3.5.1, with the exception that the difference expression in (3.5.5) is treated differently, that is

$$\begin{aligned}\left\langle \boldsymbol{A}(x, u, \boldsymbol{\xi}') - \boldsymbol{A}(x, v, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \right\rangle &\geqslant \left\langle \boldsymbol{A}(x, u, \boldsymbol{\xi}') - \boldsymbol{A}(x, u, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \right\rangle \\ &\quad + \left\langle \boldsymbol{A}(x, u, \boldsymbol{\eta}') - \boldsymbol{A}(x, v, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \right\rangle \\ &= I_1 + I_2.\end{aligned}$$

Now $I_1 \geqslant a_1|\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2$ as in (3.5.5). Also by the mean value theorem

$$I_2 = \left\langle \partial_z \boldsymbol{A}(x, t, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \right\rangle (u - v) \geqslant -c_1|\boldsymbol{\xi}' - \boldsymbol{\eta}'| \cdot |u - v|,$$

where t is in the open interval between u and v, and $c_1 = \sup_{K'} |\partial_z \boldsymbol{A}(x, z, \boldsymbol{\xi})|$. By Young's inequality this yields $I_2 \geqslant -a_1|\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2/2 - 2c_1^2(u - v)^2/a_1$. In combination, in place of (3.5.5) we now have

$$\left\langle \boldsymbol{A}(x, u, \boldsymbol{\xi}') - \boldsymbol{A}(x, v, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \right\rangle \geqslant \frac{1}{2}a_1|\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2 - a_2(u - v)^2, \tag{3.5.9}$$

where $a_2 = 2c_1^2/a_1$. In addition

$$B(x, u) - B(x, v) \leqslant 0 \quad \text{when } u > v. \tag{3.5.10}$$

Now let $w = u - v$ and define

$$\tilde{\boldsymbol{A}}(x, z, \boldsymbol{\xi}) = \boldsymbol{A}\big(x, z + v(x), \boldsymbol{\xi} + Dv(x)\big) - \boldsymbol{A}\big(x, v(x), Dv(x)\big).$$

Then proceeding as in the proof of Theorem 3.5.1 we obtain

$$\operatorname{div} \tilde{\boldsymbol{A}}(x, w, Dw) + \widetilde{B}(x, w) \geqslant 0.$$

Similarly, from the fact that $u, v \in W^{1,\infty}(\Omega)$ together with (3.5.9) and (3.5.10), we get the structural conditions

$$\big\langle \tilde{\boldsymbol{A}}(x, w, Dw), Dw \big\rangle \geqslant a_1 |Dw|^2 - a_2 w^2, \qquad \widetilde{B}(x, w) \leqslant 0$$

valid when $w > 0$ and $x \in \Omega$; the details being essentially the same as in the derivation of (3.5.9).

Hence Theorem 3.2.2 implies $w \leqslant 0$ in Ω, that is $u \leqslant v$ in Ω. □

This is essentially Theorem 10.7(ii) of [40] with the important exceptions that $\boldsymbol{A}$ and B are allowed to be singular at $\boldsymbol{\xi} = \boldsymbol{0}$, and that the class $C^1(\Omega)$ is weakened to $W^{1,\infty}(\Omega)$. Compare also Theorem 10.3 of [71].

3.6. Strongly degenerate operators

The condition of strict ellipticity in Theorem 3.5.1 can be avoided by adding suitable further hypotheses. This will allow us to cover the p-Laplace operator in the remaining case when $p > 2$, as well as general singular sets $\boldsymbol{Q}$. *We continue to assume conditions* (i) *and* (ii) *from the previous section, and furthermore*, except for Theorem 3.6.5, *the additional hypothesis*

(iii) *For all* $(x, z, \boldsymbol{\xi}), (x, z, \boldsymbol{\eta}) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ *we have*

$$\big\langle \boldsymbol{A}(x, z, \boldsymbol{\xi}) - \boldsymbol{A}(x, z, \boldsymbol{\eta}), \boldsymbol{\xi} - \boldsymbol{\eta} \big\rangle \geqslant 0.$$

An obvious case when (iii) occurs is the Euler–Lagrange equation for the variational integral

$$I[u] = \int_\Omega \mathcal{G}(x, u, Du)\, \mathrm{d}x,$$

in which the integrand $\mathcal{G}(x, u, \boldsymbol{\xi})$ is convex in $\boldsymbol{\xi}$ but not strongly convex; that is, its gradient at some places is either too "*flat*" or has corners, e.g. the integrand $|Du|^p$ for $p \neq 2$.

Condition (iii) is automatic in the important case when $\boldsymbol{Q} = \{\mathbf{0}\}$ (or $\boldsymbol{Q} = \emptyset$) and $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ is elliptic in $\Omega \times \boldsymbol{P}$, as a consequence of Proposition 2.4.3.

The main comparison theorem for degenerate elliptic inequalities is the following

THEOREM 3.6.1 (Comparison Principle). *Let $\boldsymbol{P}$ be a given open set in $\mathbb{R}^n$. Assume that $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ is independent of z and is elliptic in $\Omega \times \boldsymbol{P}$. Suppose also that $B = B(x, z, \boldsymbol{\xi})$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and is non-increasing in the variable z.*

Let u and v be solutions of (3.5.1) *and* (3.5.2) *of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, such that*

$$\operatorname*{essinf}_{\Omega}\big\{\operatorname{dist}(Du, \boldsymbol{Q}) + \operatorname{dist}(Dv, \boldsymbol{Q})\big\} > 0. \tag{3.6.1}$$

If $u \leqslant v + M$ on $\partial\Omega$, where M is constant, then $u \leqslant v + M$ in Ω.

Before giving the proof it is useful to establish the following

LEMMA 3.6.2. *Let $\boldsymbol{\xi}', \boldsymbol{\eta}'$ satisfy*

$$\boldsymbol{\xi}', \boldsymbol{\eta}' \in \boldsymbol{B}_W, \quad \operatorname{dist}(\boldsymbol{\xi}', \boldsymbol{Q}) + \operatorname{dist}(\boldsymbol{\eta}', \boldsymbol{Q}) \geqslant 4d$$

for some positive constants W and d, with $d \leqslant W$.

Let Γ be a compact subset of Ω. Then for all $x \in \Gamma \subset \Omega$ we have

$$\big\langle \boldsymbol{A}(x, \boldsymbol{\xi}') - \boldsymbol{A}(x, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \big\rangle \geqslant a_1 |\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2, \tag{3.6.2}$$

where

$$a_1 = \frac{d}{2W} \inf_{\Gamma \times \{\boldsymbol{P}^d \cap \boldsymbol{B}_W\}} \big\{\textit{min eigenvalue of } \big[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, \boldsymbol{\xi})\big]\big\}$$

and $\boldsymbol{P}^d = \{\boldsymbol{\xi} \in \mathbb{R}^n\colon \operatorname{dist}(\boldsymbol{\xi}, \boldsymbol{Q}) \geqslant d\}$.

PROOF. For $\boldsymbol{\xi}' \neq \boldsymbol{\eta}'$ we consider the line segment $[\boldsymbol{\xi}', \boldsymbol{\eta}']$, that is

$$\boldsymbol{\zeta}(t) = (1 - t)\boldsymbol{\xi}' + t\boldsymbol{\eta}', \quad t \in [0, 1].$$

By hypothesis we may suppose without loss of generality that $\operatorname{dist}(\boldsymbol{\eta}', \boldsymbol{Q}) \geqslant 2d$, so $\boldsymbol{\eta}' \in \boldsymbol{P}^d$. There are two cases:

$$\text{(I) } [\boldsymbol{\xi}', \boldsymbol{\eta}'] \not\subset \boldsymbol{P}^d; \qquad \text{(II) } [\boldsymbol{\xi}', \boldsymbol{\eta}'] \subset \boldsymbol{P}^d.$$

In case (I), let $t_0 \in (0, 1)$ be such that $\boldsymbol{\zeta}(t) \in \boldsymbol{P}^d$ for all $t \in [t_0, 1)$ while $\operatorname{dist}(\boldsymbol{\zeta}(t_0), \boldsymbol{Q}) = d$; see Figure 3. Then

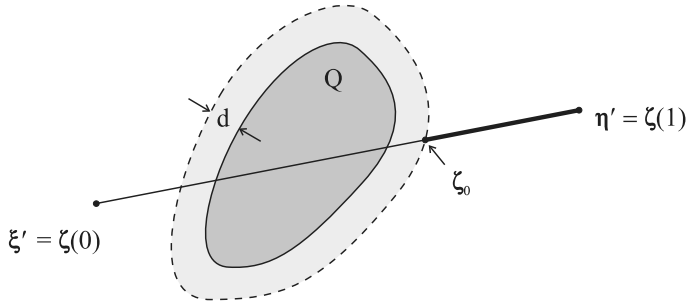

Fig. 3. The set $\boldsymbol{P}^d$ is the complement of the shaded regions. Note that $[\boldsymbol{\zeta_0}, \boldsymbol{\eta}'] \subset \boldsymbol{P}^d$ and $|\boldsymbol{\zeta_0} - \boldsymbol{\eta}'| \geqslant d$.

$$
\begin{aligned}
I &\equiv \big\langle \boldsymbol{A}(x, \boldsymbol{\xi}') - \boldsymbol{A}(x, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \big\rangle \\
&= \big\langle \boldsymbol{A}(x, \boldsymbol{\xi}') - \boldsymbol{A}(x, \boldsymbol{\zeta_0}), \boldsymbol{\xi}' - \boldsymbol{\eta}' \big\rangle + \big\langle \boldsymbol{A}(x, \boldsymbol{\zeta_0}) - \boldsymbol{A}(x, \boldsymbol{\eta}'), \boldsymbol{\xi}' - \boldsymbol{\eta}' \big\rangle \\
&= I_1 + I_2,
\end{aligned}
$$

where $\boldsymbol{\zeta_0} = \boldsymbol{\zeta}(\boldsymbol{t_0})$. By (iii)

$$
I_1 = \big\langle \boldsymbol{A}(x, \boldsymbol{\xi}') - \boldsymbol{A}(x, \boldsymbol{\zeta_0}), \boldsymbol{\xi}' - \boldsymbol{\zeta_0} \big\rangle \frac{|\boldsymbol{\xi}' - \boldsymbol{\eta}'|}{|\boldsymbol{\xi}' - \boldsymbol{\zeta_0}|} \geqslant 0.
$$

Moreover, since $\boldsymbol{A}$ is uniformly elliptic in $\Gamma \times \{\boldsymbol{P}^d \cap \boldsymbol{B}_W\}$, we have

$$
\begin{aligned}
I_2 &= \big\langle \boldsymbol{A}(x, \boldsymbol{\zeta_0}) - \boldsymbol{A}(x, \boldsymbol{\eta}'), \boldsymbol{\zeta_0} - \boldsymbol{\eta}' \big\rangle \frac{|\boldsymbol{\xi}' - \boldsymbol{\eta}'|}{|\boldsymbol{\zeta_0} - \boldsymbol{\eta}'|} \\
&\geqslant a|\boldsymbol{\zeta_0} - \boldsymbol{\eta}'|^2 \frac{|\boldsymbol{\xi}' - \boldsymbol{\eta}'|}{|\boldsymbol{\zeta_0} - \boldsymbol{\eta}'|} = a|\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2 \frac{|\boldsymbol{\zeta_0} - \boldsymbol{\eta}'|}{|\boldsymbol{\xi}' - \boldsymbol{\eta}'|},
\end{aligned}
$$

where

$$
a = \inf_{\Gamma \times \{\boldsymbol{P}^d \cap \boldsymbol{B}_W\}} \big\{ \text{min eigenvalue of } \big[\partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, \boldsymbol{\xi})\big] \big\}.
$$

Finally, $|\boldsymbol{\zeta_0} - \boldsymbol{\eta}'| \geqslant d$ and $|\boldsymbol{\zeta_0} - \boldsymbol{\eta}'|/|\boldsymbol{\xi}' - \boldsymbol{\eta}'| \geqslant d/2W$, so that

$$
I \geqslant I_2 \geqslant \frac{ad}{2W} |\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2,
$$

proving (3.6.2) when (I) holds.

Case (II) is obvious, with $I \geqslant a|\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2$ ($\geqslant (ad/W)|\boldsymbol{\xi}' - \boldsymbol{\eta}'|^2$). □

In the special case of the p-Laplace operator we can take $a_1 = d^{p-1}/2W$ when $p > 2$, while we have already shown that $a_1 = (p-1)W^{p-2}$ when $1 < p \leqslant 2$.

PROOF OF THEOREM 3.6.1. With $M = 0$, and following Step 2 of the proof of Theorem 3.5.1 almost word-for-word, we see first that the function $w = u - v$ satisfies the inequality (3.5.8) with $w = 0$ on $\partial\Omega$. Also by (3.6.1) there is a number $d > 0$ such that

$$\operatorname*{essinf}_{\Omega}\big\{\operatorname{dist}(Du, \boldsymbol{Q}) + \operatorname{dist}(Dv, \boldsymbol{Q})\big\} \geqslant 4d.$$

Then since $u, v \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ it follows from Lemma 3.6.2 that the operator $\tilde{\boldsymbol{A}}$ in (3.5.8) satisfies the first structural condition of (3.2.1) along the solution w, that is with $\boldsymbol{\xi} = Dw$, and with $a_2 = 0$. Also as in Step 1 of Theorem 3.5.1 the function $\widetilde{B}$ satisfies the second condition of (3.2.1) with $z = w$, $\boldsymbol{\xi} = Dw$, and with $b_2 = 0$.

The proof is now completed by applying Theorem 3.2.1. □

COROLLARY 3.6.3. *Assume that $B = B(x, z, \boldsymbol{\xi})$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and is non-increasing in z. Let u and v be solutions of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of the inequalities*

$$\Delta_p u + B(x, u, Du) \geqslant 0, \quad \Delta_p v + B(x, v, Dv) \leqslant 0 \quad \text{in } \Omega,$$

where $p > 1$. Suppose that

$$\operatorname*{essinf}_{\Omega}\big\{|Du| + |Dv|\big\} > 0.$$

If $u \leqslant v + M$ on $\partial\Omega$, where $M \geqslant 0$ is constant, then $u \leqslant v + M$ in Ω.

The next result is similar to Theorem 3.5.3 with the exception that $\boldsymbol{A}$ is not assumed to be uniformly elliptic and may depend on z.

THEOREM 3.6.4 (Comparison Principle). *Let $\boldsymbol{A}$ be elliptic in $\Omega \times \mathbb{R} \times \boldsymbol{P}$. Assume additionally that $B = B(x, z)$ does not depend on $\boldsymbol{\xi}$ and is non-increasing in the variable z. Let u and v be solutions of* (3.5.1) *and* (3.5.2) *of class $W^{1,\infty}(\Omega)$, with*

$$\operatorname*{essinf}_{\Omega}\big\{\operatorname{dist}(Du, \boldsymbol{Q}) + \operatorname{dist}(Dv, \boldsymbol{Q})\big\} > 0.$$

If $u \leqslant v$ on $\partial\Omega$, then $u \leqslant v$ in Ω.

PROOF. The proof is a combination of the ideas of Theorems 3.5.3 and 3.6.1. □

When the solutions u and v of the inequalities (3.5.1) and (3.5.2) are of class $C^1(\Omega)$, rather than $W^{1,\infty}_{\mathrm{loc}}(\Omega)$, the hypotheses of Theorem 3.6.1 can be weakened, giving the second main result of the section.

THEOREM 3.6.5 (Comparison Principle). *Let $\boldsymbol{P}$ be a given open set in $\mathbb{R}^n$. Assume that $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ is independent of z, obeys the conditions* (i), (ii) *stated in the previous section,*

and is elliptic in $\Omega \times \boldsymbol{P}$. *Suppose also that* $B = B(x, z, \boldsymbol{\xi})$ *is locally Lipschitz continuous with respect to* $\boldsymbol{\xi}$ *in* $\Omega \times \mathbb{R}^+ \times \mathbb{R}^n$ *and is non-increasing in the variable* z.

Let $u, v \in C^1(\Omega)$ *be respectively solutions of the inequalities* (3.5.1) *and* (3.5.2) *in the subsets*

$$\Omega_u = \{x \in \Omega\colon\ Du(x) \in \boldsymbol{P}\}, \qquad \Omega_v = \{x \in \Omega\colon\ Dv(x) \in \boldsymbol{P}\}.$$

Assume finally that $\Omega_u \cup \Omega_v = \Omega$ *and that* $u \leqslant v + M$ *on* $\partial\Omega$, M *constant. Then* $u \leqslant v + M$ *in* Ω.

PROOF. It is enough to consider the case $M = 0$. Thus suppose for contradiction that $u > v$ at some point in Ω. Let

$$V = \max_{\Omega}(u - v) > 0$$

be the supremum of $u - v$ in Ω, this being attained at an interior point y since $u - v \leqslant 0$ on $\partial\Omega$. Of course $D(u - v) = \mathbf{0}$ at y, so from the condition $\Omega_u \cup \Omega_v = \Omega$ it follows that $Du(y) = Dv(y) \in \boldsymbol{P}$.

Also let

$$\Sigma = \{x \in \Omega\colon\ \ell < u - v \leqslant V\}, \quad \ell \in (0, V),$$

be a neighborhood of the critical point y. Since Du and Dv can be made arbitrarily near $Du(y)$ in Σ by fixing ℓ sufficiently near V, we obtain $Du \in \boldsymbol{P}$, $Dv \in \boldsymbol{P}$ in Σ. In particular $\Sigma \subset \Omega_u \cap \Omega_v$, so u and v are solutions of (3.5.1) and (3.5.2) in the set Σ. In turn the comparison Theorem 3.5.1 can be applied to the solutions u and v in Σ. In fact Du, Dv can be supposed to lie in a compact subset $\boldsymbol{N}$ of $\boldsymbol{P}$, with the consequence that $\boldsymbol{A}$ is strictly elliptic in $\Sigma \times \boldsymbol{N}$ and the regular case of Theorem 3.5.1 is applicable.

Since $u = v + \ell$ on $\partial\Sigma$, it follows that $u \leqslant v + \ell$ in Σ. That is, $u - v \leqslant \ell < V$ in Σ, which contradicts the fact that $u - v = V$ at y. □

3.7. Maximum principles for non-homogeneous elliptic inequalities

Consider the differential operator

$$L[u] = \operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du),$$

where

$$\boldsymbol{A}\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \qquad B\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

and $\boldsymbol{A}$ satisfies the hypotheses (i)–(iii) of Sections 3.5 and 3.6. Additionally we assume

(iv) $\partial_z \boldsymbol{A}(x, z, \boldsymbol{\xi})$ *is locally Lipschitz continuous with respect to* $\boldsymbol{\xi}$ *in* $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

Theorem 3.6.1 has as one of its main consequence the following maximum principle for non-homogeneous elliptic inequalities. It is interesting that for this result the function $B(x, z, \boldsymbol{\xi})$ *need not* be monotone in the variable z.

THEOREM 3.7.1 (Maximum Principle). *Assume that* $\boldsymbol{A} = \boldsymbol{A}(x, z, \boldsymbol{\xi})$ *is elliptic in* $\Omega \times \mathbb{R}^+ \times \boldsymbol{P}$ *and that* $B = B(x, z, \boldsymbol{\xi})$ *is locally Lipschitz continuous with respect to* $\boldsymbol{\xi}$ *in* $\Omega \times \mathbb{R}^+ \times \mathbb{R}^n$.

Define $\mathcal{L}[z, v] : \mathbb{R}^+ \times C^1(\Omega) \to \mathbb{R}$ *pointwise by*

$$\mathcal{L}[z, v](x) = \operatorname{div} \boldsymbol{A}(x, z, Dv) + \mathcal{B}(x, z, Dv), \tag{3.7.1}$$

where

$$\mathcal{B}(x, z, \boldsymbol{\xi}) = \big\langle \partial_z \boldsymbol{A}(x, z, \boldsymbol{\xi}), \boldsymbol{\xi} \big\rangle + B(x, z, \boldsymbol{\xi})$$

for all $x \in \Omega$, $z \in \mathbb{R}^+$ *and* $\boldsymbol{\xi} \in \mathbb{R}^n$.

Let $v = v(x) \in C^1(\Omega)$ *be a* non-negative comparison function *for the operator* L, *in the sense that* $v(x) \geqslant 0$ *and* $Dv(x) \in \boldsymbol{P}$ *for* $x \in \Omega$; *and* $\mathcal{L}[z, v] \leqslant 0$ *for all* $z > 0$. *If* $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ *is a solution of the inequality* $L[u] \geqslant 0$ *in* Ω *and* $u \leqslant v$ *on* $\partial\Omega$, *then* $u(x) \leqslant v(x)$ *for all* $x \in \Omega$.

PROOF. Define

$$\tilde{\mathcal{L}}[v] \equiv \operatorname{div} \boldsymbol{A}\big(x, u(x), Dv\big) + \big\langle \partial_z \boldsymbol{A}(x, u(x), Dv), Dv - Du \big\rangle + B\big(x, u(x), Dv\big).$$

By direct calculation one gets

$$\operatorname{div} \boldsymbol{A}\big(x, u(x), Dv\big) = \operatorname{div} \boldsymbol{A}(x, z, Dv) + \big\langle \partial_z \boldsymbol{A}(x, z, Dv), Du \big\rangle$$

evaluated at $z = u(x)$ in Ω. Hence in Ω

$$\tilde{\mathcal{L}}[v] = \operatorname{div} \boldsymbol{A}(x, z, Dv) + \big\langle \partial_z \boldsymbol{A}(x, z, Dv), Dv \big\rangle + B(x, z, Dv) = \mathcal{L}[z, v].$$

By hypothesis, then, $\tilde{\mathcal{L}}[v] \leqslant 0$ whenever $u(x) > 0$. On the other hand, clearly $\tilde{\mathcal{L}}[u] \geqslant 0$ in Ω. From its definition we see that $\tilde{\mathcal{L}}[v]$ can be written in the form

$$\tilde{\mathcal{L}}[v] = \operatorname{div} \tilde{\boldsymbol{A}}(x, Dv) + \widetilde{B}(x, Dv), \tag{3.7.2}$$

where

$$\tilde{\boldsymbol{A}}(x, \boldsymbol{\xi}) = \boldsymbol{A}\big(x, u(x), \boldsymbol{\xi}\big),$$
$$\widetilde{B}(x, \boldsymbol{\xi}) = \big\langle \partial_z \boldsymbol{A}\big(x, u(x), \boldsymbol{\xi}\big), \boldsymbol{\xi} - Du \big\rangle + B\big(x, u(x), \boldsymbol{\xi}\big).$$

Of course both $\tilde{\boldsymbol{A}}$ and $\widetilde{B}$ are independent of z. Therefore in view of (i)–(iv) the functions $\tilde{\boldsymbol{A}}$ and $\widetilde{B}$ satisfy conditions (i)–(iii), while $\widetilde{B}$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $\Omega \times \mathbb{R}^+ \times \mathbb{R}^n$.

Finally $\tilde{\boldsymbol{A}}(x, \boldsymbol{\xi})$ is elliptic when $x \in \Omega$, $u(x) > 0$ and $\boldsymbol{\xi} \in \boldsymbol{P}$, since $\boldsymbol{A}(x, z, \boldsymbol{\xi})$ is elliptic in $\Omega \times \mathbb{R}^+ \times \boldsymbol{P}$.

Let $\Omega' = \{x \in \Omega\colon\ u(x) > 0\}$ where it is easy to see that $u \leqslant v$ on $\partial\Omega'$. We can apply Theorem 3.6.1 to any component $\mathcal{C}$ of Ω', with u and v satisfying $\tilde{\mathcal{L}}[u] \geqslant 0$, $\tilde{\mathcal{L}}[v] \leqslant 0$, $Dv \in \boldsymbol{P}$ in $\mathcal{C}$ (so $\operatorname{dist}(Dv, \boldsymbol{Q}) > 0$). Hence $u \leqslant v$ in $\mathcal{C}$ and in turn $u \leqslant v$ in Ω'. This finally gives $u \leqslant v$ in Ω, completing the proof. □

Theorem 3.7.1 is somewhat abstract, in that it depends on the *existence* of the comparison function v. As in Theorem 2.3.2, when $\boldsymbol{A}$ and B are more specialized we can avoid this difficulty. In particular, consider the case where $\boldsymbol{Q} \subset \boldsymbol{B}_\varrho$ for some $\varrho \geqslant 0$ (the possibility $\boldsymbol{P} = \mathbb{R}^n$ is included when $\varrho = 0$). Assume that

$$\begin{cases} \boldsymbol{A}(x, z, \boldsymbol{\xi}) \text{ is elliptic} \\ B(x, z, \boldsymbol{\xi}) + \boldsymbol{A}^*(x, z, \boldsymbol{\xi}) \leqslant \alpha|\boldsymbol{\xi}|E(x, z, \boldsymbol{\xi}) + \gamma \end{cases} \tag{3.7.3}$$

in $\Omega \times \mathbb{R}^+ \times \boldsymbol{P}$, where α and γ are non-negative constants, and

$$\begin{aligned} &\boldsymbol{A}^*(x, z, \boldsymbol{\xi}) = \operatorname{Trace}\big[\partial_x \boldsymbol{A}(x, z, \boldsymbol{\xi})\big] + \big\langle \partial_z \boldsymbol{A}(x, z, \boldsymbol{\xi}), \boldsymbol{\xi} \big\rangle, \\ &E(x, z, \boldsymbol{\xi}) = \big\langle \boldsymbol{\xi},\ \partial_{\boldsymbol{\xi}} \boldsymbol{A}(x, z, \boldsymbol{\xi})\, \boldsymbol{\xi} \big\rangle / |\boldsymbol{\xi}|^2. \end{aligned} \tag{3.7.4}$$

Note that $\boldsymbol{A}^* = \boldsymbol{0}$ in the important case when $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{\xi})$.

THEOREM 3.7.2 (Maximum Principle). *Let $\boldsymbol{A}$ and B satisfy* (3.7.3), *and suppose that*

$$|\boldsymbol{\xi}|E(x, z, \boldsymbol{\xi}) \geqslant \Psi\big(|\boldsymbol{\xi}|\big) \quad \text{in } \Omega \times \mathbb{R}^+ \times \boldsymbol{P}, \quad \boldsymbol{P} = \mathbb{R}^n \setminus \boldsymbol{Q}, \tag{3.7.5}$$

where $\Psi = \Psi(t)$ is a strictly increasing function on (ϱ, ∞), $\varrho \geqslant 0$.

Let u be a solution of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of the boundary value problem

$$\begin{aligned} &\operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du) \geqslant 0 \quad && \text{in } \Omega, \\ &u \leqslant 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.7.6}$$

where $\Omega \subset \{x \in \mathbb{R}^n\colon\ 0 < x_1 < R\}$. Then there holds

$$u(x) \leqslant R \max\{\rho, C\}\big(\mathrm{e}^k - 1\big), \tag{3.7.7}$$

where[8]

$$\begin{aligned} &C = \Psi^{-1}(R\gamma), \quad k = 1 + \alpha R, && \text{when } \lim_{t\to\infty} \Psi(t) > 2\gamma R, \\ &C = \Psi^{-1}(\ell), \quad k = 1 + (\alpha + \gamma/\ell)R, && \text{when } \lim_{t\to\infty} \Psi(t) = 2\ell \leqslant 2\gamma R. \end{aligned} \tag{3.7.8}$$

[8]If $\Psi(\varrho) = \lim_{t\to\varrho+} \Psi(t) = \ell' > 0$ then we define $\Psi^{-1}(s) = \varrho$ when $s \leqslant \ell'$. Note that the case $\lim_{t\to\infty} \Psi(t) < \infty$ is possible. That is, take $\boldsymbol{A}(\boldsymbol{\xi}) = 2\ell \log(|\boldsymbol{\xi}| + 1)\boldsymbol{\xi}/|\boldsymbol{\xi}|$ and use the computation of footnote 3 of Section 2.2.

Theorem 3.7.2 has almost exactly the formulation of the earlier Theorem 2.3.2. For completeness the full proof is given here, even though it is essentially the same as for the earlier result.

PROOF. It is enough to construct a comparison function $v = v(x)$ such that $v(x) > 0$ in Ω and $\mathcal{L}[z, v] \leqslant 0$ for all $z > 0$. Accordingly, we choose

$$v(x) = K\left(\mathrm{e}^{mR} - \mathrm{e}^{mx_1}\right), \quad x \in \Omega,$$

where $m = k/R$, $K > R\max\{\varrho, C\}$. Then

$$\partial_{x_1} v(x) = -Km\mathrm{e}^{mx_1}$$

so $|Dv| \geqslant mK$ and $Dv \in \boldsymbol{P}$, since $m > 1/R$. Also

$$\partial^2_{x_1} v(x) = -Km^2\mathrm{e}^{mx_1} = -m|Dv|.$$

In view of (3.7.1) and (3.7.3), a direct calculation then shows that $\mathcal{L}[z, v] \leqslant 0$ in Ω provided

$$m|Dv|\partial_{\xi_1} A_1(x, z, Dv) \geqslant \alpha|Dv|E(x, z, Dv) + \gamma. \tag{3.7.9}$$

But $E(x, z, Dv) = \partial_{\xi_1} A_1(x, z, Dv)$, so (3.7.9) becomes

$$m|Dv|E(x, z, Dv) \geqslant \alpha|Dv|E(x, z, Dv) + \gamma. \tag{3.7.10}$$

Obviously (3.7.10) is satisfied if $(m - \alpha)|Dv|\, E(x, z, Dv) \geqslant \gamma$ for all $z > 0$. At the same time

$$|Dv|E(x, z, Dv) \geqslant \Psi\left(|Dv|\right) \geqslant \Psi(mK) \geqslant \Psi(C) \geqslant \min\{\gamma R, \ell\},$$

since $mK > (k/R)R\max\{\varrho, C\} \geqslant C$. Therefore (3.7.10) holds when k and C are given as in (3.7.8), and in turn we get $\mathcal{L}[z, v] \leqslant 0$ in Ω, as required.

We now apply Theorem 3.7.1, giving

$$u(x) \leqslant v(x) \leqslant K\left(\mathrm{e}^k - 1\right) \quad \text{in } \Omega.$$

Letting $K \to R\max\{\varrho, C\}$ completes the proof. □

The remarks after Theorem 2.3.2 apply equally to the previous result. When B is homogeneous the global condition (3.7.3) need be assumed only for $|\boldsymbol{\xi}|$ small. We state this result as

THEOREM 3.7.3. *Assume $\boldsymbol{P} = \mathbb{R}^n$ or $\boldsymbol{P} = \mathbb{R}^n \setminus \{\mathbf{0}\}$. Let the hypotheses of Theorem* 3.7.2 *hold, with the exceptions that $\gamma = 0$, and* (3.7.3) *and* (3.7.5) *are assumed to be valid only in $\Omega \times \mathbb{R}^+ \times \boldsymbol{R}_1$. Let u be a solution of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of the boundary value problem* (3.7.6) *where Ω is a bounded domain in $\mathbb{R}^n$. Then $u \leqslant 0$ in Ω.*

In the generality of the present hypotheses, this seems to be a new result.

PROOF. Since $\gamma = 0$ only the first case of (3.7.8) applies and so $C = \Psi^{-1}(0) = \varrho = 0$. In this case the constant $K > 0$ in the proof of Theorem 3.7.2 can be chosen arbitrarily small, and in particular so that $|Dv(x)| \leqslant Kme^{mR} \leqslant 1$ in Ω. The rest of the proof of Theorem 3.7.2 then applies without change, giving $u \leqslant 0$, independent of R. Since Ω is bounded we get $u \leqslant 0$ in Ω. □

Theorem 3.7.3 is false if one weakens condition (3.7.3); see the comment after Theorem 2.3.3 and example (2.3.9).

Theorem 3.7.2 has a further direct application.

THEOREM 3.7.4 (Maximum Principle). *Let $A \in C^1(\mathbb{R}^+)$, $A(s) > 0$ and $\Lambda(s) = s[A(s) + sA'(s)] > 0$ for $s > 0$. Assume that Λ is strictly increasing, $\Lambda(0) = 0$, and, for simplicity, also that $\Lambda(s) \to \infty$ as $s \to \infty$. Suppose finally that*

$$B(x, z, \boldsymbol{\xi}) \leqslant \alpha \Lambda\big(|\boldsymbol{\xi}|\big) + \gamma \quad \text{in } \Omega \times \mathbb{R}^+ \times \big(\mathbb{R}^n \setminus \{\mathbf{0}\}\big),$$

where α and γ are non-negative constants.

Let u be a solution of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of the boundary value problem

$$\begin{aligned} &\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + B(x, u, Du) \geqslant 0 \quad && \text{in } \Omega, \\ &u \leqslant 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.7.11}$$

where $\Omega \subset \{x \in \mathbb{R}^n\colon\ 0 < x_1 < R\}$. Then there holds

$$u(x) \leqslant R\Lambda^{-1}(R\gamma)\big[e^{1+\alpha R} - 1\big].$$

Furthermore, when $\gamma = 0$ then $u \leqslant 0$ in Ω, where Ω can be any bounded domain in $\mathbb{R}^n$.

PROOF. In the present case $\boldsymbol{Q} = \{\mathbf{0}\}$, and

$$\boldsymbol{A}(\boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\boldsymbol{\xi}, \quad \partial_{\boldsymbol{\xi}}\boldsymbol{A}(\boldsymbol{\xi}) = A\big(|\boldsymbol{\xi}|\big)\mathbb{I}_n + A'\big(|\boldsymbol{\xi}|\big)\frac{\boldsymbol{\xi} \otimes \boldsymbol{\xi}}{|\boldsymbol{\xi}|}, \quad \boldsymbol{A}^*(\boldsymbol{\xi}) = \mathbf{0}.$$

The eigenvalues of the Jacobian matrix are $A(s)$ and $A(s) + sA'(s)$, with $s = |\boldsymbol{\xi}|$. Therefore by hypothesis equation (3.7.11) is elliptic for $\boldsymbol{\xi} \neq \mathbf{0}$. It is easy to see moreover that $E(\boldsymbol{\xi}) = A(s) + sA'(s)$, and in turn $|\boldsymbol{\xi}|E(\boldsymbol{\xi}) = \Lambda(s)$. The conclusion is now immediate from Theorem 3.7.2, with $\boldsymbol{Q} = \{\mathbf{0}\}$ and $\Psi(s) = \Lambda(s)$.

The final statement of the theorem is obvious from the previous proof. □

REMARKS.

1. When $A(s) = s^{p-2}$, $p > 1$, we get the important subcase of the p-Laplace operator, for which

$$E(s) = (p-1)s^{p-2}, \quad \Lambda(s) = (p-1)s^{p-1} \quad \text{and}$$
$$R\Lambda^{-1}(Rs) = \big[s/(p-1)\big]^{1/(p-1)} R^{p'}.$$

See also the comments at the end of Section 3.9, and [73].

2. The possibility that $\boldsymbol{Q} \supsetneq \{\mathbf{0}\}$, say $\boldsymbol{Q} = \boldsymbol{B}_\varrho$, $\varrho > 0$, in Theorem 3.7.2 can be illustrated by the example

$$\boldsymbol{A}(\boldsymbol{\xi}) = \begin{cases} \mathbf{0}, & \text{if } |\boldsymbol{\xi}| \leqslant 1, \\ |\boldsymbol{\xi}|^{p-2}\boldsymbol{\xi} - |\boldsymbol{\xi}|^{-1}\boldsymbol{\xi}, & \text{if } |\boldsymbol{\xi}| \geqslant 1, \end{cases}$$

with $p > 1$. Clearly $\boldsymbol{A}$ satisfies the basic conditions (i), (ii) and (iv), together with the hypothesis (3.7.3) of Theorem 3.7.2 with $\varrho = 1$. In (3.7.4) we have $\boldsymbol{A}^* = \mathbf{0}$ and

$$E(x, z, \boldsymbol{\xi}) = E(\boldsymbol{\xi}) = (p-1)|\boldsymbol{\xi}|^{p-2}, \quad \text{if } |\boldsymbol{\xi}| \geqslant 1.$$

Thus in turn

$$\Psi(s) = (p-1)s^{p-1}, \quad \text{if } s \geqslant 1,$$

which is strictly increasing in $[1, \infty)$ and tends to ∞ as $s \to \infty$. The principal condition (3.7.3) then becomes

$$B(x, z, \boldsymbol{\xi}) \leqslant \alpha|\boldsymbol{\xi}|^{p-1} + \gamma, \quad \text{if } |\boldsymbol{\xi}| \geqslant 1,$$

with no restriction assigned when $|\boldsymbol{\xi}| \leqslant 1$, namely in $\boldsymbol{B}_1$. Of course for the applicability of Theorem 3.7.2 the remaining assumption (iii) must be required on B.

The conclusion is

$$u(x) \leqslant \max\big\{1, \gamma^{1/(p-1)}\big\} \cdot \big[\mathrm{e}^{1+\alpha R} - 1\big].$$

Obviously results of this kind do not follow from the theory in [40].

THEOREM 3.7.5. *Let the hypotheses of Theorem* 3.7.2 *be satisfied, with the exception that* (3.7.3) *is replaced by the condition that*

$$B(x, z, \boldsymbol{\xi}) + \boldsymbol{A}^*(x, z, \boldsymbol{\xi}) \leqslant \big(\alpha|\boldsymbol{\xi}| + \beta|\boldsymbol{\xi}|^q\big)E(x, z, \boldsymbol{\xi}) + \gamma, \quad 0 < q < 1,$$

in $\Omega \times \mathbb{R}^+ \times \boldsymbol{P}$, *where* α, β, γ *are non-negative constants.*

Then (3.7.7) *holds with the previous constant* C *replaced by* $C + \beta^{1/(1-q)}$ *and the previous constant* k *replaced by* $k + 1$.

The proof is essentially the same as before. The additional term $\gamma|\boldsymbol{\xi}|^q$ (in the case $q = 0$) was first introduced by Gilbarg and Trudinger ([40], Theorem 10.3).

3.8. Uniqueness of the singular Dirichlet problem

The structure built up in the earlier parts of this chapter allows one to present a number of uniqueness theorems for distribution solutions of the Dirichlet problem

$$\begin{aligned} &\operatorname{div} \boldsymbol{A}(x, u, Du) + B(x, u, Du) = 0 \quad &&\text{in } \Omega, \\ &u = u_0 &&\text{on } \partial\Omega, \end{aligned} \tag{3.8.1}$$

where $u_0 \in C(\partial\Omega)$ and Ω is a bounded domain of $\mathbb{R}^n$. We assume that $\boldsymbol{A}$ and B satisfy the hypotheses (i)–(iv) of Sections 3.5–3.7.

THEOREM 3.8.1. *Suppose that $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ is independent of z and strictly elliptic in $\Omega \times \boldsymbol{R}_1$.*

Assume additionally that $B = B(x, z, \boldsymbol{\xi})$ is locally Lipschitz continuous with respect to $\boldsymbol{\xi}$ in $\Omega \times \mathbb{R}^+ \times \boldsymbol{P}$ and is non-increasing in the variable z.

Then problem (3.8.1) *can have at most one solution of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$.*

This is an immediate consequence of Theorem 3.5.1.

THEOREM 3.8.2. *Assume that $\boldsymbol{A} = \boldsymbol{A}(x, \boldsymbol{\xi})$ is independent of z and is elliptic in $\Omega \times \boldsymbol{P}$. Suppose also that B is non-increasing in z. Let u and v be solutions of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of* (3.8.1), *with*

$$\operatorname*{essinf}_{\Omega} \big\{ \operatorname{dist}(Dv, \boldsymbol{Q}) + \operatorname{dist}(Du, \boldsymbol{Q}) \big\} > 0.$$

Then $u = v$ in Ω.

This is a corollary of Theorem 3.6.1

In the same way the Comparison Theorems 3.5.3 and 3.6.4 allow corresponding uniqueness results, whose statements can be left to the reader. The special case of the p-Laplace operator is of particular interest.

COROLLARY 3.8.3. *Let $B = B(x, z, \boldsymbol{\xi})$ be non-increasing in the variable z. Let u and v be solutions of class $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of the Dirichlet problem*

$$\begin{aligned} &\Delta_p u + B(x, u, Du) = 0 \quad &&\textit{in } \Omega, \\ &u = u_0 &&\textit{on } \partial\Omega, \end{aligned} \tag{3.8.2}$$

where $u_0 \in C(\partial\Omega)$.

Then $u = v$ if $1 < p \leqslant 2$ and B is regular. The same conclusion holds when $p > 2$ (without the condition that B be regular), provided that either $\operatorname{essinf}_{\Omega} |Du| > 0$ *or* $\operatorname{essinf}_{\Omega} |Dv| > 0$.

This is an obvious consequence of Corollaries 3.5.2 and 3.6.3.

REMARKS.

1. The second part of Corollary 3.8.3 fails when $\operatorname{essinf}_\Omega\{|Du|+|Dv|\}=0$. Indeed the problem

$$\Delta_4 u + |Du|^2 = 0 \quad \text{in } B_R \subset \mathbb{R}^2, \qquad u = 0 \quad \text{on } \partial B_R,$$

admits the two solutions $u(x)=0$ and $v(x)=\frac{1}{8}(R^2-|x|^2)$ in B_R. Here $|Du|+|Dv|=0$ at 0, and in turn $\operatorname{essinf}_\Omega\{|Du|+|Dv|\}=0$.

2. As an application of the second part of Corollary 3.8.3, consider the problem (3.8.2), with $B(x,z,\boldsymbol{\xi})=|\boldsymbol{\xi}|^2-1$ and $u_0(x)=x_1$, $x=(x_1,\dots,x_n)$. This admits only the single solution $u(x)=x_1$ whatever the bounded domain Ω may be, since $|Du|>0$ in $\mathbb{R}^n$.

When the boundary data takes the canonical form $u=0$ on $\partial\Omega$, then the condition in Theorem 3.8.1 that $\boldsymbol{A}$ be strictly elliptic can be dropped. The result is as follows.

THEOREM 3.8.4. *Let $\boldsymbol{A}(x,z,\boldsymbol{\xi})$ be elliptic in $\Omega\times\mathbb{R}\times\boldsymbol{P}$, where $\boldsymbol{Q}=\emptyset$ or $\{\mathbf{0}\}$. Assume that*

$$[\operatorname{sign} z]\cdot B(x,z,\boldsymbol{\xi}) \leqslant \alpha\Psi\big(|\boldsymbol{\xi}|\big), \tag{3.8.3}$$

with $|\boldsymbol{\xi}|E(x,z,\boldsymbol{\xi})\geqslant\Psi(|\boldsymbol{\xi}|)$, where Ψ is strictly increasing in $\mathbb{R}$, $\Psi(0)=0$, and E is given by (3.7.4).

If $u\in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is a solution of the Dirichlet problem

$$\begin{aligned} &\operatorname{div}\boldsymbol{A}(x,u,Du)+B(x,u,Du)=0 \quad && \text{in } \Omega,\\ &u=0 && \text{on } \partial\Omega, \end{aligned} \tag{3.8.4}$$

then $u\equiv 0$.

PROOF. This follows immediately from Theorem 3.7.3, when we observe that the function $v(x)=-u(x)$ also satisfies an equation of the form (3.8.4), with the corresponding inequality (3.8.3) equally valid; that is, the only possible solution of (3.8.4) is $u\equiv 0$. □

3.9. Maximum principles for structured inequalities

As in Section 3.2 we consider the quasilinear differential inequality

$$\operatorname{div}\boldsymbol{A}(x,u,Du)+B(x,u,Du)\geqslant 0 \quad \text{in } \Omega, \tag{3.9.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, and $\boldsymbol{A}$ and B satisfy the generic assumptions of Section 3.1.

The earlier maximum principles Theorems 3.2.1 and 3.2.2 can be extended to the case when (3.9.1) is inhomogeneous, that is, there are constants a_2, b_1, b_2, a, $b \geqslant 0$ such that for all $(x, z, \boldsymbol{\xi}) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n$ there holds, for $p > 1$,

$$\begin{aligned} &\big\langle \boldsymbol{A}(x, z, \boldsymbol{\xi}), \boldsymbol{\xi} \big\rangle \geqslant |\boldsymbol{\xi}|^p - a_2 z^p - a^p, \\ &B(x, z, \boldsymbol{\xi}) \leqslant b_1 |\boldsymbol{\xi}|^{p-1} + b_2 z^{p-1} + b^{p-1}. \end{aligned} \tag{3.9.2}$$

The apparently more general situation when the principal term $|\boldsymbol{\xi}|^p$ in (3.9.2) is replaced by $a_1|\boldsymbol{\xi}|^p$, $a_1 > 0$, in fact immediately reduces to (3.9.2) by rescaling.

We now have the following maximum principles for p-regular solutions of (3.9.1)

THEOREM 3.9.1 (Semi-maximum principle). *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p \geqslant 1$, be a solution of the inequality* (3.9.1) *in Ω, with $u \leqslant M$ on $\partial\Omega$ for some $M \geqslant 0$. If* (3.9.2) *holds, then $u^+ \in L^\infty(\Omega)$ and*

$$u \leqslant C\big[\|u^+\|_p + k + \big(a_2^{1/p} + b_2^{1/(p-1)}\big)M\big] + M \quad \text{a.e. in } \Omega, \tag{3.9.3}$$

where $k = a + b \geqslant 0$ and the constant C depends only on p, n, $|\Omega|$, b_1 and $a_2 + b_2$.

THEOREM 3.9.2. *Theorem* 3.9.1 *continues to be valid if the coefficients a, b, a_2, b_1 and b_2 are functions in the Lebesgue spaces*:

$$\begin{aligned} &a, b_1 \in L^{p\alpha}(\Omega), \quad b \in L^{(p-1)\alpha}(\Omega), \quad a_2, b_2 \in L^{\alpha}(\Omega), \\ &\alpha = \frac{\max\{n/p, 1\}}{1 - \varepsilon}, \quad \varepsilon \in (0, 1] \end{aligned} \tag{3.9.4}$$

and (3.9.3) *is replaced by*

$$u \leqslant C\big[\|u^+\|_p + k + \big(\|a_2\|^{1/p} + \|b_2\|^{1/(p-1)}\big)M\big] + M, \quad \text{a.e. in } \Omega, \tag{3.9.5}$$

where $k = \|a + b\|$ and the constant C now depends also on ε.

Here and in the sequel, we understand by $\|a\|$, $\|b\|$, $\|a_2\|$, $\|b_1\|$ and $\|b_2\|$ the norms of a, b, a_2, b_1, b_2 in the respective Lebesgue spaces (3.9.4), *or, in the limit case $\varepsilon = 1$, the Lebesgue spaces $L^\infty(\Omega)$.*

THEOREM 3.9.3 (Maximum principle). *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > 1$, be a solution of* (3.9.1) *in Ω, where $\boldsymbol{A}$ and B satisfy* (3.9.2) *with $b_1 = b_2 = 0$. Suppose $u \leqslant M$ on $\partial\Omega$ for some $M \geqslant 0$. Then $u^+ \in L^\infty(\Omega)$ and*

$$u \leqslant C\big(a + b + a_2^{1/p} M\big) + M \quad \text{a.e. in } \Omega,$$

where C can be taken in the form $\exp\{C(p, n, |\Omega|)(1 + a_2)^\nu\}$ *where $\nu = (n + p)/p^2$ when $1 < p < n$ and $\nu = 5/p$ when $p \geqslant n$.*

THEOREM 3.9.4 (Maximum principle). *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $p > 1$, be a solution of* (3.9.1) *in Ω, where $\boldsymbol{A}$ and B satisfy* (3.9.2) *with $a_2 = b_2 = 0$. Suppose $u \leqslant M$ on $\partial\Omega$ for some $M \geqslant 0$. Then $u^+ \in L^\infty(\Omega)$ and*

$$u \leqslant C(a+b) + M \quad \text{a.e. in } \Omega,$$

where C can be taken in the form $\exp\{C(p, n, |\Omega|)(1+b_1)^\nu\}$.

THEOREM 3.9.5. *Theorems* 3.9.3 *and* 3.9.4 *continue to be valid if the coefficients a, b, a_2 and b_1 are functions in the Lebesgue spaces*:

$$\begin{gathered} a, b_1 \in L^{p\beta}(\Omega), \quad b \in L^{(p-1)\beta}(\Omega), \quad a_2 \in L^{\beta}(\Omega), \\ \beta = \begin{cases} n/p(1-\varepsilon), & \text{if } 1 < p \leqslant n, \\ 1, & \text{if } p > n, \end{cases} \qquad \varepsilon \in (0, 1]. \end{gathered} \tag{3.9.6}$$

Note the difference in the Lebesgue spaces allowed for the coefficients in Theorems 3.9.2 and 3.9.5. In passing we comment that in [40] the spaces are correctly stated on page 276 for the analogue of Theorem 3.9.5, but seem to be too weak for Theorem 3.9.2 in the case $p > n$.

Theorem 3.9.5 applies in particular to the linear elliptic inequality

$$\partial_{\xi_i}\{a_{ij}(x)\partial_{\xi_j} u\} + b_i(x)\partial_{\xi_i} u + c(x)u \geqslant f(x),$$

provided that the coefficients $[a_{ij}]$, b_i are bounded, the coefficient c is non-positive, and $f \in L^q(\Omega)$ for some $q > n/2$. In fact here

$$B(x, z, \boldsymbol{\xi}) = b_i(x)\boldsymbol{\xi}_i + c(x)z - f(x) \leqslant b_1|\boldsymbol{\xi}| + |f(x)|,$$

when $z \geqslant 0$, so the required hypotheses are satisfied with $p = 2$.

For the special case of the p-Laplace inequality

$$\Delta_p u + B(x, u, Du) \geqslant 0, \quad \text{with } B(x, z, \boldsymbol{\xi}) \leqslant b_1|\boldsymbol{\xi}|^{p-1} + b^{p-1}, \ p > 1, \tag{3.9.7}$$

the above results can usefully be compared with Theorem 3.7.4, or Theorem 2.3.2 when $u \in C^2(\Omega)$. Indeed, when $\Omega = \{x \in \mathbb{R}^n\colon -R < x_1 < R\}$ and $M = 0$, Theorem 3.7.4 gives

$$u \leqslant (p-1)^{-1/(p-1)}\left[\mathrm{e}^{1+Rb_1/(p-1)} - 1\right] bR^{p'} \quad \text{a.e. in } \Omega. \tag{3.9.8}$$

On the other hand, when $\Omega = B_R$ we find from Theorem 3.9.4

$$u \leqslant C(p, n, R, b_1) bR^{p'} \quad \text{a.e. in } B_R. \tag{3.9.9}$$

The estimate (3.9.8) is considerably better than (3.9.9), but of course the class of equations covered by Theorem 3.9.4 contains inequalities not included in Theorem 3.7.4 (and vice versa).

When $b_1 = 0$ the explicit solution

$$u(x) = \left[n^{-1/(p-1)} b/p'\right] \cdot \left(R^{p'} - |x|^{p'}\right), \quad x \in B_R,$$

of (3.9.7) shows that the optimal estimate is $u \leqslant n^{-1/(p-1)} b R^{p'}/p'$.

A second point of comparison can be made with the estimate of Alexandrov (Theorem 9.1 of [40]). For simplicity, consider the inhomogeneous Laplace equation $\Delta u + f(x) = 0$ in the ball $\Omega = B_R$, $n \geqslant 2$, with $u \leqslant 0$ on ∂B_R. From Theorem 3.9.3 or Theorem 3.9.4, and Theorem 3.9.5, we see that, for $u \in W^{1,2}_{\mathrm{loc}}(B_R)$,

$$u \leqslant C(n, \varepsilon) R^{\varepsilon/2} \| f \|_{n/2(1-\varepsilon), B_R} \quad \text{a.e. in } B_R.$$

On the other hand, Theorem 9.1 of [40] for this case states that, for $u \in W^{2,n}(B_R)$,

$$u \leqslant C(n, R) \| f \|_{n, B_R} \quad \text{in } B_R.$$

Clearly the first estimate is better for the case in question. On the other hand, the difference in the range of operators allowed here and in Alexandrov's theory is considerable.

Finally the Lebesgue spaces (3.9.6) are in all probability best possible. For definiteness, consider the p-Laplace operator with $1 < p < n$. One can check that the function $u(x) = [\log(1/|x|)]^{\gamma}$, $\gamma > 0$, is in $W^{1,p}(B_1)$ and that u is a solution of the equation

$$\Delta_p u + b^{p-1} = 0$$

with $b \in L^{n/p'}(B_1)$, provided only that $n > p \max\{p - 1, 1/(p-1)\}$ and $\gamma < [(n-1)p - n]/(n-1)p$, while nevertheless u is unbounded as $x \to 0$.

Proofs of the results of this section are beyond the scope of the present article. For these the reader is referred to [73], Chapter 6.

3.10. Appendix: Sobolev's inequality

Here we review various results which are needed in the earlier parts of the chapter. We begin with the standard Sobolev inequality.

THEOREM 3.10.1 (Theorem 7.10 of [40]). *Let $1 \leqslant p < n$. Then there exists a constant $S(p, n)$ such that for every function $u \in W^{1,p}_0(\Omega)$, $\Omega \subset \mathbb{R}^n$, such that*

$$\|u\|_{p^*, \Omega} \leqslant S(p, n) \|Du\|_p, \quad p^* = \frac{np}{n-p}.$$

An explicit bound for $S(p, n)$ is given in [40], that is $S(p, n) \leqslant (n-1)p/\sqrt{n}(n-p)$. This is less than 1 for p suitably near 1. The case $p = 1$ is particularly simple: $S(1, n) = n^{-1}\omega^{-1/n}$, see [93] and also [34], where the result is indicated rather obscurely.

THEOREM 3.10.2 (Sobolev inequality, II). *Let $p \geqslant n$, $s > 1$. There exists a constant $Q(s,n)$ such that*

$$\|u\|_s \leqslant Q(s,n)\|Du\|_p|\Omega|^{1/n-1/p+1/s}.$$

PROOF. Define $q = ns/(n+s)$. Then $q < n \leqslant p$, $q^* = s$. Hence by Sobolev's inequality

$$\|u\|_s \leqslant S(q,n)\|Du\|_q \leqslant S(q,n)\|Du\|_p|\Omega|^{1/q-1/p}.$$

Setting $Q(s,n) = S(q,n)$ completes the proof. □

Note that if $s \to \infty$, then $q \to n$ and $Q(s,n) \to \infty$. Also, both previous results can be combined into a single one.

THEOREM 3.10.3. *Let $p, n, s \geqslant 1$, with $s < p^*$ if $p < n$. Define*

$$S(n, p; s) = \begin{cases} S(p,n), & \text{if } p < n, \\ Q(s,n), & \text{if } p \geqslant n. \end{cases}$$

Then

$$\|u\|_{s^*} \leqslant S(n, p; s)\|Du\|_p|\Omega|^{1/n-1/p+1/s}.$$

Finally, we have the simplest case of Poincaré's inequality.

THEOREM 3.10.4 (Poincaré's inequality). *Let $p \geqslant 1$. Then there exists a constant $Q = Q(n) = \omega_n^{-1/n}$ such that every function $u \in W_0^{1,p}(\Omega)$ obeys*

$$\|u\|_p \leqslant Q\|Du\|_p|\Omega|^{1/n}.$$

Notes

The early results of this chapter, Theorem 3.2.1 when $p > 1$ and Theorem 3.2.2, are special cases of the later Theorems 3.9.3 and 3.9.4, but along with their proofs are of interest in themselves. The importance of thin sets theorems such as Theorem 3.3.1 seems to have been first pointed out by Berestycki and Nirenberg [11]. Thin set theorems, however, already appear in the work of Gilbarg and Trudinger [40], Theorem 10.10. It is worth adding that by using the differencing technique of Section 2.5 one can obtain thin set comparison theorems without monotonicity conditions on the function B.

Theorems 3.5.1 and 3.5.3 generalize the corresponding Theorem 10.7 (i) and (ii) of [40], in that we treat solutions in $W^{1,\infty}_{\text{loc}}(\Omega)$ rather than $C^1(\overline{\Omega})$, and also allow the operator $\boldsymbol{A}$ and the nonlinearity B to be singular (degenerate). Theorems 3.5.1 and 3.6.1 appear in weaker forms as Theorems 10.3 and 10.1 of [71]. Theorems 3.6.1, 3.6.4, 3.7.1–3.7.4 are new; it is

interesting that they are the direct analogues of Theorems 2.2.3–2.3.3 for non-divergence operators.

The semi-maximum principle given in Theorems 3.9.1 and 3.9.2 is due to Serrin (Theorem 3 of [81]), based on earlier work for homogeneous linear equations by Stampacchia [92], Maz'ya [51], and, particularly, Moser [56]. Theorems 3.9.3–3.9.5 are combined work of Gilbarg, Serrin, Trudinger. Theorems 3.9.3 and 3.9.4 (with constant coefficients) were first stated in Theorem 9.7 of [40].

SECTION 6.4

The Strong Maximum Principle and the Compact Support Principle

4.1. Introduction

This chapter is concerned with the Strong Maximum Principle and the Compact Support Principle for singular quasilinear differential inequalities. Since these results can be less known to the reader, and at the same time are of recent research interest, we shall pay special attention to them here. Consider in the first instance the canonical divergence structure inequality

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} - f(u) \leqslant 0 \tag{4.1.1}$$

in a domain (connected open set) Ω in $\mathbb{R}^n$, $n \geqslant 2$. To begin with, the following conditions on the operator $A = A(s)$ and the nonlinearity $f = f(u)$ will be imposed.

(A1) $A \in C(\mathbb{R}^+)$,

(A2) $s \mapsto sA(s)$ *is strictly increasing in* $\mathbb{R}^+$ *and* $sA(s) \to 0$ *as* $s \to 0$;

(F1) $f \in C(\mathbb{R}_0^+)$,

(F2) $f(0) = 0$ *and* f *is non-decreasing on some interval* $(0, \delta)$, $\delta > 0$.

Condition (A2) is a minimal requirement for ellipticity of (4.1.1), allowing moreover singular and degenerate behavior of the operator A at $s = 0$, that is at critical points ($Du = 0$) of u. No assumptions of differentiability are made on either A or f when dealing with the canonical model.

For the Laplace operator, that is when (4.1.1) takes the classical form

$$\Delta u - f(u) \leqslant 0, \quad u \geqslant 0,$$

we have $A(s) \equiv 1$. Similarly, for the degenerate p-Laplace operator $\operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, here denoted by Δ_p, we have $A(s) = s^{p-2}$, while for the mean curvature operator $A(s) = 1/\sqrt{1+s^2}$. Another operator of interest is $A(s) = s^{p-2} + s^{p_1-2}$, $1 < p < p_1$, occurring in quantum physics; see [9]. Note also that (4.1.1), when equality holds, is precisely the Euler–Lagrange equation for the variational integral

$$I[u] = \int_\Omega \big\{\mathcal{G}\big(|Du|\big) + F(u)\big\}\,\mathrm{d}x, \qquad F(u) = \int_0^u f(s)\,\mathrm{d}s, \tag{4.1.2}$$

where $\mathcal{G}$ and A are related by $A(s) = \mathcal{G}'(s)/s$, $s > 0$.

Condition (A2) implies that $s \mapsto \mathcal{G}'(s)$ should be strictly increasing, so that $\mathcal{G}(|\boldsymbol{\xi}|)$ must be a symmetric strictly convex function of $\boldsymbol{\xi}$. In particular, for the area integrand $\mathcal{G}(s) = \sqrt{1+s^2} - 1$ we have $A(s) = 1/\sqrt{1+s^2}$.

In what follows, by a *classical solution* (or a classical distribution solution) of (4.1.1) in Ω we mean a function $u \in C^1(\Omega)$ which satisfies (4.1.1) in the distribution sense, as described in Section 3.1.

In order to state the Strong Maximum Principle for the inequality (4.1.1), we shall need a further definition. With the notation $\Phi(s) = sA(s)$ when $s > 0$, and $\Phi(0) = 0$, we introduce the function

$$H(s) = s\Phi(s) - \int_0^s \Phi(s)\,\mathrm{d}s, \quad s \geqslant 0. \tag{4.1.3}$$

This is easily seen to be strictly increasing, as follows from the inequality

$$s_1\Phi(s_1) - s_0\Phi(s_0) > (s_1 - s_0)\Phi(s_1) > \int_{s_0}^{s_1} \Phi(s)\,\mathrm{d}s > 0$$

when $s_1 > s_0 \geqslant 0$.

For the Laplace operator, the p-Laplace operator and the mean curvature operator, respectively, we have $H(s) = \frac{1}{2}s^2$, $H(s) = s^p/p'$ and $H(s) = 1 - 1/\sqrt{1+s^2}$. In the last example, note the anomalous behavior $\Phi(\infty) = H(\infty) = 1$, a possibility which occasionally requires extra care in the statement and treatment of results. Finally, for the variational problem (4.1.2) one has $H(s) = s\mathcal{G}'(s) - \mathcal{G}(s)$, the pre-Legendre transform of $\mathcal{G}$.

By the strong maximum principle for (4.1.1) *we mean the statement that if u is a nonnegative classical solution of* (4.1.1) *with $u(x_0) = 0$ at some point $x_0 \in \Omega$, then $u \equiv 0$ in Ω.*

THEOREM 4.1.1 (Strong Maximum Principle). *In order for the strong maximum principle to hold for* (4.1.1) *it is necessary and sufficient that either $f(s) \equiv 0$ for $s \in [0, d]$, $d > 0$, or that $f(s) > 0$ for $s \in (0, \delta)$ and*

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(F(s))} = \infty. \tag{4.1.4}$$

The choice of the base level zero for the statement of the principle is of course a matter only of convenience, as is whether we deal with minimum or maximum values at the base point x_0.

In the next result we consider the situation when the integral in (4.1.4) is convergent. Here the appropriate hypotheses are that u satisfies the converse inequality

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} - f(u) \geqslant 0, \tag{4.1.5}$$

and also "*vanishes*" at ∞, rather than at some finite point $x_0 \in \Omega$. We formalize this in the following definition.

By the compact support principle for (4.1.5) *we mean the statement that if u is a nonnegative classical solution of* (4.1.5) *in an exterior domain Ω, with $u(x) \to 0$ as $|x| \to \infty$, then u has compact support in Ω.*

THEOREM 4.1.2 (Compact Support Principle). *Let $f(s) > 0$ for $0 < s < \delta$. Then in order for the compact support principle to hold for* (4.1.5), *it is necessary and sufficient that*

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(F(s))} < \infty. \tag{4.1.6}$$

If Theorem 4.1.2 were an exact analogue of Theorem 4.1.1, the conclusion would be that $u \equiv 0$ in Ω, but this would be incorrect since (4.1.5) admits non-negative, non-trivial compact support solutions under assumption (4.1.6), see Theorem 4.3.2 below.

The existence of compact support solutions for quasilinear equations was studied extensively in the 80's, as well as other properties of the set where the solution vanishes. In chemical models, for example, when the values of a solution represent the density of a reactant, the vanishing of a solution then delineates a region, called the *dead core*, where no reactant is present (see [5,6,27,71,72,91,97]). In Section 5.4 we give an extended discussion of this phenomenon.

4.2. A more general inequality

The results described above can be extended to a wider class of differential inequalities by replacing $f(u)$ by $-B(x, u, Du)$ in (4.1.1) or (4.1.5). Here B is continuous and satisfies

$$-\text{Const.}\,\Phi\big(|\xi|\big) - g(u) \leqslant B(x, u, \xi) \leqslant \text{Const.}\,\Phi\big(|\xi|\big) - f(u) \tag{4.2.1}$$

for $x \in \Omega$, $u \geqslant 0$, with f and g obeying (F1) and (F2). See Theorem 4.7.1 below.

Some special cases of the above results are worth note. When $\Delta_p u - u^q \leqslant 0$, $p > 1$, $q > 0$, the strong maximum principle holds if and only if $q \geqslant p - 1$, while the compact support principle holds for $\Delta_p u - u^q \geqslant 0$ if and only if $0 < q < p - 1$. In particular, by the main results of [71] and [72], see Section 5.4 below, there exist C^2 non-negative radially symmetric compact support solutions of $\Delta_p u - u^q = 0$ when $0 < q < p - 1$, this being an explicit case of the earlier comment after Theorem 4.1.2.

When $q = 0$ the above analysis cannot be applied. Indeed the equation $\Delta u - 2n = 0$ in any domain Ω containing the origin admits the non-trivial solution $u(x) = |x|^2$, but $u(0) = 0$. We also note that the equation $\Delta u - c = 0$ admits no non-negative compact support solutions for any $c \in \mathbb{R}$.

An important prototype of the general situation is the equation

$$\Delta_p u - |Du|^q - f(u) = 0, \quad p > 1,\ q > 0. \tag{4.2.2}$$

Since $\Phi(s) = s^{p-1}$ for this case, condition (4.2.1) applies with $f = g$ and requires $q \geqslant p - 1$. In turn, the strong maximum principle holds for (4.2.2) when $q \geqslant p - 1$ and either $f \equiv 0$ in $[0, \mu]$, $\mu > 0$, or f obeys (4.1.4). On the other hand, when $q \in (0, p - 1)$ the strong maximum principle can fail, even when $f \equiv 0$, e.g. the C^1 function $u(x) = C|x|^k$ satisfies

$$\Delta_p u - |Du|^q = 0, \tag{4.2.3}$$

where

$$k = \frac{p-q}{s}, \quad \frac{1}{C} = k\left[\frac{(p-1)n-(n-1)q}{s}\right]^{1/s}, \quad s = p-1-q > 0$$

(for $p = 2$, this example is due to Barles, Diaz and Diaz [8]; for general $p > 1$ is due to [74]). It is of further interest in connection with this example that the compact support principle can fail even if (4.2.1) is satisfied, namely when $q > p-1$! Indeed, the function $u(x) = L|x|^{-l}$ satisfies (4.2.3) in $\Omega_R = \mathbb{R}^n \setminus \overline{B_R}$, with $l = (p-q)/t > 0$, provided that

$$q > \frac{n(p-1)}{n-1}, \quad L = \frac{1}{l}\left[\frac{(n-1)q-(p-1)n}{t}\right]^{1/t}, \quad t = q-p+1.$$

As was shown in Section 2.1, the maximum principle implies the Comparison Principle, Theorem 2.1.4. On the other hand, for singular equations, even if they are smooth, the situation is more delicate. Consider for example the equation

$$\Delta_4 u + |Du|^2 = 0, \quad n = 2. \tag{4.2.4}$$

The Strong Maximum Principle continues to hold (see Theorem 4.7.1), while on the other hand (4.2.4) admits two unequal solutions $u \equiv 0$ and $u(x) = \frac{1}{8}(R^2 - |x|^2)$ in B_R, both with the same boundary values. Thus a comparison theorem must fail.

4.3. Existence theorems

The proof of Theorem 4.1.1 will be based on a preliminary existence result for the Dirichlet problem for the equation

$$\operatorname{div}\{A(|Dv|)Dv\} - f(v) = 0. \tag{4.3.1}$$

Here it is enough to assume a weakened version of condition (F2), namely

(F3) $f(0) = 0$ *and* f *is non-negative on some interval* $[0, \delta)$*, with* δ *possibly infinite.*

Of course, in addition to (F3), conditions (A1), (A2), (F1) will be maintained throughout the section.

LEMMA 4.3.1. *Let* B_R *be an arbitrary open ball of radius* R *in* $\mathbb{R}^n$ *and let* $E_R = B_R \setminus \overline{B_{R/2}}$. *If* $\Phi(\infty) = \infty$, *then for every* $m \in (0, \delta)$ *there exists a non-negative function* $v \in C^1(\overline{E_R})$ *which is a solution of* (4.3.1) *in the annulus* E_R *with boundary values*

$$v = 0 \quad \text{on } \partial B_R, \qquad v = m \quad \text{on } \partial B_{R/2}.$$

If $\Phi(\infty) = \omega < \infty$, *the conclusion remains valid provided* $m \in (0, \delta)$ *is so small that*

$$2^{n-1}\left[\frac{R\bar{f}(m)}{2} + \Phi\left(\frac{2m}{R}\right)\right] < \omega, \tag{4.3.2}$$

where $\bar{f}(m) = \max_{u\in[0,m]} f(u)$.

Furthermore, if (F2) *holds and either* $f \equiv 0$ *in* $[0, d]$, $d > 0$, *or* (4.1.4) *is valid, then* $|Dv| > 0$ *in* $\overline{E_R}$ *and in particular* $\partial_{\boldsymbol{\nu}} v < 0$ *on* ∂B_R, *where* $\boldsymbol{\nu}$ *is the exterior unit normal to* B_R.

This lemma is proved in [71, Propositions 4.1 and 4.4]; see also [73, Propositions 4.2.1 and 4.2.2]. In particular, in these results choose $q(t) = (R-t)^{n-1}$, $t \in [0, R/2]$, and $v(x) = w(t)$, $t = R - r$, $r = |x - x_0|$, where x_0 denotes the center of B_R. Moreover, the solution v is a radial function in the annulus. Lemma 4.3.1 can be compared with Lemma 2.7.2.

The proof of Theorem 4.1.2 will similarly be based on an existence result for the Dirichlet problem in an exterior domain Ω_R, a result which is moreover of independent interest. In place of (F3) we shall assume the slightly stronger condition

(F2′) $f(0) = 0$ *and* f *is positive and non-decreasing in* $(0, \delta)$, $\delta > 0$ *finite*.

Clearly (F2′) implies (F2) which implies (F3).

THEOREM 4.3.2 (Exterior Dirichlet Problem). *For all* $R > 0$ *and* $m \in (0, \delta)$, *with* m *sufficiently small if* $\Phi(\infty) = \omega < \infty$, *there is a* C^1 *radial solution* $u(x) = u(r)$ *of the problem*

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} - f(u) = 0, \quad u \geqslant 0 \tag{4.3.3}$$

in $\Omega_R = \{x \in \mathbb{R}^n\colon |x| > R\}$, *such that*

$$u(R) = m, \qquad u(x) \to 0 \quad \text{as } |x| \to \infty. \tag{4.3.4}$$

Moreover $u' < 0$ *whenever* $u > 0$.

Furthermore, if (4.1.4) *holds then* $u > 0$ *in* Ω_R, *while conversely if* (4.1.6) *is satisfied then* u *is compactly supported in* Ω_R.

The required smallness condition on m when $\omega < \infty$ is given by

$$\left(\frac{R+1}{R}\right)^{n-1}\big[f(m) + \Phi(m)\big] = \widehat{\omega} < \omega. \tag{4.3.5}$$

The theorem is proved in [71, Theorems 5.1 and 5.2]; see also [73, Theorems 4.3.1 and 4.3.3].

4.4. The dead core lemma

The failure of the strong maximum principle, Theorem 4.1.1, when the function f satisfies (4.1.6) is due to the existence of solutions of (4.1.1) which have a minimum value, say 0, in the domain Ω, but are not identically 0. In the following lemma we consider this possibility in more detail.

Here we maintain the conditions (A1), (A2), (F1) and (F2).

LEMMA 4.4.1 (Dead core lemma). *Suppose* $f(z) > 0$ *for* $z > 0$ *and that* (4.1.6) *is satisfied. For fixed* $\sigma > 0$, *define*

$$C_\sigma = \int_0^\delta \frac{\mathrm{d}s}{H^{-1}(\sigma F(s))}. \tag{4.4.1}$$

Then for every $C \in (0, C_\sigma)$ *there exists a number* $\gamma = \gamma(C) \in (0, \delta)$ *and a function* $w \in C^1[0, C]$ *such that*

(i) $\gamma \to 0$ *as* $C \to 0$,
(ii) $w(0) = w'(0) = 0$, $w(C) = \gamma$; $0 \leqslant w' \leqslant H^{-1}(F(\gamma))$,
(iii) $[\Phi(w'(t))]' = \sigma f(w(t))$ *for* $t \in (0, C)$,
(iv) $\Phi(w'(t)) \leqslant \sigma t f(w(t))$ *for* $t \in (0, C)$.

[*If* $H(\infty)$ *is finite we take* $\delta > 0$ *so small that* $\sigma F(\delta) < H(\infty)$.]

In order to prove this lemma it is convenient first to have two preliminary results.

LEMMA 4.4.2.

(i) *For any constant* $\sigma \in [0, 1]$ *there holds*

$$F(\sigma u) \leqslant \sigma F(u), \quad u \in [0, \delta); \qquad F(u) = \int_0^u f(v)\,\mathrm{d}v.$$

(ii) *Let* $w = w(t)$ *be of class* $C^1(0, T)$, *and write* $' = \mathrm{d}/\mathrm{d}t$. *If* $\Phi \circ w'$ *is of class* $C^1(0, T)$ *then* $H \circ w'$ *is of class* $C^1(0, T)$, *and in this case*

$$\big[H\big(w'(t)\big)\big]' = w'(t)\big[\Phi\big(w'(t)\big)\big]' \quad \text{in } (0, T). \tag{4.4.2}$$

Conversely, if $H \circ w'$ *is of class* $C^1(0, T)$ *and* $w' > 0$, *then* $\Phi \circ w'$ *is of class* $C^1(0, T)$ *and* (4.4.2) *continues to be satisfied.*

To obtain (i), observe that $\sigma f(\sigma u) \leqslant \sigma f(u)$ for $u \in [0, \delta]$, since f is non-decreasing by (F2). Integrating this relation from 0 to u yields the result.

The first statement of (ii) is an immediate consequence of the monotonicity of H, a result which follows also from the representation

$$H(s) = \int_0^{\Phi(s)} \Phi^{-1}(\omega)\,\mathrm{d}\omega, \quad s \geqslant 0, \tag{4.4.3}$$

this being a consequence of the Stieltjes formula $H(s) = \int_0^s \sigma\,\mathrm{d}\Phi(\sigma)$.

The second part is also a consequence of (4.4.3) together with a small lemma:

Let I *be any interval of* $\mathbb{R}$ *and let*

$$B(t) = \int_{a(t_0)}^{a(t)} b(s)\,\mathrm{d}s, \quad t, t_0 \in I.$$

Suppose $a, b \in C(I)$, $B \in C^1(I)$ and $b > 0$. Then $a \in C^1(I)$ and $a' = B'/(b \circ a)$.

This is easily demonstrated by using difference coefficients and the integral mean value theorem to get $\Delta B/\Delta t = b(a + \theta\Delta a)\Delta a/\Delta t$, $0 \leqslant \theta \leqslant 1$. The lemma then follows by dividing by $b(a + \theta\Delta a)$ and letting $\Delta t \to 0$. □

LEMMA 4.4.3. *Suppose $f(z) > 0$ for $z > 0$ and (in case $H(\infty) < \infty$) that $F(\delta) < H(\infty)$. Let $\sigma > 0$. If* (4.1.6) *is satisfied, then*

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(\sigma F(s))} < \infty,$$

while if (4.1.4) *holds, then*

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(\sigma F(s))} = \infty.$$

PROOF. To show the first part of the lemma it is obviously enough to consider values $\sigma < 1$. In this case, by Lemma 4.4.2(i),

$$\int_\varepsilon^\delta \frac{\mathrm{d}s}{H^{-1}(\sigma F(s))} \leqslant \int_\varepsilon^\delta \frac{\mathrm{d}s}{H^{-1}(F(\sigma s))} = \frac{1}{\sigma}\int_{\varepsilon\sigma}^{\delta\sigma} \frac{\mathrm{d}t}{H^{-1}(F(t))}$$

and the first part now follows by letting $\varepsilon \to 0$ and applying (4.1.6).

On the other hand, for the second part of the lemma it is enough to consider only values $\sigma > 1$. Then, for small $\varepsilon > 0$, we have by Lemma 4.4.2(i)

$$\int_{\varepsilon/\sigma}^{\delta/\sigma} \frac{\mathrm{d}s}{H^{-1}(\sigma F(s))} \geqslant \int_{\varepsilon/\sigma}^{\delta/\sigma} \frac{\mathrm{d}s}{H^{-1}(F(\sigma s))} = \frac{1}{\sigma}\int_\varepsilon^\delta \frac{\mathrm{d}t}{H^{-1}(F(t))}.$$

Letting $\varepsilon \to 0$ and applying (4.1.4) gives the second result. □

PROOF OF LEMMA 4.4.1. First note that the integral in (4.4.1) is convergent, in view of Lemma 4.4.3 and (4.1.6). For given $C \in (0, C_\sigma)$, we take $\gamma \in (0, \delta)$ so that

$$0 < C = \int_0^\gamma \frac{\mathrm{d}s}{H^{-1}(\sigma F(s))};$$

clearly $\gamma = \gamma(C)$ is uniquely determined by C, and of course $\gamma \to 0$ as $C \to 0$.

Now define $w : [0, C] \to \mathbb{R}$ by

$$t = \int_0^{w(t)} \frac{\mathrm{d}s}{H^{-1}(\sigma F(s))}.$$

Hence

$$\frac{w'(t)}{H^{-1}(\sigma F(w(t)))} = 1,$$

that is $H(w') = \sigma F(w)$ and in turn $[H(w')]' = \sigma f(w)w'$. Obviously part (ii) of the lemma is satisfied; moreover, since $w' > 0$ on $(0, C]$, from Lemma 4.4.2(ii) we obtain part (iii).

An integration using parts (ii), (iii) and the monotonicity of f in (F2) shows that also $\Phi(w'(t)) \leqslant \sigma t f(w(t))$. This completes the proof. □

There is a slightly stronger result, proved in the same way.

LEMMA 4.4.4. *The conclusions* (i)–(iii) *of Lemma* 4.4.1 *are valid if* (F2), *together with the positivity of* f, *is replaced by the weaker condition that* $f(0) = 0$ *and* F *is positive in some interval* $(0, \delta)$.

4.5. Proof of the Strong Maximum Principle

With the work of the preceding Sections 4.1–4.4 available, we can turn to the proofs of the Strong Maximum Principle, Theorem 4.1.1, and the Compact Support Principle, Theorem 4.1.2, stated above.

PROOF OF SUFFICIENCY IN THEOREM 4.1.1. We proceed exactly as in Section 2.8, with the two exceptions that (a) the weak maximum principle, Theorem 2.8.1, is replaced by Theorem 2.4.1 and Proposition 2.4.2, and (b) Lemma 2.8.2 is replaced by Lemma 4.3.1 under condition (4.1.4). □

PROOF OF NECESSITY IN THEOREM 4.1.1. As remarked in Section 4.1, the necessity is due to Diaz [26]. Hence Theorem 4.1.1 is proved. □

ANOTHER PROOF OF THE NECESSITY OF (4.1.4). Suppose that $F > 0$ in some interval $(0, \delta)$, and that (4.1.4) fails, that is (4.1.6) holds. By the dead core Lemma 4.4.4 we can then introduce the function $w = w(t)$, of class $C^1[0, C]$, $C \in (0, C_1)$, $\sigma = 1$. Let $\Omega = \{x \in \mathbb{R}^n\colon\ x_n < C\}$ and define $u(x) = 0$ if $x_n \leqslant 0$, $u(x) = w(x_n)$ if $x_n \in [0, C)$. Hence $u \in C^1(\Omega)$ is non-negative by Lemma 4.4.4(ii), and is also a solution of (4.1.1), *with the equality sign*, by Lemma 4.4.4(iii). Clearly $u(\mathbf{0}) = w(0) = 0$ and at the same time $u \not\equiv 0$. Hence the strong maximum principle fails. □

REMARKS.

1. The proof of sufficiency we have given is in fact not different in its underlying ideas from those in [10,20,28,76,97], the principal improvements being the direct approach, the generality of the equation and of the solution class, and the clarification of the method. The proof here uses only standard calculus, and the elementary Leray–Schauder theorem (see [40], Theorem 11.6), but requires neither monotone operator theory (as [97,27–29]), nor Orlicz–Sobolev space theory, nor viscosity solution theory (as [44]), nor probabilistic methods.

   We note also that Diaz, Saa and Thiel have stated a version of Theorem 4.1.1, see Theorem 6 of [29], but with partially insufficient proof.

2. The necessity of condition (4.1.4) for the Strong Maximum Principle can be obtained under a weaker hypothesis than (F2). In fact, it is enough to suppose only

$$f(0) = 0 \quad \text{and either} \quad f \equiv 0 \quad \text{or} \quad F(s) > 0 \quad \text{for } s \in (0, \delta).$$

   This is because the principal construction required for Diaz' proof uses only this condition; see also the second proof of necessity of Theorem 4.1.1 given above.
3. The second proof of necessity for the Strong Maximum Principle also yields a direct and simple counterexample to the unique continuation question for the equation $\operatorname{div}\{A(|Du|)Du\} - f(u) = 0$, when (4.1.6) holds. That is, the function $u(x) = w(x_n)$ shows that a solution in a domain Ω may vanish in a subdomain without vanishing throughout Ω.

EXAMPLE: THE DEGENERATE LAPLACIAN. The strong maximum principle can be treated more simply in the case of the canonical p-Laplacian inequality, $p > 1$,

$$\Delta_p u - f(u) \leqslant 0, \quad u \geqslant 0.$$

For our present purpose, we assume that

$$f(u) \leqslant cu^{p-1}, \tag{4.5.1}$$

the borderline case for (4.1.4).

An appropriate comparison function $v = v(r)$, $r = |x|$, can be taken in the form

$$v(r) = \alpha\left[\left(\frac{R}{r}\right)^{\vartheta} - 1\right], \quad \frac{R}{2} \leqslant r \leqslant R, \tag{4.5.2}$$

where $\alpha = m/(2^{\vartheta} - 1)$ and ϑ, R are to be determined. Then

$$\Phi\left(|v'|\right) = |v'|^{p-1} = \left(\frac{\alpha\vartheta}{R}\right)^{p-1}\left(\frac{R}{r}\right)^{(p-1)(\vartheta+1)};$$

and, after a short calculation, there results

$$\begin{aligned}&\left[|v'|^{p-1}\right]' + \frac{n-1}{r}|v'|^{p-1} + f(v)\\ &\quad\leqslant (\alpha\vartheta)^{p-1}\left(\frac{R}{r}\right)^{(p-1)\vartheta}\left\{\frac{n-1-(p-1)(\vartheta+1)}{r^p} + \frac{c}{\vartheta^{p-1}}\right\}.\end{aligned}$$

This again will be $\leqslant 0$ provided that

$$\vartheta = \frac{2(n-1)}{p-1} - 1, \qquad R \leqslant \left[\frac{(n-1)}{c}\right]^{1/p}\vartheta^{1/p'}.$$

That is, $\Delta_p v - f(v) \geqslant 0$ for $R/2 \leqslant |x| \leqslant R$, and the proof of the strong maximum principle, Theorem 4.1.1, now applies unchanged, but avoiding the delicate arguments of Lemma 4.3.1, or of [97].

In summary, for the borderline case (4.5.1) of the p-Laplacian inequality, we get an elementary proof of Vázquez' strong maximum principle.

The simple comparison function (4.5.2) does not suffice for general operators or for more complicated nonlinearities. This observation indicates the need for the deeper lying construction of v in Lemma 4.3.1, used in the proof of Theorem 4.1.1.

4.6. Proof of the Compact Support Principle

PROOF OF SUFFICIENCY IN THEOREM 4.1.2. Let u be a (non-negative) solution of (4.1.5) in an exterior domain $\Omega \supset \Omega_R$ with $u(x) \to 0$ as $|x| \to \infty$. We must show that u has compact support in Ω. To begin with, clearly there exists $R_0 \geqslant R$ such that $u(x) \leqslant \delta' < \delta$ if $|x| \geqslant R_0$. Let $w = w(t)$ be the function introduced in the alternative proof of the necessity of Theorem 4.1.1, with $\sigma = 1$ and with C chosen so near C_1 that $\gamma(C) \geqslant \delta'$. Define $\Omega_0 = \{x \in \mathbb{R}^n\colon\ |x| > R_0\}$ and $v(x) = w(C + R_0 - |x|)$ for $R_0 < |x| \leqslant C + R_0$. We extend the definition of v to all $x \in \Omega_0$ by setting $v(x) = 0$ when $|x| > C + R_0$. Clearly $v \in C^1(\Omega_0)$ by Lemma 4.4.4(ii). Moreover, for $x \in \Omega_0$ and $r = |x|$, we have

$$\begin{aligned} \operatorname{div}\{A(|Dv|)Dv\} - f(v) &= -\big[\Phi\big(|v'|\big)\big]' - \frac{(n-1)}{r}\Phi\big(|v'|\big) - f(v) \\ &\leqslant \big[\Phi(w_t)\big]_t - f(w) \leqslant 0 \end{aligned} \tag{4.6.1}$$

in view of Lemma 4.4.4(iii) and (A2). Since $u(x) \leqslant \delta' \leqslant v(x)$ on $\partial\Omega_0$, and since $u(x)$, $v(x) \to 0$ as $|x| \to \infty$, we can apply the comparison Theorem 2.4.1 and Lemma 2.4.2 to obtain $0 \leqslant u(x) \leqslant v(x)$ in Ω_0. In particular, $u(x) \equiv 0$ when $|x| \geqslant R_1 = R_0 + C$, as required. □

PROOF OF NECESSITY IN THEOREM 4.1.2. To prove necessity, suppose (4.1.6) fails, that is (4.1.4) holds. By Theorem 4.3.2 there exists a *positive* classical solution u of (2.4.1) (and thus also of (4.1.5) with equality) in the exterior domain Ω_R, such that $u(x) \to 0$ as $|x| \to \infty$. This violates the compact support principle. Hence (4.1.6) is necessary. □

REMARKS.

1. The sufficiency part of Theorem 4.1.2 is closely related to Theorem 4 of [76], by specializing the results there to the matrix $a_{ij} = A(|\boldsymbol{\xi}|)\delta_{ij} + [A'(|\boldsymbol{\xi}|)/|\boldsymbol{\xi}|]\xi_i\xi_j$ which arises by expansion of the divergence term in (4.1.5). This specialization requires, however, two assumptions which are not needed here, first that the operator A be of class $C^1(\mathbb{R}^+)$, and second, that the solutions in consideration should be of class C^2 at points of Ω where $Du \neq \mathbf{0}$. In the proof of Theorem 4 of [76] it is also not evident that an appropriate comparison principle can be applied without the further

assumption that the nonlinearity f be non-decreasing for small $u > 0$ – that is, for the validity of Theorem 4 of [76] this additional assumption, which is exactly (F2) above, seems to be required as well. For the special case of the degenerate Laplacian, see also [28].

2. The last sentence of the proof of the sufficiency of Theorem 4.1.2 gives an a priori estimate for the support of the solution u.
3. The sufficiency of condition (4.1.6) for the Compact Support Principle can be obtained under a weaker hypothesis than (F2). In fact, it is enough to suppose only

$$f(0) = 0 \quad \text{and either} \quad f \equiv 0 \quad \text{or} \quad F(s) > 0 \quad \text{for } s \in (0, \delta),$$

this condition in fact being all that it is needed for the application of Lemma 4.4.4.
4. For the case of maximal monotone graphs f see [28,97].

4.7. Strong Maximum Principle: Generalized version

Consider the differential inequality

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + B(x, u, Du) \leqslant 0 \tag{4.7.1}$$

in a domain $\Omega \subset \mathbb{R}^n$. We suppose that A satisfies (A2) and a slightly stronger condition than (A1), that is

(A1′) $A \in C^1(\mathbb{R}^+)$,

and that $B(x, z, \boldsymbol{\xi})$ subject to one or the other of the conditions (B1), (B2) below:

There exist a constant $\kappa > 0$ and nonlinearities f and g, continuous in $\mathbb{R}_0^+$, such that

(B1) $B(x, z, \boldsymbol{\xi}) \geqslant -\kappa\Phi(|\boldsymbol{\xi}|) - f(z)$,

(B2) $B(x, z, \boldsymbol{\xi}) \leqslant \kappa\Phi(|\boldsymbol{\xi}|) - g(z)$,

for $x \in \Omega$, $z \geqslant 0$, and all $\boldsymbol{\xi} \in \mathbb{R}^n$ with $|\boldsymbol{\xi}| \leqslant 1$.

Moreover f and g are assumed to satisfy

(F2) $f(0) = 0$ *and f is non-decreasing on some interval* $(0, \delta)$, $\delta > 0$;

(G2) $g(0) = 0$ *and g is non-decreasing on some interval* $(0, \delta)$, $\delta > 0$.

In the following results $B(x, z, \boldsymbol{\xi})$ *itself need not* be explicitly non-decreasing in the variable z; this corresponds to the situation of Theorem 2.1.2 where the coefficient $c(x)$ is *not* required to satisfy a sign condition for the validity of the conclusion.

THEOREM 4.7.1 (Strong maximum principle). *Let* (B1) *and* (F2) *be satisfied. For the strong maximum principle to be valid for* (4.7.1) *it is sufficient that either $f \equiv 0$ in $[0, d]$, $d > 0$, or that* (4.1.4) *holds.*

Assume (B2) *and* (G2). *For the strong maximum principle to hold for* (4.7.1) *it is necessary that either $g \equiv 0$ for $u \in [0, d]$, $d > 0$, or that*

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(G(s))} = \infty, \tag{4.7.2}$$

where $G(u) = \int_0^u g(s)\,\mathrm{d}s$.

PROOF. *Sufficiency*. Assume that (4.7.2) is valid. As in the proof of Theorem 4.1.1, we apply the Hopf comparison technique. Assume, contrary to the validity of the strong maximum principle, that there is a non-negative solution $u \in C^1(\Omega)$ of (4.7.1) which vanishes at some point, but is not identically zero. As in the demonstration of the Hopf Maximum Principle, Section 2.8, there is a ball B_R, with closure in Ω, such that $u > 0$ in B_R and $u = 0$ at some point $y \in \partial B_R \cap \Omega_0$, where $\Omega_0 = \{x \in \Omega\colon\ u(x) = 0\}$. Clearly $u(y) = |Du(y)| = 0$ and R can be taken arbitrarily small so that

$$0 < u < \delta, \quad |Du| \leqslant 1 \quad \text{in } B_R.$$

Hence by (B1) the function u is also a solution of

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} - \kappa\Phi\big(|Du|\big) - f(u) \leqslant 0 \quad \text{in } E_R, \tag{4.7.3}$$

where $E_R = B_R \setminus \overline{B}_{R/2}$.

Call x_0 the center of B_R. Also let $m' > 0$ be the minimum of u on $\partial B_{R/2}$ and choose

$$k = n + \kappa R.$$

As comparison function we take the non-negative radial solution $v : E_R \to \mathbb{R}^+$ of (4.3.1) given by Lemma 4.3.1, *in the space dimension* k rather than n, that is v as a function of r, $r = |x - x_0|$, satisfies the ordinary differential equation

$$\big[r^{k-1}\Phi\big(|v'|\big)\big]' + r^{k-1} f(v) = 0, \quad v \geqslant 0$$

in $(R/2, R)$. For later purposes one take the corresponding boundary value parameter m so small that $m \leqslant m'$. In turn, in contrast with (4.7.3), v becomes a solution of the inequality:

$$\begin{aligned}
&\operatorname{div}\big\{A\big(|Dv|\big)Dv\big\} - \kappa\Phi\big(|Dv|\big) - f(v) \\
&\quad = -r^{1-n}\big\{\big[r^{n-1}\Phi\big(|v'|\big)\big]'\big\} - \kappa\Phi\big(|v'|\big) - f(v) \\
&\quad = -r^{1-k}\big\{\big[r^{k-1}\Phi\big(|v'|\big)\big]'\big\} + \kappa\left(\frac{R}{r} - 1\right)\Phi\big(|v'|\big) - f(v) \\
&\quad \geqslant -r^{1-k}\big\{\big[r^{k-1}\Phi\big(|v'|\big)\big]'\big\} - f(v) = 0
\end{aligned} \tag{4.7.4}$$

in E_R with, see Lemma 4.3.1,

$$\begin{aligned}
&v = 0 \quad \text{on } \partial B_R, \qquad v = m \quad \text{on } \partial B_{R/2}, \\
&\partial_{\boldsymbol{\nu}} v < 0 \quad \text{on } \partial B_R, \qquad |Dv| > 0 \quad \text{in } \overline{E}_R.
\end{aligned}$$

We can now apply Theorem 3.6.5 to the solutions u of (4.7.3) and v of (4.7.4) in the set E_R. In making this application one must of course check that the principal hypotheses (i)–(ii), see Section 3.5, are verified for $\boldsymbol{A}(\boldsymbol{\xi}) = A(|\boldsymbol{\xi}|)\boldsymbol{\xi}$, with $\boldsymbol{A}(\mathbf{0}) = \mathbf{0}$ and with the regular set $\boldsymbol{P} = \mathbb{R}^n \setminus \{\mathbf{0}\}$. This, however, follows directly from (A1′) and (A2).

To verify the further assumptions of Theorem 3.6.5, we see by (F2) that the function $-\kappa\Phi(|\boldsymbol{\xi}|) - f(z)$ is non-increasing in the variable z in the solution range $[0, \delta)$, while by (A1′) it is locally Lipschitz continuous when $\boldsymbol{\xi}$ is in $\boldsymbol{P}$. Finally, since $|Dv| > 0$ it is evident that $(E_R)_v = E_R$, where $(E_R)_v = \{x \in E_R\colon\ Dv(x) \in \boldsymbol{P}\}$.

Because $u \geqslant v$ on ∂E_R, we then obtain from Theorem 3.6.5 that $u \geqslant v$ in E_R. In particular $0 = \partial_{\boldsymbol{\nu}} u(y) \leqslant \partial_{\boldsymbol{\nu}} v(y) < 0$, which is a contradiction. The sufficiency part of the theorem is therefore proved.

Necessity. For each $x_0 \in \Omega$ we shall exhibit a subdomain Ω', with $x_0 \in \Omega'$, and a solution u of (4.7.1) in Ω' such that $u(x_0) = 0$ but $u \not\equiv 0$ in Ω'. The assumption to be made for this purpose is that (B2) and (G2) hold, with $g(z) > 0$ for $z > 0$, together with the negation of (4.7.2), namely

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(G(s))} < \infty. \tag{4.7.5}$$

Thus fix $x_0 \in \Omega$ and let $B_R \subset \Omega$ be a ball centered at x_0 with radius R. Define

$$\sigma = (n + \kappa R)^{-1},$$

where κ is given by (B2). Let C_σ be given by (4.4.1), with F replaced by G. Then choose $C < \min\{R, C_\sigma\}$, also so small that $H^{-1}(G(\gamma)) \leqslant 1$, where the parameter $\gamma = \gamma(C) > 0$ is defined in Lemma 4.4.1.

Put $\varepsilon = R - C$ and consider the function w given by Lemma 4.4.1 corresponding to the given value σ. For $x \in B_R$ we define the radial function $u(r) = w(r - \varepsilon)$ when $r \in [\varepsilon, R]$, $r = |x - x_0|$, and extend u as a non-negative C^1 function to all of B_R by putting $u \equiv 0$ for $0 \leqslant r < \varepsilon$. Then $|Du| = u' \leqslant 1$ in B_R by (ii) of Lemma 4.4.1, and so by (B2)

$$\begin{aligned}
\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + B\big(x, u(x), Du(x)\big) &\leqslant \operatorname{div}\big\{A\big(|Du|\big)Du\big\} + \kappa\Phi(u') - g(u) \\
&\leqslant \big[\Phi(u')\big]' + \left(\kappa + \frac{n-1}{r}\right)\Phi(u') - g(u) \\
&\leqslant \sigma g(u) + \left(\kappa + \frac{n-1}{r}\right)(r - \varepsilon)\sigma g(u) - g(u) \\
&\leqslant \big[\sigma(n + \kappa R) - 1\big]g(u) = 0;
\end{aligned}$$

here we use (iii) and (iv) of Lemma 4.4.1.

But u vanishes in $B_\varepsilon(x_0)$, while $u(x) = w(C) = \gamma > 0$ when $|x| = R$, that is $u \not\equiv 0$ in $\Omega' = B_R$, contradicting the validity of the strong maximum principle. □

It is exactly in the application of Theorem 3.6.5 at the end of the proof of sufficiency that the strengthened condition (A1′) is needed.

Theorem 4.7.1 implies as well a necessary and sufficient criterium for the validity of the strong maximum principle.

COROLLARY 4.7.2. *Assume* (B1), (B2), (F2) *and* (G2). *Suppose that there exists* $\nu \in (0, 1]$ *such that* $g(z) \geqslant \nu f(z)$ *for* $z \in [0, \delta)$. *Then the strong maximum principle is valid for* (4.8.1) *if and only if either* $f \equiv 0$ *in* $[0, d]$, $d > 0$, *or* (4.1.4) *holds*.

When $A(s) = 1/\sqrt{1+s^2}$, i.e., the mean curvature operator, then Theorem 4.7.1 yields

COROLLARY 4.7.3. *Assume* (B1), (F2). *Then the strong maximum principle is valid for the mean curvature differential inequality*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + B(x, u, Du) \leqslant 0 \quad \text{in } \Omega, \tag{4.7.6}$$

if either $f \equiv 0$ *in* $[0, d)$, $d > 0$, *or* (4.1.4) *is satisfied*.

Assume (B2), (G2). *For the strong maximum principle to hold for* (4.7.6) *it is necessary that either* $g \equiv 0$ *in* $[0, d]$, $d > 0$, *or that* (4.7.2) *holds*.

Equation (4.8.1) also has a corresponding boundary point lemma. Remarkably, in contrast with Hopf's boundary point lemma the basic equation *need not be uniformly elliptic*, this ultimately being due to the strong result of Lemma 4.3.1.

THEOREM 4.7.4 (Boundary Point Lemma). *Suppose that* (B1), (F2) *hold and that either* $f \equiv 0$ *in* $[0, d]$, $d > 0$, *or that* (4.1.4) *is satisfied*.

Let u *be a* C^1 *solution of* (4.8.1) *in* $\overline{\Omega}$, *with* $u > 0$ *in* Ω *and* $u(y) = 0$, *where* $y \in \partial\Omega$. *If* Ω *satisfies an interior sphere condition at* y, *then* $\partial_\nu u < 0$ *at* y.

PROOF. By the interior sphere condition there is an open ball $B_R = B_R(x_0) \subset \Omega$ with $y \in \partial B_R$. If R is suitably small, then there exists, exactly as in the proof of the sufficiency of Theorem 4.7.1, a comparison function v in the annular region $E_R = B_R \setminus \overline{B}_{R/2}$. Continuing as in the proof of Theorem 4.7.1 it follows that $u \geqslant v$ in E_R, which immediately supplies the conclusion $\partial_\nu u(y) \leqslant \partial_\nu v(y) = v'(R) < 0$. □

There is a related strong maximum principle for the converse inequality

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + B(x, u, Du) \geqslant 0 \tag{4.7.7}$$

in a domain $\Omega \subset \mathbb{R}^n$.

THEOREM 4.7.5 (Strong Maximum Principle). *Suppose that* (4.7.7) *holds and that* (B2) *applies with* $g(z) \geqslant 0$ *for all* $z \geqslant 0$. *If* u *is a non-negative solution of* (4.7.7), *then* u *cannot attain a maximum value* M *in the interior of* Ω *unless* $u \equiv M$.

PROOF. Suppose u reaches a maximum value M in Ω. Define $\bar{u} = M - u$. Then $\bar{u}$ is non-negative and obeys the inequality

$$\operatorname{div}\big\{A\big(|D\bar{u}|\big)D\bar{u}\big\} - \kappa\Phi\big(|D\bar{u}|\big) \leqslant 0$$

at all points of Ω where $|D\bar{u}| < 1$. This has exactly the form (4.8.1) with

$$B(x, z, \xi) = -\kappa \Phi\big(|\xi|\big).$$

That is (B1), (F2) hold with $f \equiv 0$.

We can therefore apply the first part of Theorem 4.7.1 to $\bar{u}$. That is, since $\bar{u} = 0$ at a point of Ω, then $\bar{u} \equiv 0$ in Ω, i.e. $u \equiv M$ in Ω. □

COROLLARY 4.7.6. *Under the assumptions of Theorem* 4.7.5, *if u is a non-negative solution of* (4.7.7) *in Ω and $u \leqslant m$ on $\partial\Omega$, then $u \leqslant m$ in Ω.*

PROOF. If $u > m$ at some point in Ω, then define $M = \max u$ in Ω. In turn, Theorem 4.7.7 yields $u \equiv M$, which is absurd. □

4.8. Further extensions of the Strong Maximum Principle

Consider the differential inequality

$$\partial_{x_j}\big\{a_{ij}(x, u)A\big(|Du|\big)\partial_{x_j}u\big\} + B(x, u, Du) \leqslant 0 \tag{4.8.1}$$

in a domain $\Omega \subset \mathbb{R}^n$, where the symmetric coefficient matrix $\boldsymbol{a}(x, z) = [a_{ij}(x, z)]$, $i, j = 1, \ldots, n$, is defined and continuously differentiable in $\Omega \times [0, \delta']$ for some $\delta' > 0$, and furthermore is such that

$$\lambda(z)|\zeta|^2 \leqslant a_{ij}(x, z)\zeta_i\zeta_j \leqslant \Lambda(z)|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^n, \tag{4.8.2}$$

where λ and Λ are positive and continuous in $[0, \delta']$. We suppose that $A = A(s)$ satisfies of (A1′) and (A2) of Section 4.7. Moreover $B(x, z, \xi) \in L^{\infty}_{\text{loc}}(\Omega \times \mathbb{R}^+ \times \mathbb{R}^n)$ is subject to one or the other of the conditions (B1) or (B2) of the previous section, while f verifies (F2) and g the analogous condition (G2). The function $B(x, z, \xi)$ need not be explicitly non-decreasing in the variable z.

THEOREM 4.8.1 (Strong Maximum Principle). *Suppose that*

$$\lim_{s \downarrow 0} \frac{sA'(s)}{A(s)} = c > -1 \tag{4.8.3}$$

and, when $c \neq 0$, assume also that the positive definite matrix $[a_{ij}]$ satisfies (4.8.2) *and*

$$\sqrt{\frac{\Lambda(0)}{\lambda(0)}} < \phi(c), \quad \phi(c) = \frac{2 + c + 2\sqrt{1 + c}}{|c|}. \tag{4.8.4}$$

Let (B1) *and* (F2) *be satisfied. For the strong maximum principle to be valid for* (4.8.1) *it is sufficient that either $f \equiv 0$ in $[0, d]$, $d > 0$, or that* (4.1.4) *holds.*

Assume (B2) *and* (G2). *For the strong maximum principle to hold for* (4.8.1) *it is necessary that either* $g \equiv 0$ *in* $[0, d]$, $d > 0$, *or that* (4.7.2) *is satisfied.*

Consider also the converse inequality

$$\partial_{x_j}\{a_{ij}(x)A(|Du|)\partial_{x_j}u\} + B(x, u, Du) \geqslant 0, \tag{4.8.5}$$

the domain Ω being an exterior set, say with $\Omega \supset \Omega_R = \{|x| > R\}$. Conditions (A1′), (A2) and (4.8.2) are assumed to be valid, as for the Strong Maximum Principle. On the other hand, we restrict the matrix $[a_{ij}]$ to depend only on x, with the coefficients $a_{ij}(x)$ having uniformly bounded derivatives in Ω. The functions λ and Λ in (4.8.2) are now purely positive constants. (A corresponding boundedness condition on the derivatives of a_{ij} is unneeded for the Strong Maximum Principle because the arguments there are purely local.)

THEOREM 4.8.2 (Compact Support Principle[9]). *If* (B1) *and* (F2) *are satisfied, with* $f(z) > 0$ *for* $z > 0$, *then for the compact support principle to hold for* (4.8.5) *it is necessary that* (4.1.6) *be valid.*

On the other hand, assume (4.8.3), *and when* $c \neq 0$ *that* (4.8.4) *is also verified. Then for the compact support principle to hold for* (4.8.5) *it is sufficient that* (B2) *and* (G2) *are satisfied, with* $g(z) > 0$ *for* $z > 0$ *and*

$$\int_{0^+} \frac{ds}{H^{-1}(G(s))} < \infty. \tag{4.8.6}$$

As in the case of the strong maximum principle, Theorem 4.8.2 implies a necessary and sufficient criterium for the validity of the compact support principle.

COROLLARY 4.8.3. *Assume* (B1), (B2), (F2), (G2). *Suppose that there exists* $\nu \in (0, 1]$ *such that* $g(z) \geqslant \nu f(z) > 0$ *for* $z > 0$. *Then the compact support principle is valid for* (4.8.5) *if and only if* (4.1.6) *holds.*

We have also the following consequence of Theorem 4.8.2 for the mean curvature case.

COROLLARY 4.8.4. *Assume* (B2), (G2) *are satisfied, with* $g(z) > 0$ *for* $z > 0$, *and that* (4.8.6) *holds. Then the compact support principle is valid for the mean curvature differential inequality*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) - B(x, u, Du) \geqslant 0 \quad \text{in } \Omega. \tag{4.8.7}$$

On the other hand, if (B1), (F2) *are valid, with* $f(z) > 0$ *for* $z > 0$, *then for the compact support principle to hold for* (4.8.7) *it is necessary that* (4.1.6) *is satisfied.*

[9] For the definition of the compact support principle, see the first paragraph before Theorem 4.1.2 in Section 4.1.

It is clear from the proof of the *necessity* of the Compact Support Principle that the matrix $[a_{ij}]$ in this case can depend on z as well as x, since the solution u considered there, together with its gradient, is a priori bounded, see (5.6.4) of [73]. That is $\alpha = \sup_{x\in\Omega_R} |\partial_{x_i} a_{ij}(x, u(x))|$ is still finite.

It is an open problem whether the *sufficiency* of the Compact Support Principle for (4.8.5) remains valid when the matrix $[a_{ij}]$ is also allowed to depend on the solution variable z.

The following counterexample [74] shows the importance of the boundedness condition (B2) for the validity of the compact support principle. Consider the inequality

$$\Delta_p u + |Du|^{q_1} - u^{q_2} \geqslant 0, \quad u \geqslant 0,\ p > 1,\ q_1, q_2 > 0. \tag{4.8.8}$$

Clearly conditions (4.8.6) and (B2) are satisfied if and only if $q_1 \geqslant p-1$ and $q_2 < p-1$. The compact support principle then holds for (4.8.8). On the other hand, for any $q_1 \in (0, p-1)$ we can take $q_1 < q_2 < p-1$. One easily checks that (4.8.8) then has positive solutions $u(x) = \text{const.}|x|^{-\kappa}$ on Ω_R for κ and R suitably large. Hence the compact support principle fails even though condition (4.8.6) is fulfilled!

The reader is referred to [73] for proofs of the theorems of this section.

Notes

The background and literature for Theorem 4.1.1 is fairly complicated and deserves a number of comments.

The necessity of (4.1.4) for the case of the Laplace operator is due to Benilan, Brezis and Crandall [10], while for the p-Laplacian it is due to Vázquez [97]. In these cases (4.1.4) reduces respectively to

$$\int_{0^+} \frac{\mathrm{d}s}{\sqrt{F(s)}} = \infty \quad \text{and} \quad \int_{0^+} \frac{\mathrm{d}s}{[F(s)]^{1/p}} = \infty.$$

For general operators satisfying (A1), (A2), necessity is due to Diaz ([26], Theorem 1.4).

Sufficiency for the case of the Laplace operator and also for the p-Laplacian is again due to Vázquez [97], see also [26,45] and [61]. For a general A-Laplace operator satisfying (A1), (A2), sufficiency was proved in Theorem 1 of [74] under an additional technical assumption; in Theorem 1 of [69] without the technical assumption; and in Theorem 1.1 of [71] with a simplified proof. For the vectorial case see [33].

The case when $f \equiv 0$ was studied by Cellina [18] for non-negative minimizers of the integral $\int_\Omega \mathcal{G}(|Du|)\,\mathrm{d}x$. An alternative abstract approach to the strong maximum principle appears in [20].

As in the case of the strong maximum principle it is worth commenting on the background and literature for the compact support principle Theorem 4.1.2.

Necessity was first shown in Corollary 2 of [74] under an additional technical assumption as noted above, and in [69], with a proof which is not at all easy. The proof given in [71] is simpler and at the same time provides an existence theorem for radial solutions of exterior Dirichlet problems; see Theorem 4.3.2.

The sufficiency of (4.1.6) is Theorem 2 of [74]. For radially symmetric solutions of (4.1.5) sufficiency was proved in Proposition 1.3.1 of [36] under the weaker assumption that $F(z) > 0$ for $z \in (0, \delta)$.

Theorem 4.7.1 is an improved version of a result originally given in [74]. For the further results of Section 4.8 see [73], Chapter 5.

SECTION 6.5

Applications

5.1. Cauchy–Liouville theorems

A Cauchy–Liouville type theorem is a statement that under appropriate circumstances an entire solution (a solution defined over $\mathbb{R}^n$) of an elliptic equation must be constant.[10] For the Laplace equation in particular, it is enough that a solution u should be bounded, or even, at a minimum, that $u(x) = o(|x|)$ as $|x| \to \infty$. For quasilinear equations, and even for equations of the form

$$\Delta u + B(u, Du) = 0, \quad x \in \mathbb{R}^n, \tag{5.1.1}$$

the same question is more delicate than might at first be expected, since a number of different kinds of behavior can be seen even for relatively simple examples.

Consider first the simple Poisson equation

$$\Delta u = f(u), \quad u \in C^2\big(\mathbb{R}^n\big), \tag{I}$$

in which $f(u)$ is a non-decreasing function. If $u(x) = o(|x|)$ as $|x| \to \infty$, then $u \equiv$ constant. For the equation

$$\Delta u = \big(|Du|^2 - 1\big)^2 u, \quad u \in C^2\big(\mathbb{R}^n\big), \tag{II}$$

the same result holds, and indeed, more precisely $u \equiv 0$. On the other hand, in contrast to the Laplace equation, a one sided bound on u is not enough to make $u \equiv$ constant, since one can check that both

$$u(x) = \sqrt{1 + x_1^2}, \qquad u(x) = -\sqrt{1 + x_1^2}$$

are solutions. In a third case

$$\Delta u = |Du|^2, \quad u \in C^1\big(\mathbb{R}^n\big), \tag{III}$$

the *only* entire solutions are constants, without placing any bound on the solution itself. Even more the equation

$$\Delta u = |Du|^2 + a, \quad a = \text{constant} \neq 0, \tag{IV}$$

[10] Frequently called Liouville theorems in the literature. For a discussion of the relative contributions of Cauchy and Liouville, see reference [87].

has no bounded entire solutions whatsoever.

Case (III) is proved by making the substitution $v = \mathrm{e}^{-u}$, whence $\Delta v = 0$, $v > 0$, so that v and hence u must be constants.

Cases (I), (II) and (IV) rely on the following subtle lemma, which we state in greater generality than initially needed.

LEMMA 5.1.1. *Consider the quasilinear equation*

$$a_{ij}(x, u, Du)\partial^2_{x_i x_j} u + B(x, u, Du) = 0, \quad x \in \mathbb{R}^n, \tag{5.1.2}$$

in which $[a_{ij}(x, z, \boldsymbol{\xi})]$ *is an* $n \times n$ *non-negative definite matrix, uniformly bounded in* $\mathbb{R}^n \times \mathbb{R} \times \boldsymbol{B}_\delta$ *for some* $\delta > 0$, *where* $\boldsymbol{B}_\delta$ *denotes the* δ*-ball of* $\mathbb{R}^n$. *Assume also that*

$$B(x, z, \boldsymbol{\xi}) \leqslant -f(z) \quad \text{in } \mathbb{R}^n \times \mathbb{R} \times \boldsymbol{B}_\delta, \tag{5.1.3}$$

where f *is non-decreasing in* $\mathbb{R}$.

If $u \in C^2(\mathbb{R}^n)$ *is an entire solution of* (5.1.2) *such that* $u(x) = \mathrm{o}(|x|)$ *as* $|x| \to \infty$, *then* $f(c) = 0$ *for all values* c *in the range of* u.

Equation (5.1.1) is obviously covered by Lemma 5.1.1.

PROOF. Let $x_0 \in \mathbb{R}^n$ and $c = u(x_0)$. For $\varepsilon \in (0, \delta)$ put

$$v(x) = u(x) - c - \varepsilon h(x), \quad h(x) = \sqrt{1 + |x - x_0|^2} - 1.$$

Then $v(x_0) = 0$, while $v(x) \to -\infty$ as $|x| \to \infty$. Consequently v takes a non-negative maximum at some point y. Then $v(y) = u(y) - c - \varepsilon h(y) \geqslant 0$ and equally

$$Dv(y) = Du(y) - \varepsilon Dh(y) = \mathbf{0}, \qquad a_{ij}\big(y, u(y), Du(y)\big)\partial^2_{x_i x_j} v(y) \leqslant 0.$$

Since $|Du(y)| = \varepsilon |Dh(y)| \leqslant \varepsilon < \delta$, by evaluating (5.1.2) at y and using (5.1.3) we get

$$\begin{aligned} f\big(u(y)\big) &\leqslant -B\big(y, u(y), Du(y)\big) = a_{ij}\big(y, u(y), Du(y)\big)\partial^2_{x_i x_j} u(y) \\ &\leqslant \varepsilon a_{ij}\big(y, u(y), Du(y)\big)\partial^2_{x_i x_j} h(y) \\ &\leqslant \varepsilon \sum_{i=1}^{n} a_{ii}\big(y, u(y), \varepsilon Dh(y)\big), \end{aligned} \tag{5.1.4}$$

since

$$\partial^2_{x_i x_j} h(x) = \big(1 + |x - x_0|\big)^{-1/2}\delta_{ij} - \big(1 + |x - x_0|\big)^{-3/2}(x_i - x_{0,i})(x_j - x_{0,j})$$

and a_{ij} is non-negative definite.

But $c \leqslant u(y) - \varepsilon h(y) < u(y)$, so $f(c) \leqslant f(u(y))$. Thus, letting $\varepsilon \to 0$ in (5.1.4) yields $f(c) \leqslant 0$. In the same way we find $f(c) \geqslant 0$, completing the proof. □

To prove that (I) has no entire solutions which are $o(|x|)$ as $|x| \to \infty$, observe from the lemma with $B = -f(z)$ that $f(c) = 0$ for all c in the range of u. Thus in fact $\Delta u = 0$. But then (making use of the spherical harmonic expansion of u about a given origin) we see that $u \equiv$ constant, as required.

To prove (II) we find in the same way that $c = f(c) = 0$, so $u \equiv 0$ is the only entire solution. To obtain (IV), let u be a bounded solution, or even a solution such that $u(x) = o(|x|)$ as $|x| \to \infty$. Then by Lemma 5.1.1, with $f(z) \equiv a$ and $B(x, z, \boldsymbol{\xi}) = |\boldsymbol{\xi}|^2 + a$ when $a > 0$, we find that $f(c) = 0$ for all c in the range of u, which is impossible. When $a < 0$ we set $v = -u$, so that v satisfies the equation $\Delta v = -|Dv|^2 - a$. With $f(z) = |a|/2$ ($\varepsilon^2 < |a|/2$), we again reach a contradiction.

As the examples (I)–(IV) make clear, there seems no simple overall Liouville theorem for quasilinear elliptic equations, even in cases in which the principal part consists of the Laplace operator. Nevertheless, there are several further useful results which can be obtained without difficulty.

A first case of interest occurs if f is strictly monotone in $\mathbb{R}$. Then $f(c) = 0$ implies that c must have a unique value $c = a$, and so in turn every entire solution which is $o(|x|)$ as $|x| \to \infty$ is constant, that is $u \equiv a$. An important example is the *capillary surface equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \kappa u, \quad \kappa > 0. \tag{5.1.5}$$

In particular, the only entire solution which is $o(|x|)$ as $|x| \to \infty$ is $u \equiv 0$. Even more, the result of Lemma 5.1.1 extends to solutions u defined in exterior domains, the result being again that $f(c) = 0$ for all values c which the solution u can attain at ∞. For (5.1.5), this means that any exterior capillary surface solution must approach the limit 0 as $|x| \to \infty$, if it is $o(|x|)$ as $|x| \to \infty$.

When $f(z)$ is non-decreasing but not strictly monotone in z, the situation is more complicated. In this case (5.1.2) reduces to the equation

$$a_{ij}(x, u, Du)\partial^2_{x_i x_j} u = 0, \tag{5.1.6}$$

and we must ask whether entire solutions of (5.1.6) are constants (possibly assuming that u is $o(|x|)$ as $|x| \to \infty$). For the Laplace equation there is of course no problem – all solutions which are $o(|x|)$ as $|x| \to \infty$, or are bounded either above or below, are constants.

Otherwise the simplest case concerning entire solutions of (5.1.6) occurs when it can be written in the canonical divergence structure form

$$\operatorname{div} \boldsymbol{A}(Du) = 0,$$

where $\boldsymbol{A}$ is elliptic and of class $C^1(\mathbb{R}^n)$. Here, if $\boldsymbol{\xi}$ is a bounded vector, then as in the proof of Theorem 3.4.1 we get

$$\big\langle \boldsymbol{A}(\boldsymbol{\xi}), \boldsymbol{\xi} \big\rangle \geqslant \text{Const.}\, |\boldsymbol{\xi}|^2, \qquad \big|\boldsymbol{A}(\boldsymbol{\xi})\big| \leqslant \text{Const.}\, |\boldsymbol{\xi}|,$$

where we assume without loss of generality that $\boldsymbol{A}(\mathbf{0}) = \mathbf{0}$. Then by the Harnack inequality (Theorem 5 of [81] or Corollary 7.2.3 of [73]), if $u \in W^{1,\infty}$ is an entire solution of (5.1.6), bounded either above or below, then $u \equiv$ constant in $\mathbb{R}^n$.

Another case is the mean curvature equation in the form

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f(u), \tag{5.1.7}$$

for which the following result seems to be new.

THEOREM 5.1.2. *Let $u \in C^2(\mathbb{R}^n)$ be an entire solution of* (5.1.7), *where f is a non-decreasing function. If $n \leqslant 7$ and $u(x) = o(|x|)$ as $|x| \to \infty$, then $u \equiv$* constant.

This is a consequence of a result of J. Simon [89], stating that when $n \leqslant 7$ all entire solutions of the minimal surface equation are linear functions.[11]

Lemma 5.1.1 can be extended to apply to certain operators in which $[a_{ij}]$ is unbounded. Suppose for example that $\|\boldsymbol{a}(x, z, \boldsymbol{\xi})\| \leqslant \alpha|\boldsymbol{\xi}|^{p-2}$, $1 < p < 2$, when $\boldsymbol{\xi} \neq \mathbf{0}$. In this case at the final step in the proof of Lemma 5.1.1 we would get

$$\varepsilon \sum_{i=1}^{n} a_{ii}\big(y, u(y), \varepsilon Dh(y)\big) \leqslant \alpha\varepsilon^{p-1},$$

and the proof is then completed as before. In particular, the following result holds for the p-Laplace operator for all $p > 1$.

THEOREM 5.1.3. *Let $u \in C^2(\mathbb{R}^n)$ be a bounded entire solution of*

$$\Delta_p u = f(u), \quad p > 1,$$

where f is a non-decreasing function. Then $u \equiv$ constant.

If f is strictly increasing then the condition that u be bounded can be replaced by $u(x) = o(|x|)$ as $|x| \to \infty$.

Notes

The conclusions of this section are in most respects new, though based originally on [84]. Other related results can be found in [62] and [90].

It has been assumed throughout the section that f is a non-decreasing function of u. When this is not the case, for example for the equation

$$\Delta u + |u|^{q-2}u = 0, \quad q > 1,$$

[11] The corresponding result in $n = 2$ dimensions is a famous theorem of S. Bernstein; a particularly simple proof of Bernstein's theorem was given by J.C.C. Nitsche [59] in a beautiful paper, one of the pearls of mathematics.

the situation is entirely different and the results much more delicate (moreover, for the most part, being independent of maximum principle techniques). There is a large literature concerning this case, cf. [12,13,38,54], and particularly [64] and [87], to which the reader can be referred.

5.2. Radial symmetry

Let B be a ball in $\mathbb{R}^n$, for definiteness centered at the origin, and consider the Dirichlet problem

$$\begin{aligned} &\Delta u + f(u) = 0, \quad u > 0 \quad \text{in } B, \\ &u = 0 \quad \text{on } \partial B. \end{aligned} \tag{5.2.1}$$

One may expect the existence of radial solutions $u = u(r)$ of this problem, coming from the ordinary differential equation

$$u'' + \frac{n-1}{r}u' + f(u) = 0.$$

The question then arises whether solutions are *necessarily* radial. Delicate examples show that this in fact may not be the case, see for example [35, page 104]. On the other hand, if the function f is, say, of class C^1 then the answer is yes, as a consequence of the following

THEOREM 5.2.1 (Radial Symmetry). *Let B be an open ball in $\mathbb{R}^n$, $n \geqslant 1$. Assume $u \in C^2(B) \cap C(\overline{B})$ satisfies* (5.2.1), *where f is locally Lipschitz continuous in $\mathbb{R}_0^+$. Then u is radially symmetric, that is can be written in the form $u = u(r)$, $r = |x|$.*

This result is due to Gidas, Ni and Nirenberg [37] for solutions of class $C^2(\overline{B})$ and to Berestycki and Nirenberg [11] for the stated case. A short proof of Theorem 5.2.1 was given by Brezis [15].

Theorem 5.2.1 allows extension to radially symmetric quasilinear equations, moreover without the assumption of positivity of the solution, or the full Lipschitz continuity of the nonlinearity f. There are two main cases, first when the solution $u \in C^1(\overline{B})$, and second for $u \in C^1(B) \cap C(\overline{B})$.

In the second result, which we state as Theorem 5.2.3, less regularity is required of u near the boundary of B. This however leads to stronger regularity hypotheses being needed for the operator A and nonlinearity f. At the same time, it is easy to see that these extra hypotheses automatically hold for the problem (5.2.1), where $A(z, s) \equiv 1$ and $f = f(u)$. Thus Theorem 5.2.1 is a special case of Theorem 5.2.3.

THEOREM 5.2.2 (Radial Symmetry, I). *Let B be an open ball in $\mathbb{R}^n$, $n \geqslant 1$. Assume $u \in C^1(\overline{B})$ is a distribution solution of the problem*

$$\begin{aligned} &\operatorname{div}\big\{A\big(u, |Du|\big)Du\big\} + f\big(u, |Du|\big) = 0, \quad u \geqslant 0 \quad \text{in } B, \\ &u = 0 \quad \text{on } \partial B. \end{aligned} \tag{5.2.2}$$

Here $A = A(z, s) : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$ *is assumed continuously differentiable with*

$$sA'(z, s) + A(z, s) > 0 \quad (' = \partial_s); \tag{5.2.3}$$

while the function $f = f(z, s)$ *is locally Lipschitz continuous in* $\mathbb{R}_0^+ \times \mathbb{R}_0^+$.

Then u *is radially symmetric about the origin in* B *and is of class* $C^2(B)$. *When* $n \geqslant 2$ *then either* $u \equiv 0$ *or* $u > 0$ *in* B *with* $u'(r) < 0$ *for* $0 < r < R$.

The principal operator in (5.2.2) is closely related to the variational integral

$$I[u] = \int_\Omega \mathcal{G}\big(u, |Du|\big)\, \mathrm{d}x,$$

where $\mathcal{G}$ and A are related by $A(z, s) = \mathcal{G}'(z, s)/s$, $s > 0$. Ellipticity then is equivalent to $\mathcal{G}''(z, s) > 0$. Theorem 5.2.2 applies in particular to the mean curvature equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = f\big(u, |Du|\big)$$

Here $A = A(s) = (1 + s^2)^{-1/2} > 0$ and $A(s) + sA'(s) = (1 + s^2)^{-3/2} > 0$, that is the equation is elliptic. Thus *every solution in* $C^1(\overline{B})$ *with boundary condition* $u = 0$ *on* ∂B *is radially symmetric*.

THEOREM 5.2.3 (Radial Symmetry, II). *Let* B *be an open ball in* $\mathbb{R}^n$, $n \geqslant 1$. *Assume* $u \in C^1(B) \cap C(\overline{B})$ *is a distribution solution of the problem* (5.2.2). *Here the operator* $A = A(z, s) : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^+$ *is assumed to be uniformly continuously differentiable in* $\Gamma \times \mathbb{R}_0^+$, *where* Γ *is any compact subset of* $\mathbb{R}_0^+$, *with both quantities*

$$A(z, s), \qquad sA'(z, s) + A(z, s) \tag{5.2.4}$$

uniformly bounded away from zero in $\Gamma \times \mathbb{R}_0^+$; *while the function* $f = f(z, s)$ *is uniformly Lipschitz continuous in* $\Gamma \times \mathbb{R}_0^+$. *Then the conclusion of Theorem* 5.2.2 *continues to hold*.

Condition (5.2.4) can be expressed alternatively as stating that the differential equation is uniformly elliptic.

REMARK. When the restriction $u \geqslant 0$ in B in Theorems 5.2.2 and 5.2.3 is strengthened to $u > 0$ in B, it is not hard to see from the proofs below that $f(z, s)$ does not need to be lower Lipschitz continuous in the variable z at $z = 0$, though upper Lipschitz continuity is still required. This allows for example the interesting class of nonlinearities $f(z, s)$ having asymptotic form $-z^q$ near $z = 0$, with $0 < q < 1$, not previously noted in the literature.

The possibility of radial symmetry on annuli is the concern of the next result.

THEOREM 5.2.4. *Let B be a ball or an annulus $B = B_2 \setminus \overline{B_1}$, centered at the origin. Assume that $u \in C^1(B) \cap C(\overline{B})$ is a solution of the problem*

$$\begin{aligned} &\operatorname{div}\big\{\rho(r)A\big(|Du|\big)Du\big\} + f(r,u) = 0 \quad \text{in } B, \\ &u = \text{constant} \quad \text{on any component of } \partial B. \end{aligned} \tag{5.2.5}$$

Here the function A is assumed to be positive and $sA(s)$ strictly increasing in $\mathbb{R}^+$, with $sA(s) \to 0$ as $s \to 0$; while $f(r,z)$, $r = |x|$, is locally bounded in $B \times \mathbb{R}$, and non-increasing in z; finally the function ρ is positive and locally bounded in $B \setminus \{0\}$.

Then u is unique and radially symmetric.

In contrast with Theorem 5.2.2, no restriction on the sign of u is required in Theorem 5.2.4, and even more in Theorem 5.2.4 *the operator A can be singular*, e.g. $A(s) = s^{p-2}$, $p > 1$, whereas in Theorem 5.2.2 necessarily $A(z,0) > 0$. On the other hand, the monotonicity condition on f, replacing locally Lipschitz continuity in Theorem 5.2.2, is itself a strong requirement.

Proof of Theorems 5.2.2–5.2.4

PROOF OF THEOREM 5.2.2. We use the technique of moving planes, introduced in [2] and [83].

Write $x = (x_1, x')$ with $x' = (x_2, \dots, x_n)$. For $\lambda \in (0, R)$, where R is the radius of B, we set

$$B_\lambda = \{x \in B\colon\ x_1 > \lambda\} \quad \text{and} \quad \tilde{x} = \tilde{x}^\lambda = (2\lambda - x_1, x');$$

$\tilde{x}$ is the reflection of the point x in the hyperplane T with equation $x_1 = \lambda$. Clearly $\tilde{x} \in B$ when $x \in B_\lambda$, so we can define

$$v = v^\lambda(x) = u(\tilde{x}).$$

It is easy to see that v, along with u, satisfies

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + f\big(u, |Du|\big) = 0 \quad \text{in } B_\lambda.$$

Hence in B_λ

$$\operatorname{div}\big\{A\big(v, |Dv|\big)Dv - A\big(u, |Du|\big)Du\big\} + f\big(v, |Dv|\big) - f\big(u, |Du|\big) = 0. \tag{5.2.6}$$

Put

$$w = w^\lambda = v^\lambda - u \in C^1(B_\lambda) \cap C\big(\overline{B_\lambda}\big).$$

Then $w \geqslant 0$ on ∂B_λ, that is both on $\partial B_\lambda \cap \{x_1 > \lambda\}$ and on $B \cap \{x_1 = \lambda\}$.

It follows from (5.2.3) that the matrix $[\partial_{\boldsymbol{\xi}}(A(z, |\boldsymbol{\xi}|)\boldsymbol{\xi})]$ is locally positive definite in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$; moreover $\partial_z A$ and $|\partial_{\boldsymbol{\xi}} A|$ are locally bounded in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$. In turn, using the fact that Du is bounded in B, we see that for $x \in B_\lambda$ there holds

$$\begin{aligned} &\big\langle A\big(v, |Dv|\big)Dv - A\big(u, |Du|\big)Du, Dw\big\rangle \geqslant a_1|Dw|^2 - a_2 w^2, \\ &\big|A\big(v, |Dv|\big)Dv - A\big(u, |Du|\big)Du\big| \leqslant a_3|Dw| + a_4|w|, \end{aligned} \tag{5.2.7}$$

for appropriate constants $a_1, a_3 > 0$ and $a_2, a_4 \geqslant 0$: see (2.5.9). Also since $f(z, s)$ is locally Lipschitz continuous in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$ we have similarly

$$\big|f\big(v, |Dv|\big) - f\big(u, |Du|\big)\big| \leqslant b_1|Dw| + b_2 w \tag{5.2.8}$$

for appropriate constants $b_1, b_2 \geqslant 0$; the constants in the inequalities (5.2.7)and (5.2.8) obviously depend only on bounds for u and Du in B.

For λ near R, the set B_λ has small measure. We are therefore in position to apply Theorem 3.3.1. In particular, let w be considered as a solution of equation (5.2.6), which we write in the form (3.1.1) with w replacing u. Then in view of (5.2.7), (5.2.8) and the fact that $w \geqslant 0$ on ∂B_λ, it follows that $w = w^\lambda \geqslant 0$ in B_λ. Let

$$\Lambda = \big\{\lambda \in (0, R):\ w^\lambda \geqslant 0 \text{ in } B_\lambda\big\},$$

so Λ is non-empty and relatively closed in $(0, R)$.

Let $\lambda \in \Lambda$. Remembering that f is locally *lower* Lipschitz continuous, from the tangency principle Theorem 2.5.2 applied to the pair of solutions u and $v = v^\lambda$ in the set B_λ, we see that either

$$w^\lambda \equiv 0 \quad \text{or} \quad w^\lambda > 0 \quad \text{in } B_\lambda.$$

In the sequel we will need the following result.

LEMMA 5.2.5. *If $w^\lambda > 0$ in B_λ for all $\lambda \in \Lambda$, then $\Lambda = (0, R)$.*

PROOF. It is enough to show that Λ is open. Let $\lambda \in \Lambda$. We must show that $\mu \in \Lambda$ when μ is sufficiently near λ. Suppose first that $0 < \mu < \lambda$, and let K be any compact subset of B_λ with the further property that the set $B_\mu \setminus K$ has measure so small that Theorem 3.3.1 applies (this can be accomplished by making at the same time μ suitably near λ). Obviously $w = w^\lambda \geqslant \delta$ in K for a suitable constant $\delta > 0$. Then for the function w^μ, we have *when* $x \in K$,

$$\begin{aligned} w^\mu(x) &= v^\mu(x) - u(x) = u\big(\tilde{x}^\mu\big) - v^\lambda(x) + w^\lambda(x) \\ &= u\big(\tilde{x}^\mu\big) - u\big(\tilde{x}^\lambda\big) + w^\lambda(x) \geqslant 0, \end{aligned}$$

since $|\tilde{x}^\mu - \tilde{x}^\lambda| = 2(\lambda - \mu)$ can be made as small as we wish by taking μ even nearer λ if necessary (and since u is uniformly continuous in $\overline{B_\lambda}$).

In particular, $w^\mu \geqslant 0$ on ∂K, so in turn

$$w^\mu \geqslant 0 \quad \text{on } \partial(B_\mu \setminus K) = \partial K \cup \partial B_\mu.$$

Hence by Theorem 3.3.1 we get $w^\mu \geqslant 0$ in $B_\mu \setminus K$, and in combination $w^\mu \geqslant 0$ in B_μ. Hence $\mu \in \Lambda$ for all $\mu < \lambda$ which are sufficiently near λ, as required. Essentially the same argument applies when $\mu > \lambda$. Thus Λ is open and $\Lambda = (0, R)$. □

The proof now divides into three cases.

Case 1. $u > 0$ in B. It is easy to see that $w^\lambda > 0$ in B_λ for all $\lambda \in \Lambda$: otherwise, $w_\lambda \equiv 0$ for some $\lambda \in \Lambda$, so in particular we would have $w^\lambda = v^\lambda = 0$ on $\partial B_\lambda \cap \partial B$. But this requires that $u = 0$ on the reflection of ∂B in the hyperplane $x_1 = \lambda$, contradicting the assumption that $u > 0$ in B. It now follows from Lemma 5.2.5 that $\Lambda = (0, R)$ and so $w^\lambda \geqslant 0$ in B_λ for $0 < \lambda < R$.

By continuity $u(\tilde{x}) - u(x) \geqslant 0$ for $\lambda = 0$, that is

$$u(x_1, y) \leqslant u(-x_1, y), \quad x_1 > 0.$$

The same argument applies with a moving plane $x_1 = \lambda < 0$, with $\lambda \in (-R, 0)$. Thus $u(x_1, y) \leqslant u(-x_1, y)$, $x_1 < 0$. Consequently $u(x_1, y) = u(-x_1, y)$, and u is symmetric across the hyperplane $x_1 = 0$. By rotation of coordinates the same conclusion applies in all directions and u is symmetric across any hyperplane through the origin. Thus u is radially symmetric.[12]

Case 2. $u \not> 0$ in B, and $n \geqslant 2$. We assert that there is some $\lambda \in \Lambda$ such that $w^\lambda \equiv 0$ in B_λ. Otherwise, if $w^\lambda > 0$ for all $\lambda \in \Lambda$, then by Lemma 5.2.5 we would have $\Lambda = (0, R)$. In fact, this is impossible: let $x_0 \in B$ be such that $u(x_0) = 0$, and choose λ so that x_0 lies in the reflection of B_λ across the hyperplane $x_1 = \lambda$. Then at the reflected point $\tilde{x}_0 \in B_\lambda$ there would hold

$$0 < w^\lambda(\tilde{x}_0) = u(x_0) - u(\tilde{x}_0) = -u(\tilde{x}_0) \leqslant 0,$$

a contradiction.

Let $\lambda_0 \in \Lambda$ be such that $w^{\lambda_0} \equiv 0$ in B_{λ_0}. Then necessarily $u = 0$ on the reflection L of ∂B across the hyperplane T_0: $x_1 = \lambda_0$. Let y be a point in $\partial B \cap T_0$. We reapply the previous moving planes argument, but now with hyperplanes parallel to the tangent hyperplane to ∂B at y. See Figure 4. The previous reflection and thin set arguments then supply the conclusion that the (new) functions w^λ are identically zero for all λ suitably near R; that is, for these functions the inequality $w^\lambda > 0$ is incompatible with the condition $u = 0$ on L. But then $u = 0$ on the boundary of any "*lens*" set Σ for which λ is near R. Hence in turn $u \equiv 0$ at *all* points in any sufficiently small "*lens*" set adjacent to y. Evaluating the main equation (5.2.2) in this set then yields $f(0, 0) = 0$.

In turn, (5.2.8) gives

$$f\big(u, |Du|\big) \geqslant -b_1|Du| - b_2 u \quad \text{for } u \geqslant 0.$$

[12] If one assumes to begin with that $u > 0$ in Ω, as in Theorem 5.2.1, then one can skip the delicate Cases 2 and 3 which follow.

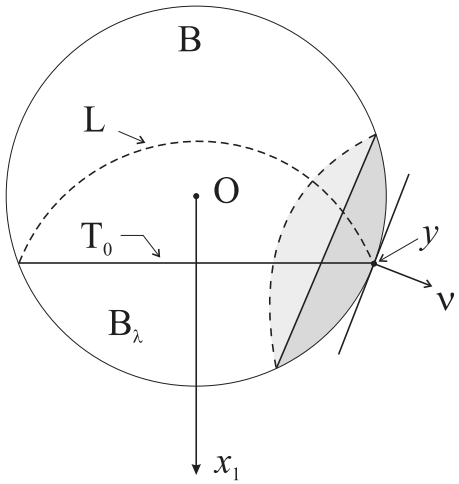

Fig. 4. The dashed set L is the reflection in the hyperplane T_0 of ∂B. By construction $u = 0$ on L and consequently by the moving plane argument also $u = 0$ in the shaded "*lens*" set Σ.

The tangency principle Theorem 2.5.2 then implies $u \equiv 0$ in B; that is, u is (trivially) radially symmetric.

Case 3. $u \not> 0$ in B and $n = 1$. In this case there exists some point in the interval $B = (-R, R)$ where $u = u' = 0$. There are two subcases.

First, if $f(0, 0) \geqslant 0$, then again by the strong maximum principle one gets $u \equiv 0$ in B.

The remaining case $f(0, 0) < 0$ is more complicated. Since this condition makes it impossible to have any subintervals of B where $u \equiv 0$, necessarily B must consist of a finite or denumerable set of open intervals I on which $u > 0$, separated by points where $u = u' = 0$. Consider any such subinterval $I = (a, b)$. On I, u must be a solution of the ordinary differential equation

$$\big\{A\big(u, |u'|\big)u'\big\}' + f\big(u, |u'|\big) = 0. \tag{5.2.9}$$

Then by Case 1 it follows that u must be symmetric about the midpoint of I, with $u' \leqslant 0$ to the right of the midpoint. Using Lemma 5.2.6 below, for the case $n = 1$ and with $J = (\frac{1}{2}(a + b), b)$, we get $u' < 0$ in J; even more, the function u, being a solution in J of the end value problem $u(b) = u'(b) = 0$ with $u' < 0$, must equal, up to translation, a unique function $U(x)$, and the interval J must have a unique length, say d. That is, the difference $b - a = 2d$ must be independent of I, and the solutions for different subintervals I must be identical following translation.

It follows that there are only a finite number of subintervals I, that $2R$ must be a multiple of $b - a$, and finally that u is symmetric on B, though of course consisting of more than a single "*hill*".

To complete the proof of Theorem 5.2.2, it thus remains only to show that when $u > 0$ in B then the solution $u = u(r)$ obeys $u'(r) < 0$ for $0 < r < R$. To accomplish this, we first

observe, since $\Lambda = (0, R)$, that necessarily $u = u(r)$ is non-increasing, hence $u'(r) \leqslant 0$. That equality cannot occur is a consequence of the following

LEMMA 5.2.6. *Let J denote the interval $(0, S)$ and let $u \in C^1(J) \cap C(J)$ be a solution of the ordinary differential equation*

$$\left\{A\left(u, |u'|\right)u'\right\}' + \frac{n-1}{r} A\left(u, |u'|\right)u' + f\left(u, |u'|\right) = 0, \quad u > 0, \tag{5.2.10}$$

where A and f satisfy the hypotheses of Theorem 5.2.2. *Suppose $u' \leqslant 0$ in J and $u(R) = 0$. Then $u \in C^2(J)$ and $u' < 0$.*

Moreover, when $n = 1$ there cannot be more than one value S and one solution $u \in C^1(\bar{J})$ such that $u(S) = u'(S) = 0$ and $u'(r) \leqslant 0$ in J.

PROOF. Define $\Phi(z, s) = sA(z, s)$ for $z > 0$, $s \geqslant 0$. By (5.2.3) the function $\Phi(z, \cdot)$ has a continuously differentiable inverse $\Phi^{-1}(z, \cdot)$. Put

$$v = v(r) = \Phi\left(u(r), \left|u'(r)\right|\right), \quad r \in J. \tag{5.2.11}$$

Then we can rewrite (5.2.10) in the form (where v' is a weak derivative)

$$\begin{aligned} u' &= -\Phi^{-1}(u, v) \\ v' &= -\tfrac{n-1}{r}\Phi\left(u, \Phi^{-1}(u, v)\right) + f\left(u, \Phi^{-1}(u, v)\right). \end{aligned} \tag{5.2.12}$$

Since $v \in C(J)$ it follows from the second equation of (5.2.12) that in fact $v' \in C(J)$, and in turn, from the first equation of (5.2.12), that $u' \in C^1(J)$ and $u \in C^2(J)$.

If at some point $c \in J$ we have $u(c) = u_0$ and $u'(c) = 0$, then also $u''(c) = 0$ since u' has a maximum at c. But then (5.2.11) gives $v'(c) = 0$, and by (5.2.12) also

$$f(u_0, 0) = 0.$$

This being shown, by the uniqueness of the initial value problem for (5.2.12), for the initial point $r = c$, we get $u \equiv u_0$, $v \equiv 0$, a contradiction since $u(R) = 0$ and $u_0 > 0$. That is, $u'(r) > 0$ in J.

The final part of the lemma follows from the uniqueness of the initial (end) value problem together with the translation invariance of (5.2.12) for the case $n = 1$. □

PROOF OF THEOREM 5.2.3. This is almost the same as for Theorem 5.2.2, the only difference being in the derivation of the estimates (5.2.7) and (5.2.8). The uniformity hypotheses however imply that the matrix $[\partial_{\boldsymbol{\xi}}(A(z, |\boldsymbol{\xi}|)\boldsymbol{\xi})]$ is uniformly positive definite in $\mathbb{R}_0^+ \times \Gamma$. Then with the help of the uniform differentiability of A, the estimates (5.2.7) and (5.2.8) are obtained as before, with the constants in both inequalities depending only on bounds for u in B. □

The key technical components in the proof of Theorems 5.2.2 and 5.2.3 are the tangency principle Theorem 2.5.2 and the thin set Theorem 3.3.1. The latter result is relatively straightforward and even applies for solutions in $W^{1,2}(\Omega) \cap C(\overline{\Omega})$. Theorem 2.5.2, on the other hand, is based on the Harnack inequality (2.5.3), and consequently is a considerably deeper result. At the same time, (2.5.3) also applies when the solution is in $W^{1,2}(\Omega) \cap C(\overline{\Omega})$.

From these comments, it follows that Theorem 5.2.3 continues to hold for solutions in $W^{1,2}(\Omega) \cap C(\overline{\Omega})$. As a special case, Theorem 5.2.1 remains valid when u is of class $W^{1,2}(\Omega) \cap C(\overline{\Omega})$, as observed by Dancer [25].

More elementary proofs of Theorems 5.2.2 and 5.2.3 can be given if $u \in C^2(\Omega) \cap C(\overline{\Omega})$, for then we can use the tangency principle Theorem 2.2.1, based strictly on the Hopf strong maximum principle.

PROOF OF THEOREM 5.2.4. Consider a second solution $v(x) = u(-x_1, y)$. Since v is equally a solution of (5.2.5), it follows from Theorem 2.6.2 that $u \equiv v$ in B. That is, u must be symmetric across the plane $x_1 = 0$. But then as in the proof of Theorem 5.2.2, the solution must be radial. □

Notes

For the problem (5.2.1), Fraenkel [35, Theorem 3.6] gives conditions on f closely related to those indicated in the remark after Theorem 5.2.3. Castro and Shivaji [17] removed the positivity condition on the solution u in (5.2.1) in the case $n \geqslant 2$. Theorem 5.2.4 is Theorem 1.1 of [68].

The (complete) symmetry results of Theorems 5.2.2 and 5.2.4 can easily be extended to unidirectional symmetry for domains which exhibit symmetry in only one (or several) directions, the proofs being essentially unchanged from the radial case. A summary of results of this type is given in [15]; see also [11,35,65].

Other work of interest, e.g. for radial symmetry when $\Omega = \mathbb{R}^n$, or for degenerate operators, is contained in the papers [23,24,31,86].

The reader can also be referred to the Notes for Chapter 3 of [35].

5.3. Symmetry for overdetermined boundary value problems

In this section we consider overdetermined boundary value problems on a general domain Ω, for example when the boundary conditions involve both Dirichlet and Neuman data. In this case, a natural question is whether the domain itself must be restricted.

To be specific, let Ω be a bounded domain of $\mathbb{R}^n$, $n \geqslant 2$, having a smooth boundary $\partial\Omega$. Suppose as a first prime example the Poisson differential equation

$$\Delta u + 1 = 0 \quad \text{in } \Omega, \tag{5.3.1}$$

together with the overdetermined boundary conditions

$$u = 0, \quad \partial_\nu u = \text{const.} \quad \text{on } \partial\Omega. \tag{5.3.2}$$

Must Ω be a ball? We shall show that the answer is affirmative, and that u must have the specific form $(R^2 - r^2)/2n$, where R is the radius of the ball and r denotes distance from its center. The precise result is as follows.

THEOREM 5.3.1. *Let Ω be a bounded domain with boundary of class C^2. Suppose there exists a solution $u \in C^2(\overline{\Omega})$ of the overdetermined problem* (5.3.1)–(5.3.2). *Then Ω is a ball and u has the specific form $(R^2 - r^2)/2n$ noted above.*

For the physical motivation of Theorem 5.3.1, consider a viscous incompressible fluid moving in straight parallel streamlines through a straight pipe of given cross sectional form Ω. If we fix rectangular coordinates in space with the z axis directed along the pipe, it is well known that the flow velocity u along the pipe is then a function of x, y alone, satisfying the Poisson differential equation

$$\Delta u + \kappa = 0 \quad \text{in } \Omega \subset \mathbb{R}^2,$$

where κ is a constant related to the viscosity and density of the fluid and to the pressure differential per unit length along the pipe. Supplementary to the differential equation one has the adherence condition $u = 0$ on $\partial\Omega$. Finally, the tangential stress per unit area on the pipe wall is given by the quantity $\mu\partial_\nu u$, where μ is the viscosity. Theorem 5.3.1 states that *the tangential stress on the pipe wall is the same at all points of the wall if and only if the pipe has a circular cross section.*

Exactly the same differential equation and boundary conditions arise in linear theory of torsion of a solid straight bar of cross section Ω. Theorem 5.3.1 then states that, *when a solid straight bar is subject to torsion, the magnitude of the resulting traction which occurs at the surface of the bar is independent of position if and only if the bar has a circular cross section.*

Theorem 5.3.1 is a special case of the following general result for quasilinear equations.

THEOREM 5.3.2. *Suppose the functions $A(z, s)$ and $f(z, s)$ satisfy the hypotheses of Theorem* 5.2.2, *but with A now being assumed twice continuously differentiable in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$.*

Let $u \in C^2(\overline{\Omega})$ be a solution of the problem

$$\begin{aligned} &\operatorname{div}\big\{A\big(u, |Du|\big)Du\big\} + f\big(u, |Du|\big) = 0, \quad u > 0 \quad \text{in } \Omega, \\ &u = 0, \quad \partial_\nu u = \text{constant} \quad \text{on } \partial\Omega, \end{aligned} \tag{5.3.3}$$

where Ω is a bounded domain with boundary of class C^2. Then Ω is a ball, and u is radially symmetric about its center.

The proof is given below. With the help of Theorem 5.3.2, we can consider the case of a liquid rising in a straight capillary tube of cross section $\Omega \subset \mathbb{R}^2$. The function $u = u(x, y)$ describing the upper surface of the liquid satisfies the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \kappa u,$$

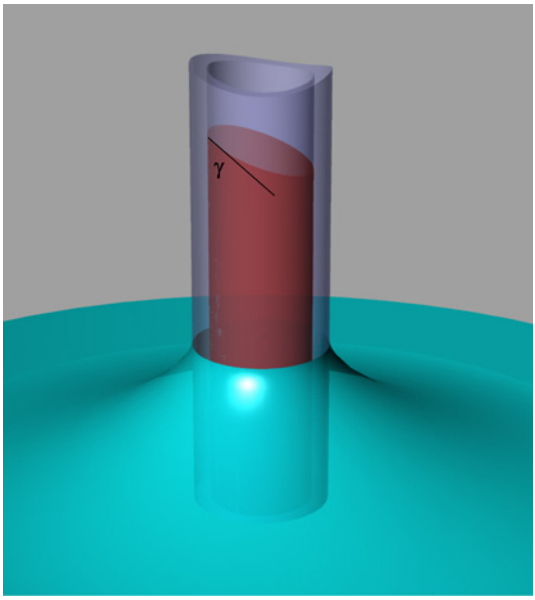

Fig. 5. Liquid rise in a non-circular capillary tube. Here γ is the wetting angle.

where κ is a positive constant, see Example 2 of the Introduction. The requirement that the wetting angle γ at the wall of the tube be constant leads to the boundary condition

$$\partial_{\boldsymbol{\nu}} u = \cot\gamma = \text{const.} \quad \text{on } \partial\Omega,$$

where $\boldsymbol{\nu}$ is the outward normal direction. Then, *provided the wetting angle γ is different from $\pi/2$, a liquid will rise to the same height at each point of the wall of a capillary tube if and only if the tube has a circular cross section*. See Figure 5. When $\gamma = \pi/2$ the unique solution is $u \equiv 0$ for any cross sectional form of the tube.

REMARK. The domain Ω in Theorem 5.3.2 need not be assumed simply connected. The conclusion that the domain must be a ball (simply connected) is unaffected.

PROOF OF THEOREM 5.3.2. The idea is the same as for Theorem 5.2.2, using the method of moving planes, but without originally knowing the location of the eventual center of Ω.

Let $\lambda \in \mathbb{R}$ and define as before $\tilde{x} = (2\lambda - x_1, y)$. In general $\tilde{x} \notin \Omega$. Let $\lambda_0 \in \mathbb{R}$ be such that the hyperplane $x_1 = \lambda_0$ is one–sidedly tangent to Ω, that is, with $\Omega \subset \{x \in \mathbb{R}^n\colon x_1 < \lambda_0\}$. Consider the set

$$\Omega_\lambda = \{x \in \Omega\colon \lambda < x_1 < \lambda_0\}.$$

Since $\partial\Omega$ is of class C^2, it is evident that at least when λ is suitably near λ_0 and $x \in \Omega_\lambda$, then $\tilde{x} \in \Omega$. Consequently for such λ and for $x \in \overline{\Omega}_\lambda$ we can define $v(x) = v^\lambda(x) = u(\tilde{x})$

and $w = w^\lambda = v^\lambda - u$. Moreover, $w \geqslant 0$ on $\partial\Omega_\lambda$ as before, and again as before if λ is even closer λ_0, if necessary, the thin set Theorem 3.3.1 gives $w \geqslant 0$ in Ω_λ.

Now define

$$\Lambda = \{\lambda < \lambda_0 \colon x \in \Omega_\lambda \text{ implies } \tilde{x} \in \Omega \text{ and } w^\lambda(x) \geqslant 0\};$$

of course Λ is non-empty and closed. Consider the set $Q_\lambda = \partial\Omega \cap T_\lambda$, where $T\lambda$ is the hyperplane $x_1 = \lambda$, and let $\boldsymbol{n}$ denote the exterior normal vector to Ω at points of Q_λ. It is evident that $\langle \boldsymbol{n}, \boldsymbol{e}_1 \rangle < 0$ when λ is near λ_0, and that as λ decreases there would be a first value λ_1 where $\langle \boldsymbol{n}, \boldsymbol{e}_1 \rangle = 0$ for some point $y \in Q_{\lambda_1}$.

Step 1. Assume $\lambda_1 < \lambda < \lambda_0$ and $\lambda \in \Lambda$. As in Case 1 of the proof of Theorem 5.2.2 we must have $w^\lambda > 0$ in B_λ.

We now consider two subcases:

(i) there is $y \in \partial\Omega \setminus T_\lambda$ such that $\tilde{y} \in \partial\Omega$;

(ii) $\tilde{y} \in \Omega$ for all $y \in \partial\Omega \setminus T_\lambda$.

For case (i) we use the overdetermined condition $\partial_{\boldsymbol{\nu}} u = c = \text{constant}$ on $\partial\Omega$. In fact $\partial_{\boldsymbol{\nu}} u(y) = c$, $\partial_{\boldsymbol{\nu}} v(y) = \partial_{\boldsymbol{\nu}} u(\tilde{y}) = c$, so that $\partial_{\boldsymbol{\nu}} w(y) = 0$. Recalling that $w \geqslant 0$ in Ω_λ, the boundary point Theorem 2.7.1 applied at the boundary point y shows that $w \equiv 0$ in Ω_λ. In turn, $u = 0$ on the reflection of $\partial\Omega_\lambda \cap \partial\Omega$. Since $\langle \boldsymbol{n}, \boldsymbol{e}_1 \rangle < 0$ on $\partial\Omega \cap T_\lambda$ it follows that $u = 0$ at a set of interior points of Ω, a contradiction. That is, case (i) cannot occur.

In case (ii), it is apparent by simple geometry that $\tilde{x} \in \Omega$ also for all $x \in \Omega_\mu$, when the value $\mu < \lambda$ is sufficiently near λ. But then we can apply Lemma 5.2.5 of Theorem 5.2.2 to show that Λ is open. That is, $\Lambda = (\lambda_1, \lambda_0)$, and in particular $w^\lambda \geqslant 0$ in B_λ when $\lambda = \lambda_1$.

Step 2. Let $\lambda = \lambda_1$ and choose $y \in Q_\lambda$ so that $\langle \boldsymbol{n}, \boldsymbol{e}_1 \rangle = 0$ at y. Since $w = 0$ on T_λ we have $\partial_{\boldsymbol{n}} w(y) = 0$; because $\partial_{\boldsymbol{t}} w(y) = 0$ for any direction $\boldsymbol{t}$ tangent to $\partial\Omega$ at y, it follows that $Dw(y) = 0$.

We now wish to apply the edge theorem stated as Lemma 2 in [83]. To this end, it is first necessary to write the difference equation (5.2.6) in non-divergence form. In fact, since the function A is twice continuously differentiable in $\mathbb{R}_0^+ \times \mathbb{R}_0^+$, we can write (5.3.3) in the form (after division by A)

$$\tilde{a}_{ij}(u, Du)\partial^2_{x_i x_j} u + \tilde{b}\big(u, |Du|\big)|Du|^2 + \tilde{f}\big(u, |Du|\big) = 0,$$

where

$$\tilde{a}_{ij}(z, \boldsymbol{\xi}) = \delta_{ij} + h\big(z, |\boldsymbol{\xi}|\big)\frac{\xi_i \xi_j}{|\boldsymbol{\xi}|},$$

and

$$h(z,s) = \frac{\partial_s A(z,s)}{A(z,s)}, \quad \tilde{b}(z,s) = \frac{\partial_z A(z,s)}{A(z,s)}, \quad \tilde{f}(z,s) = \frac{f(z,s)}{A(z,s)}.$$

A similar non-divergence equation of course holds also for v.

Then by subtraction we get, using the Lipschitz continuity of h, $\tilde{b}$ and $\tilde{f}$ in the variables z and s,

$$\tilde{a}_{ij}(v, Dv)\partial^2_{x_i x_j} w \leqslant b_1|Dw| + c_1 w$$

and equally (!)

$$\tilde{a}_{ij}(u, Du)\partial^2_{x_i x_j} w \leqslant b_2|Dw| + c_2 w.$$

Finally, adding the last two inequalities yields

$$a_{ij}(x)\partial^2_{x_i x_j} w \leqslant b|Dw| + cw, \tag{5.3.4}$$

where

$$a_{ij}(x) = 2\delta_{ij} + \left\{ h\big(u, |Du|\big)\frac{\partial_{x_i} u \partial_{x_j} u}{|Du|} + h\big(v, |Dv|\big)\frac{\partial_{x_i} v \partial_{x_j} v}{|Dv|} \right\}. \tag{5.3.5}$$

The matrix $[a_{ij}(x)]$ is bounded and strictly elliptic in Ω_λ. Moreover, it has the crucial property

$$a_{1j} = 0 \quad \text{on } T_\lambda \cap \Omega, \ \ j = 2, \ldots, n. \tag{5.3.6}$$

Indeed on T_λ we have, by the reflection construction, $\partial_{x_1} v = -\partial_{x_1} u$, $\partial_{x_j} v = \partial_{x_j} u$ for $j = 2, \ldots, n$, $|Dv| = |Du|$, whence (5.3.6) follows from (5.3.5). But also the coefficients a_{ij} are uniformly Lipschitz continuous in Ω_λ, so that (5.3.6) implies

$$\big|a_{1j}(x)\big| \leqslant \text{Const.}\, x_1 \quad \text{in } \Omega_\lambda; \tag{5.3.7}$$

here it is convenient to choose new coordinates so that T_λ is the hyperplane $x_1 = 0$, with $x_1 > 0$ in Ω_λ, while the x_n-axis is in the normal direction $-\boldsymbol{n}$ at y.

Since $w^\lambda \geqslant 0$, we are now in position to apply Lemma 2 of [83] to the inequality (5.3.4), with the single exception that the right side is no longer zero but instead has the form $b|Dw| + cw$, a case not directly covered by the lemma. In order not to obstruct the flow of the proof we defer discussion of this point until the Appendix at the end of the section. Recalling that $w(y) = Dw(y) = 0$, the conclusion of the lemma is that either $w \equiv 0$ in Ω_λ or $\partial^2_{s^2} w(y) > 0$ along any direction $\boldsymbol{s}$ which enters Ω_λ at y.

In fact, $D^2 w(y) = 0$. To see this, observe that (continuing to use the special coordinates noted above)

$$w = w^\lambda = u(-x_1, x') - u(x_1, x') \quad \text{in } \overline{\Omega}_\lambda.$$

Consequently on T_λ we have

$$\partial^2_{x_1 x_1} w = \partial^2_{x_i x_j} w = 0, \quad i, j = 2, \ldots, n.$$

Moreover, by the boundary condition $u = 0$, $\partial_\nu u =$ constant on $\partial\Omega$ there holds

$$\partial^2_{x_i x_n} u = 0, \quad i = 1, \dots, n-1.$$

and the assertion follows. Lemma 2 of [83] therefore shows that $w \equiv 0$ in Ω_λ.

Hence for $x \in \Omega_\lambda$ there holds $u(\tilde{x}) = u(x)$. In particular, by continuity

$$u(\tilde{x}) = u(x) = 0 \quad \text{for } x \in \partial\Omega_\lambda \setminus T_\lambda.$$

Consequently $\tilde{x} \in \partial\Omega$ since $u > 0$ in Ω. Otherwise stated, the boundary of Ω is symmetric across the hyperplane T_λ, and in turn Ω is symmetric across T_λ. Since, by rotation, this is also true for corresponding hyperplanes T_λ with arbitrary normal directions, it follows that Ω must be convex. But the only convex domains which have the symmetry just noted are balls. □

The condition that $u > 0$ in Ω can be weakened to $u \geqslant 0$ provided that either the Neumann constant in (5.3.3) is positive (of course it is necessarily non-negative) or $f(0, 0) \geqslant 0$. The details can be left to the reader.

Appendix to Section 5.3

The calculations involved in the proof of Lemma 2 of [83] (see lines 7–24 on page 314 of [83]) are more complicated than one might wish, but still are within reach of pencil and paper.[13] At the same time, there are three further points which need to be made.

(1) The inequality (5.3.7) takes the place of (26) of the lemma; it is used on line 13 on page 314.

(2) The terms $b|Dw| + cw$ on the right side of (5.3.4) cause no essential new difficulties, once it is observed that, in the notation of [83],

$$\begin{aligned} z(x) &= \left(\mathrm{e}^{-\alpha(x_1 - r_1)^2} - \mathrm{e}^{-\alpha r_1^2}\right)\left(\mathrm{e}^{-\alpha r^2} - \mathrm{e}^{-\alpha r_1^2}\right) \\ &\leqslant 2\alpha(r_1 - x_1)x_1 \mathrm{e}^{-\alpha r_1^2}\left(\mathrm{e}^{-\alpha r^2} - \mathrm{e}^{-\alpha r_1^2}\right) \end{aligned}$$

(by the mean value theorem as on line 17 of page 324). In turn

$$cz(x) \leqslant 2\alpha c x_1 r_1 \mathrm{e}^{-\alpha(r^2 + r_1^2)} \leqslant 2\alpha c x_1 r_1 \mathrm{e}^{-\alpha[r^2 + (x_1 - r_1)^2]}.$$

Therefore in lines 20, 21 the estimate for Lz need be changed only to include the additional term $-2c/\alpha$ in the first set of braces, which leaves the proof essentially unchanged.

[13] Fraenkel [35, page 305] remarks that results of the type of Lemma 2 unavoidably involve "*greater complexity*" than standard boundary point theorems. The proof in [83], as extended by the discussion below, should be judged in the context of Fraenkel's remark.

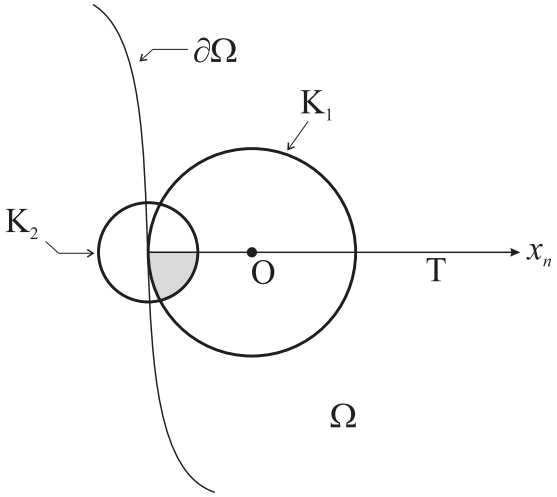

Fig. 6. The critical point y on $\partial\Omega$, where $\langle \boldsymbol{n}, \boldsymbol{e}_1 \rangle = 0$, is at the center of the small ball K_2. The hyperplane T has the equation $x_1 = 0$, with x_1 pointing downward; and the x_n-axis is taken in the direction $-\boldsymbol{n}$. (The diagram thus shows an (x_1, x_n)-plane section of $\mathbb{R}^n$ near y.) The ball K_1 has center O on the x_n-axis, is tangent to $\partial\Omega$ at y and $\overline{K_1} \subset \Omega \cup \{y\}$. The radius of K_1 is r_1 and the radius of K_2 is θr_1, with $0 < \theta < 1/2$. The shaded region is the (open) set $K = K_1 \cap K_2 \cap \{x \in \mathbb{R}^n \colon x_1 > 0\}$.

(3) For lines 22–24 we observe that, again in the notation of [83], see Figure 6;

$$z = w = 0 \quad \text{on } T; \qquad z = 0, \quad w > 0 \quad \text{on } \partial K_1 \cap \partial K',$$
$$z(x) \leqslant 2\alpha r_1 \cdot x_1 \quad \text{on } \partial K_2 \cap \partial K'.$$

By the tangency Theorem 2.5.2 either $w \equiv 0$ or $w > 0$ in Ω_λ. In the latter case, from the boundary point Theorem 2.7.1 with $B(x, z, \boldsymbol{\xi}) = -b|\boldsymbol{\xi}| - cz$, one gets $\partial_{x_1} w > 0$ on $T \cap \Omega$. But because w is continuously differentiable and $w > 0$ in Ω_λ, by compactness it follows that $w \geqslant \varepsilon x_1$ on $\partial K_2 \cap \partial K'$.

We can now compare the solutions w and mz in K', for suitably small $m > 0$. By the comparison Theorem 2.3.1, noting that the required monotonicity for the function $B(x, z, \boldsymbol{\xi})$ is satisfied since $c \geqslant 0$, it follows from the fact that $w \geqslant mw$ on $\partial K'$ (m suitably small) that $w \geqslant mw$ in K'. Because $Dw = Dz = \mathbf{0}$ at y we obtain, as stated, that $\partial^2_{\boldsymbol{s}^2} w(y) > 0$ along any direction $\boldsymbol{s}$ which enters Ω_λ. □

Notes

Theorem 5.3.2 is essentially Theorem 2 of [83], the conditions on the nonlinearity f however being weaker, and the proof improved over the original version.

The overdetermined boundary value problem for exterior domains when the principal operator is the Laplacian was studied by Reichel [77] and by Aftalion and Busca [1].

5.4. The phenomenon of dead cores

An elliptic equation or inequality is said to have a *dead core solution* u in some domain $\Omega \subset \mathbb{R}^n$ provided that there exists an open subset Ω_1 with compact closure in Ω, called the *dead core* of u, such that

$$u \equiv 0 \quad \text{in } \Omega_1, \qquad u > 0 \quad \text{in } \Omega \setminus \overline{\Omega_1}.$$

The condition $u > 0$ could be replaced by $u \neq 0$, but for definiteness (and physical reality) we prefer the condition as stated.

In chemical models, for example, when the values of a solution represent the density of a reactant, the vanishing of a solution then delineates a region (dead core) where no reactant is present (see [5,27,71,72]). We turn to an extended discussion of this phenomenon.

In particular, consider the dead core problem for the model A-Laplace equation

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} - f(u) = 0 \quad \text{in } \Omega. \tag{5.4.1}$$

The following conditions will be imposed, as in Section 4.1:

(A1) $A \in C(\mathbb{R}^+)$;

(A2) $s \mapsto sA(s)$ *is strictly increasing in* $\mathbb{R}^+$ *and* $\Phi(s) = sA(s) \to 0$ *as* $s \to 0$;

(F1) $f \in C(\mathbb{R})$,

(F2) $f(0) = 0$ *and* f *is non-decreasing in* $\mathbb{R}$.

By the strong maximum principle, Theorem 4.1.1, equation (5.4.1) can have a dead core only if (4.1.4) fails, that is if $f > 0$ for $u > 0$ and

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(F(s))} < \infty, \qquad F(u) = \int_0^u F(s)\,\mathrm{d}s, \tag{5.4.2}$$

with H given by (4.1.3). Consequently, we assume the (5.4.2) holds throughout the sequel, except for Theorems 5.4.2 and 5.4.3.

The equation $\Delta u = |u|^{q-1}u$ for example allows dead cores only if $0 < q < 1$. Actually condition (5.4.2) is not only necessary, but also sufficient for the existence of solutions with dead cores. We have the following main result, where $\Phi(s) = sA(s)$.

THEOREM 5.4.1. *Suppose* $\Phi(\infty) = H(\infty) = \infty$. *Assume the dead core condition* (5.4.2) *holds and let* u *be a solution of* (5.4.1), *with* $0 \leqslant u(x) \leqslant m$ *on* $\partial\Omega$ *for some constant* $m > 0$. *Then the following properties are valid*:

(a) $0 \leqslant u < m$ *in* Ω.

(b) *Assume that*

$$R_0 = \int_0^\infty \frac{\mathrm{d}s}{H^{-1}(F(s)/n)} < \infty, \tag{5.4.3}$$

and let B_R *be a ball with radius* $R \geqslant R_0$, *compactly contained in* Ω. *Then* u *has a dead core in* Ω *for all* $m > 0$.

(c) *If B is any ball compactly contained in Ω, then $u \equiv 0$ in B provided that $m > 0$ is sufficiently small.*

A more refined version of Theorem 5.4.1 can be obtained when $\Omega = B_R$, where B_R is any open ball in $\mathbb{R}^n$, $n \geqslant 2$, of radius $R > 0$. Until explicitly noted later, we continue to assume that $\Phi(\infty) = H(\infty) = \infty$.

THEOREM 5.4.2. *Let* (5.4.2) *hold, with $f(z) > 0$ for $z > 0$. Then the problem*

$$\begin{cases} \operatorname{div}\{A(|Du|)Du\} = f(u) & \text{in } B_R, \\ u = m > 0 & \text{on } \partial B_R, \end{cases} \tag{5.4.4}$$

admits a unique distribution solution u, necessarily radial. Moreover $u = u(r) = u(r, m)$ is of class $C^1[0, R]$ and satisfies $u \geqslant 0$, $u' \geqslant 0$ in $[0, R]$ and $u'(0) = 0$, where $' = \mathrm{d}/\mathrm{d}r$.

Finally, at any $r > 0$ where $u(r, m) > 0$ we have also $u'(r, m) > 0$.

It is easy to see that the solution $u = u(\cdot, m)$ must be of one of the following three types, see Figure 7:

(a) $u > 0$ in B_R;

(b) $u(0, m) = 0$ and $u'(r, m) > 0$ when $r > 0$;

(c) There exists $S \in (0, R)$ such that $u'(r, m) > 0$ when $S < r < R$ and $u \equiv 0$ in B_S.

That is, in case (c) the solution u of (5.4.4) has a dead core B_S. The solution u has further properties of interest.

THEOREM 5.4.3. *The function $u = u(\cdot, m)$ is continuous and non-decreasing in the variable m (> 0) and $u < m$ in B_R.*

The following theorem gives an important relation between the value m and the existence of dead core solutions of (5.4.4).

THEOREM 5.4.4. *Let $u(\cdot, m)$ be the unique solution of* (5.4.4). *Then either $u(\cdot, m)$ has a dead core for all $m > 0$, or there is a unique (finite) number*

$$m = m_0 = m_0(R) > 0$$

for which a solution $u_0 = u_0(r) = u_0(r, m_0)$ of (5.4.4) *in B_R exists, with the properties that*

(i) $u_0(0) = 0$;

(ii) $u(0, m) > 0$ *for every* $m > m_0$;

(iii) $u(\cdot, m)$ *has a dead core for every* $m \in (0, m_0)$.

For convenience we define $m_0 = m_0(R)$ to be ∞ when $u(0, m) = 0$ for all $m > 0$. The examples

$$\Delta u = (\operatorname{sign} u)\sqrt{|u|}, \tag{5.4.5}$$

$$\Delta_4 u = u \tag{5.4.6}$$

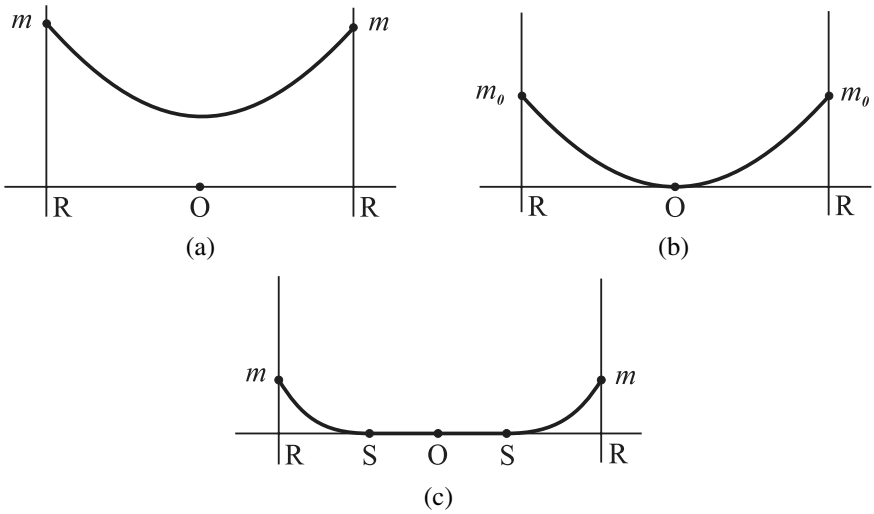

Fig. 7. Three cases of Theorem 5.4.2. The values m are decreasing from case (a) to case (c).

are particularly interesting as illustrations of the main theorems above. Indeed, both of these are included in the canonical case

$$\Delta_p u = |u|^{q-1}u, \quad p > 1,\ q > 0, \tag{5.4.7}$$

for which $F(u) = |u|^{q+1}/(q+1)$. Here the dead core condition (5.4.2) reduces exactly to

$$0 < q < p - 1.$$

For these special cases, we search for u_0 in the form cr^k, $c, k > 0$. Then from (5.4.7) one finds

$$k = \frac{p}{p-1-q}, \quad c = k^{-k/p'}(n + kq)^{-k/p}, \quad m_0 = cR^k. \tag{5.4.8}$$

For the case (5.4.5) we have $p = 2$, $q = 1/2$, $k = 4$, so that

$$m_0 = \frac{1}{(n+2)^2}\left(\frac{R}{2}\right)^4,$$

while for (5.4.6) we have $p = 4$, $q = 1$, $k = 2$ and so

$$m_0 = \frac{R^2}{2\sqrt{2(n+2)}}.$$

These reduce exactly to $m_0 = R^4/400$ and $m_0 = R^2/2\sqrt{10}$ when $n = 3$. In particular for the unit radius $R = 1$ we obtain respectively the unexpectedly small numbers $m_0 = 0.00125$ and $m_0 \cong 0.158$.

Equation (5.4.6), when written in full for $n = 2$ has the form

$$|Du|^2 \Delta u + 2u_x^2 u_{xx} + 4u_x u_y u_{xy} + 2u_y^2 u_{yy} = u,$$

which is analytic in all its variables. Thus dead core behavior is not due simply to a lack of smoothness in the basic equation. In fact (5.4.6) is an analytic partial differential equation, elliptic except when $Du = 0$, which has a non-analytic solution.

As a final example, consider the equation

$$\Delta u = (\operatorname{sign} u)\sqrt{|u|} + |u|^2 u.$$

Here $F(u) = \frac{2}{3}|u|^{3/2} + \frac{1}{4}|u|^4$ so

$$R_0 = \sqrt{\frac{n}{2}} \int_0^\infty \frac{\mathrm{d}s}{\sqrt{(2/3)s^{3/2} + s^4/4}} < \infty.$$

By numerical calculation $R_0 \cong 6.4334$ if $n = 2$. Therefore by the results of this section we have $m_0 = \infty$ whenever $R \geqslant 7$. In particular for the problem

$$\begin{cases} \Delta u = (\operatorname{sign} u)\sqrt{|u|} + |u|^2 u & \text{in } B_7 \subset \mathbb{R}^2, \\ u = m > 0 & \text{on } \partial B_7, \end{cases}$$

a dead core occurs for all $m > 0$. [This result also follows without recourse to numerical calculation, since one can write, when $n = 2$,

$$R_0 = \left(\frac{9}{2}\right)^{1/5} \int_0^\infty \frac{\mathrm{d}t}{\sqrt{t^{3/2} + t^4}} < \left(\frac{9}{2}\right)^{1/5} \left\{ \int_0^1 \frac{\mathrm{d}t}{\sqrt{t^{3/2}}} + \int_1^\infty \frac{\mathrm{d}t}{\sqrt{t^4}} \right\}$$
$$= 5(4.5)^{1/5} \cong 6.75.$$

The case $n = 3$ can be treated in the same way, with $R_0 \cong 7.879$, but here the radius $R = 7$ should be replaced by $R = 8$.

Proof of Theorems 5.4.2 and 5.4.3

PROOF OF THEOREM 5.4.2. *Existence of a radial solution u of* (5.4.4)*, with* $u \geqslant 0$*,* $u' \geqslant 0$ *and* $u'(0) = 0$. For the purpose of this proof only, we shall redefine f so that $f(v) = f(m)$ for all $v \geqslant m$, and $f(v) = 0$ when $v \leqslant 0$. This will not affect the conclusion of the theorem, since clearly any ultimate solution u of (5.4.4), with $u \geqslant 0$, $u' \geqslant 0$ in $[0, R]$, satisfies $0 \leqslant u \leqslant m$.

We shall make use of the Leray–Schauder fixed point theorem. Denote by X the Banach space $X = C[0, R]$, endowed with the usual norm $\| \cdot \|_\infty$, and let $\mathcal{T}$ be the mapping from X to X defined pointwise for all $w \in X$ and $r \in [0, R]$ by

$$\mathcal{T}[w](r) = m - \int_r^R \Phi^{-1}\left(s^{1-n} \int_0^s t^{n-1} f\big(w(t)\big)\,\mathrm{d}t \right) \mathrm{d}s. \tag{5.4.9}$$

Clearly $\mathcal{T}[w](R) = m$. Also

$$\mathcal{T}[w]'(r) = \Phi^{-1}\left(r^{1-n}\int_0^r t^{n-1} f\big(w(t)\big)\,\mathrm{d}t\right), \quad r \in (0, R]. \tag{5.4.10}$$

Obviously $\mathcal{T}[w]'$ is continuous and non-negative in $(0, R]$, since $0 \leqslant f(w) \leqslant f(m)$ for all $w \in X$. Moreover $r^{1-n}\int_0^r t^{n-1} f(w(t))\,\mathrm{d}t$ tends to zero as $r \to 0^+$. Therefore $\mathcal{T}[w]'(r)$ approaches 0 as $r \to 0^+$, since $\Phi(0) = 0$, and in turn $\mathcal{T}[w] \in C^1[0, R]$ with $\mathcal{T}[w]'(0) = 0$.

We claim that if w is a fixed point of $\mathcal{T}$ in X, then $w(0) \geqslant 0$. Otherwise $w(0) < 0$ and $w(R) = m > 0$. Thus there exists a first point $r_0 \in (0, R)$ such that $w(r) < 0$ in $[0, r_0)$ and $w(r_0) = 0$. Consequently $f(w(r)) = 0$ in $[0, r_0]$ and so $w' \equiv 0$ for $r \in [0, r_0]$ by (5.4.10). Hence $w(r_0) = w(0) < 0$ which is impossible, proving the claim.

Define the homotopy $\mathcal{H} : X \times [0, 1] \to X$ by

$$\mathcal{H}[w, \sigma](r) = \sigma m - \int_r^R \Phi^{-1}\left(\sigma s^{1-n}\int_0^s t^{n-1} f\big(w(t)\big)\,\mathrm{d}t\right)\mathrm{d}s. \tag{5.4.11}$$

By the above argument, any fixed point $w_\sigma = \mathcal{H}[w_\sigma, \sigma]$ is of class $C^1[0, R]$ and has the properties $w_\sigma \geqslant 0$, $w_\sigma' \geqslant 0$ in $[0, R]$ and $w_\sigma(R) = \sigma m$. Additionally, by (5.4.10) we find that $\Phi(w_\sigma') \in C^1[0, R]$, and then from (5.4.9) that w_σ is a classical distribution solution of the problem

$$\begin{cases} [r^{n-1}\Phi(w_\sigma'(r))]' - \sigma r^{n-1} f(w_\sigma(r)) = 0 \quad \text{in } (0, R], \\ w_\sigma'(0) = 0, \quad w_\sigma(R) = \sigma m. \end{cases} \tag{5.4.12}$$

In turn, it is evident that any function w_1 which is a fixed point of $\mathcal{H}[w, 1]$ (that is $w_1 = \mathcal{H}[w_1, 1]$) is a non-negative radial distribution solution of problem (5.4.4) in $B_R \setminus \{0\}$, with $w'(0) = 0$ and $w' \geqslant 0$ in $[0, R]$.

Since $f > 0$ for $u > 0$ it follows equally from (5.4.12) that the final statement of the theorem is valid.

We assert that such a fixed point $w = w_1$ exists, using Browder's version of the Leray–Schauder theorem for this purpose (see Theorem 11.6 of [40]).

To begin with, obviously $\mathcal{H}[w, 0] \equiv 0$ for all $w \in X$, that is $\mathcal{H}[w, 0]$ maps X into the single point $w_0 = 0$ in X. (This is the first hypothesis required in the application of the Leray–Schauder theorem.) We show next that $\mathcal{H}$ is compact from $X \times [0, 1]$ into X. First, $\mathcal{H}$ is continuous on $X \times [0, 1]$. Indeed, let $w_j \to w$, $\sigma_j \to \sigma$, $(w_j, \sigma_j) \in X \times [0, 1]$. Then in (5.4.11) clearly $\sigma_j f(w_j) \to \sigma f(w)$, since the modified function f is continuous on $\mathbb{R}$. Hence $\mathcal{H}[w_j, \sigma_j] \to \mathcal{H}[w, \sigma]$, as required.

Next let $(w_k, \sigma_k)_k$ be a bounded sequence in $X \times [0, 1]$. It is clear from (5.4.10) that

$$\big\|\mathcal{H}[w_k, \sigma_k]'\big\|_\infty \leqslant \Phi^{-1}\left(\frac{Rf(m)}{n}\right). \tag{5.4.13}$$

As an immediate consequence of the Ascoli–Arzelà theorem $\mathcal{H}$ then maps bounded sequences into relatively compact sequences in X, so $\mathcal{H}$ is compact.

To apply the Leray–Schauder theorem it is now enough to show that there is a constant $M > 0$ such that

$$\|w\|_\infty \leqslant M \quad \text{for all } (w, \sigma) \in X \times [0, 1], \text{ with } \mathcal{H}[w, \sigma] = w. \tag{5.4.14}$$

Let (w, σ) be a pair of type (5.4.14). But, as observed above, one has $w \geqslant 0$, $w' \geqslant 0$, so that $\|w\|_\infty = w(R) \leqslant \sigma m \leqslant m$. Thus we can take $M = m$ in (5.4.14).

The Leray–Schauder theorem therefore now implies that the mapping $\mathcal{T}[w] = \mathcal{H}[w, 1]$ has a fixed point $w \in X$, which is the required solution of (5.4.4) in $B_R \setminus \{0\}$, proving the assertion above.

The fixed point $u = w$ is a C^1 distribution solution of (5.4.4) *in* B_R. The proof is standard. Let $\varphi \in C_c^1(B_R)$. We have to show that

$$\int_{B_R} \big\langle A\big(|Du|\big)Du, D\varphi\big\rangle \,\mathrm{d}x = -\int_{B_R} f(u)\varphi \,\mathrm{d}x.$$

To this end, let $\psi = \varphi\eta$, where for $0 < 2\varepsilon < R$,

$$\eta(x) = \begin{cases} 0 & \text{for } |x| \leqslant \varepsilon, \\ 1 & \text{for } |x| \geqslant 2\varepsilon, \end{cases}$$

and such that $\eta \in C^1(\mathbb{R}^n)$, $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^n$, $|D\eta(x)| \leqslant 2/\varepsilon$ for all x with $\varepsilon \leqslant |x| \leqslant 2\varepsilon$. Consequently, using ψ as a test function in $B_R \setminus \{0\}$, we get

$$\begin{aligned} &\int_{B_R \setminus B_{2\varepsilon}} \big\langle A\big(|Du|\big)Du, D\varphi\big\rangle \,\mathrm{d}x + \int_{B_{2\varepsilon} \setminus B_\varepsilon} \big\langle A\big(|Du|\big)Du, \eta D\varphi + \varphi D\eta\big\rangle \,\mathrm{d}x \\ &\quad = -\int_{B_R \setminus B_{2\varepsilon}} f(u)\varphi \,\mathrm{d}x - \int_{B_{2\varepsilon} \setminus B_\varepsilon} f(u)\eta\varphi \,\mathrm{d}x. \end{aligned}$$

Now

$$\begin{aligned} \left| \int_{B_{2\varepsilon} \setminus B_\varepsilon} \big\langle A\big(|Du|\big)Du, \eta D\varphi + \varphi\eta\big\rangle \,\mathrm{d}x \right| &\leqslant \sup_{B_{2\varepsilon}} \left\{ \Phi\big(|Du|\big) \cdot \left[|D\varphi| + \frac{2}{\varepsilon}|\varphi| \right] \right\} \cdot |B_{2\varepsilon}| \\ &= \mathrm{o}\big(\varepsilon^{n-1}\big) \end{aligned}$$

since $Du(0) = \mathbf{0}$ and Φ is continuous at $\varrho = 0$ by (A2). Moreover

$$\left| \int_{B_{2\varepsilon} \setminus B_\varepsilon} f(u)\eta\varphi \,\mathrm{d}x \right| \leqslant \text{Const.}\, \varepsilon^n.$$

Letting $\varepsilon \to 0$ we get the required conclusion.

Uniqueness of C^1 distribution solutions of (5.4.4). This is an immediate consequence of the comparison Theorem 2.4.1 and Proposition 2.4.2. □

PROOF OF THEOREM 5.4.3. Monotonicity follows from comparison, as above.

Continuity. Let $0 < m_1 < m_2$ and write $u_1(r) = u(r, m_1)$ and $u_2(r) = u(r, m_2)$. We show that

$$0 \leqslant u_2(r) - u_1(r) \leqslant m_2 - m_1, \quad r \in [0, R]. \tag{5.4.15}$$

By (5.4.9), for all $r \in [0, R]$,

$$u_2(r) = m_2 - \int_r^R \Phi^{-1}\left(s^{1-n}\int_0^s t^{n-1} f\big(u_2(t)\big)\,\mathrm{d}t\right)\mathrm{d}s,$$

$$u_1(r) = m_1 - \int_r^R \Phi^{-1}\left(s^{1-n}\int_0^s t^{n-1} f\big(u_1(t)\big)\,\mathrm{d}t\right)\mathrm{d}s.$$

Then by subtraction

$$u_2(r) - u_1(r) = m_2 - m_1 - \int_r^R \left[\Phi^{-1}\left(s^{1-n}\int_0^s t^{n-1} f\big(u_2(t)\big)\,\mathrm{d}t\right) - \Phi^{-1}\left(s^{1-n}\int_0^s t^{n-1} f\big(u_1(t)\big)\,\mathrm{d}t\right)\right]\mathrm{d}s.$$

The function Φ^{-1} is strictly increasing by (A2) and f is non-decreasing in $\mathbb{R}$ by (F2). Therefore, since $u_1 \leqslant u_2$ in $[0, R]$ by monotonicity, one sees that the quantity in square brackets above is non-negative, and (5.4.15) is proved.

Proof that $u < m$ in B_R. By (5.4.9) it is enough to show that

$$I = \int_r^R \Phi^{-1}\left(s^{1-n}\int_0^s t^{n-1} f\big(u(t)\big)\,\mathrm{d}t\right)\mathrm{d}s > 0 \quad \text{for } r \in [0, R).$$

Clearly $u > 0$ in some interval $(r_0, R]$ with $r_0 \geqslant 0$, and in turn $f(u(s)) > 0$ in $(r_0, R]$ by (F2). Therefore

$$I \geqslant \int_{\max\{r_0, r\}}^R \Phi^{-1}\left(s^{1-n}\int_{r_0}^s t^{n-1} f\big(u(t)\big)\,\mathrm{d}t\right)\mathrm{d}s > 0,$$

as required. □

Proof of Theorem 5.4.4

We begin with a preliminary result, of interest in itself.

THEOREM 5.4.5. *If $u_1 = u(\cdot, m_1)$ has a dead core B_{S_1}, then $u_2 = u(\cdot, m_2)$, $m_2 < m_1$, has a dead core B_{S_2}, with $S_2 > S_1$. Similarly, if either $u_1(0) > 0$ or $u_1(0) = 0$ and $u_1(r) > 0$ for $r \in (0, R]$, then $u_2 > u_1$ in B_R when $m_2 > m_1$.*

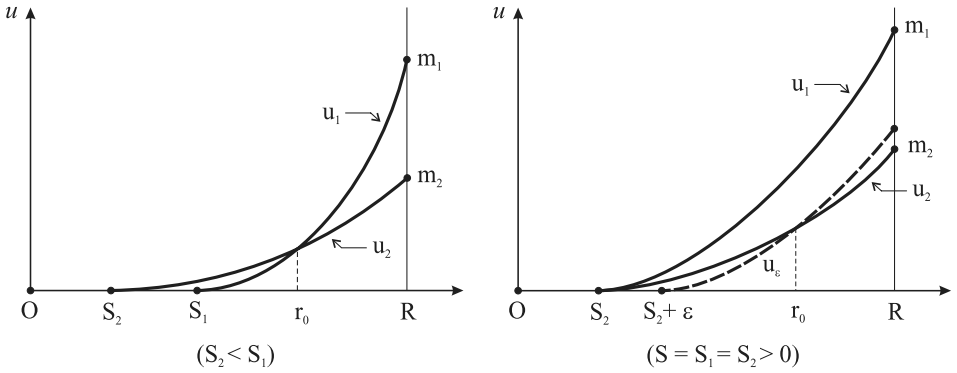

Fig. 8.

PROOF. To prove the first part of the lemma, assume for contradiction that $m_2 < m_1$, but either $u_2(r) > 0$ in $(0, R]$, or $0 < S_2 \leqslant S_1$. In the first case the solutions u_1 and u_2 must cross at some point $r_0 \in (S_1, R)$. Then, applying Theorem 2.4.1 in B_{r_0} (always with the help of Proposition 2.4.2), we find that $u_1 \equiv u_2$ in $[0, r_0]$, which is an obvious contradiction since $u_2(r) > 0$ on $(0, r_0]$, while $u_2 \equiv u_1 \equiv 0$ in $[0, S_1]$. The next case $0 < S_2 < S_1$ leads to a contradiction in the same way, see Figure 8.

The remaining case, when $S = S_2 = S_1 > 0$ needs more care. For $\varepsilon \in (0, R)$ define

$$u_\varepsilon(r) = \begin{cases} 0, & 0 \leqslant r \leqslant \varepsilon, \\ u_1(r-\varepsilon), & \varepsilon \leqslant r \leqslant R, \end{cases}$$

If $\varepsilon > 0$ is suitably small then one has $m_1 > u_\varepsilon(R) > m_2 = u_2(R)$, while at the same time

$$u_2(S+\varepsilon) > 0 = u_1(S) = u_\varepsilon(S+\varepsilon). \tag{5.4.16}$$

Thus there is a point $r_0 \in (S+\varepsilon, R)$ where u_ε and u_2 cross, see the second case of Figure 8.

We assert that u_ε is a supersolution of (5.4.1) in the annulus $B_R \setminus \overline{B_\varepsilon}$. Indeed in this set we have

$$\begin{aligned} \operatorname{div}\{A(|Du_\varepsilon|)Du_\varepsilon\} - f(u_\varepsilon) &= \{A(|u_\varepsilon'|)u_\varepsilon'\}' + \frac{n-1}{r}A(|u_\varepsilon'|)u_\varepsilon' - f(u_\varepsilon) \\ &= \left(\frac{n-1}{r} - \frac{n-1}{r-\varepsilon}\right)\Phi\big(u_1'(r-\varepsilon)\big) \\ &= -\varepsilon\frac{n-1}{r(r-\varepsilon)}\Phi\big(u_1'(r-\varepsilon)\big) \leqslant 0. \end{aligned} \tag{5.4.17}$$

Observing that $u_2(0) = u_\varepsilon(0) = 0$, we can then apply the comparison Theorem 2.4.1 in B_{r_0}. Therefore $u_2 \leqslant u_\varepsilon$ in $[0, r_0]$, which contradicts (5.4.16) for the specific vale $r = S + \varepsilon$, and completes the first part of the proof.

To obtain the second part of the theorem, first assume for contradiction that $u_2(0) = u_1(0)$ when $m_2 > m_1$. Now by the final part of Theorem 5.4.2 we have $u_1'(r) > 0$ for $r \in (0, R]$. Define

$$\tilde{u}_\varepsilon(r) = \begin{cases} u_2(0), & 0 \leqslant r \leqslant \varepsilon, \\ u_2(r-\varepsilon), & \varepsilon \leqslant r \leqslant R, \end{cases}$$

where ε is chosen so small that $m_2 > \tilde{u}_\varepsilon(R) > m_1 = u_1(R)$. On the other hand $u_1(\varepsilon) > \tilde{u}_\varepsilon(\varepsilon) = 0$. Hence there is a crossing point $r_0 \in (\varepsilon, R)$ where $u_1(r_0) = \tilde{u}_\varepsilon(r_0)$. As before u_ε is a supersolution of (5.4.1) in R_R (5.4.1) so that $u_1 \leqslant \tilde{u}_\varepsilon$ in B_{r_0} by Theorem 2.4.1. Therefore $u_1 \equiv \tilde{u}_\varepsilon \equiv 0$ in $[0, \varepsilon]$, which is impossible, since $u_1(r) > 0$ for all $r \in (0, R]$.

That $u_2 > u_1$ in all B_R now follows at once, since otherwise u_2 and u_1 would cross at some value $r = r_0$ in which case comparison would lead to the absurd result $u_2 \equiv u_1$ in B_{r_0}. □

PROOF OF THEOREM 5.4.4. For the purpose of this proof, we suppose that there is some $m > 0$ for which $u(0, m) > 0$.

Existence of u_0. Define

$$m_0 = \inf\{m > 0\colon\ u(0, m) > 0\}. \tag{5.4.18}$$

We claim first that $m_0 > 0$. Choose $\mu > 0$ so small that

$$R_{0,\mu} = \int_0^\mu \frac{\mathrm{d}s}{H^{-1}(F(s)/n)} < R, \tag{5.4.19}$$

which of course is possible by assumption (5.4.2), see Lemma 4.4.3. Define

$$v(r) = w(r - S), \quad r \in [S, R],\ S = R - C,$$

where w is the function constructed in the dead core Lemma 4.4.1, with $\sigma = 1/n$ and $C = R_{0,\mu}$.

We assert that v is a supersolution of (5.4.1) in the set $B_R \setminus \overline{B}_S$. In fact

$$\operatorname{div}\{A(|Dv|)Dv\} = [\Phi(v')]' + \frac{n-1}{r}\Phi(v') \leqslant \left\{1 + \frac{n-1}{r}(r - S)\right\}\sigma f(v)$$

by (iii) and (iv) of Lemma 4.4.1. Thus

$$\operatorname{div}\{A(|Dv|)Dv\} \leqslant \left\{1 - \frac{n-1}{nr}S\right\} f(v) \leqslant f(v),$$

as required. Then, since $v(S) = v'(S) = 0$, by defining v to be zero in B_S, the extended function v is a C^1 supersolution of (5.4.1) in all B_R. while also $v(R) = \mu$. By the comparison Theorem 2.4.1 we find that $u(\cdot, \mu) \equiv 0$ in B_S. Therefore $m_0 \geqslant \mu > 0$ by (5.4.18) and Theorem 5.4.5, and the claim is proved.

Next, if (i) would be false, then $u_0(0) > 0$ and by Theorem 5.4.3 also $u(0, m) > 0$ for all values m sufficiently close to m_0, which would contradict (5.4.18). Property (ii) is again a direct consequence of the definition (5.4.18) of m_0 and Theorem 5.4.3. Finally if there is $m \in (0, m_0)$ such that the corresponding solution $u(\cdot, m)$ of (5.4.4) has no dead core, then $u(0, m) \geqslant 0$ and $u(r, m) > 0$ for $r \in (0, R]$. Thus by Theorem 5.4.5, with $m_1 = m$ and $m_2 = m_0$, we get $u_0(0) > u(0, m) \geqslant 0$, contradicting (i) and proving (iii).

Uniqueness of u_0. Suppose both m_0 and $\overline{m}_0$ have the properties (i)–(iii) of the theorem. Then $u_0(0) = u_0(0, m_0) = 0$ by (i), while $u(0, m) > 0$ when $m > \overline{m}_0$ by (ii). Hence $m_0 \leqslant \overline{m}_0$. Similarly $\overline{m}_0 \leqslant m_0$. Therefore $\overline{m}_0 = m_0$, as desired.

The case $m_0 = \infty$. If $u(0, m) = 0$ for all $m > 0$, then $u(\cdot, m)$ has a dead core for all $m > 0$. Otherwise there would be a value $\overline{m} > 0$ for which $u(0, \overline{m}) = 0$ and $u(r, \overline{m}) > 0$ for $r \in (0, R]$. Hence $u(0, m) > 0$ for $m > \overline{m}$ by Theorem 5.4.5, contradicting the assumption. This also justifies the earlier agreement that $m_0 = \infty$ in this case. □

REMARK. In summary, if m_0 is finite and $m > m_0$, then the solution $u = u(\cdot, m)$ of (5.4.4) is positive, namely $u(r, m) > 0$ for all $r \in [0, R]$. On the other hand, if $m < m_0 \leqslant \infty$, then the solution $u = u(\cdot, m)$ of (5.4.4) has a dead core $B_S \subset B_R$, $0 < S < R$.

The size of a dead core and proof of Theorem 5.4.1

Recall the assumption that $\Phi(\infty) = H(\infty) = \infty$, and let

$$R_0 = \int_0^\infty \frac{\mathrm{d}s}{H^{-1}(F(s)/n)}. \tag{5.4.20}$$

Clearly $0 < R_0 \leqslant \infty$ since the integral is convergent at 0 by Lemma 4.4.3 with $\sigma = 1/n$. Of course the integral can possibly be divergent at ∞.

We prove two preliminary results.

THEOREM 5.4.6. *We have*

$$m_0 = \infty \quad \text{if } R_0 < \infty \text{ and } R \geqslant R_0, \tag{5.4.21}$$

while

$$m_0 \geqslant \overline{m} \quad \text{if } R < R_0, \tag{5.4.22}$$

where $\overline{m}$ *is defined by the relation*

$$R = \int_0^{\overline{m}} \frac{\mathrm{d}s}{H^{-1}(F(s)/n)}.$$

PROOF. The proof of (5.4.21) is essentially the same as the proof of the first part of Theorem 5.4.4, the only exception being that $C_{n,\mu}$ is replaced by R_0.

To obtain (5.4.22), we define $v(r) = w(r)$ as in the proof of Theorem 5.4.4 but with $S = 0$. Then by Lemma 4.4.1 there holds $v(0) = v'(0) = 0$, $v(R) = w(R) = \overline{m}$. Moreover v is a supersolution of (5.4.1). By virtue of Theorem 2.4.1, it follows that $0 \leqslant u(r, m) \leqslant v(r)$. Hence $u(0, \overline{m}) = 0$, and in turn from the definition (5.4.18) of m_0 we get $m_0 \geqslant \overline{m}$, as required in (5.4.22). □

THEOREM 5.4.7. *Let $m < m_0$, so that a dead core exists by Theorem* 5.4.4(iii). *In particular the solution $u = u(\cdot, m)$ satisfies*

$$u \equiv 0 \quad \text{in } B_S \subset B_R,$$

where

$$R - \int_0^m \frac{\mathrm{d}s}{H^{-1}(F(s)/n)} < S < R.$$

If $R \geqslant R_0$, then for all $m > 0$ one has

$$R - R_0 < S < R.$$

PROOF. The proof is the same as the first part of the proof of Theorem 5.4.4. □

REMARK. For any $\varepsilon > 0$, if m is suitably small (depending on ε) we have $R - \varepsilon < S < R$.

PROOF OF THEOREM 5.4.1. *Part* (a). That $u \geqslant 0$ follows by Theorem 2.4.1 by comparing the given solution u with the trivial solution 0.

The constant function m is a supersolution of (5.4.1), so that again by Theorem 2.4.1 we have $u \leqslant m$ in Ω. In fact $u < m$ in Ω. To see this, let y be any point of Ω and B a ball in Ω centered at y. Let $v(\cdot, m)$ be the radial solution of (5.4.1) in B constructed in Theorem 5.4.2, with $v(|x - y|, m) = m$ for $x \in \partial B$. Therefore $u(x) \leqslant m = v(|x - y|, m)$ for $x \in \partial B$, and in turn $u(x) \leqslant v(|x - y|, m) < m$ for $x \in B$ by the final part of Theorem 5.4.3.

Part (b). This is a direct consequence of Theorem 5.4.6.

Part (c). Clearly there exists $R > 0$ such that $\overline{B} \subset B_R \Subset \Omega$, with B and B_R centered at the same point of Ω. By (a) we know that $u < m$ on ∂B_R. Denoting by $R - \varepsilon$ the radius of B, then by comparison, together with the remark after Theorem 5.4.7, we have $u \equiv 0$ in B when $m > 0$ is suitably small. □

The case $\Phi(\infty) < \infty$

This is the case, for example, for the mean curvature operator for which

$$\Phi(\infty) = H(\infty) = 1.$$

The proof of the principal Theorem 5.4.2 requires only the modification that the parameter m in (5.4.4) should be restricted so that

$$Rf(m) < n\Phi(\infty), \tag{5.4.23}$$

so that $\mathcal{T}$ in (5.4.9) is well defined. Moreover, for the application of the dead core Lemma 4.4.1 in the proof of Theorem 5.4.4 we also need the further restriction

$$F(m) < nH(\infty). \tag{5.4.24}$$

Denote by m_∞ the supremum of all $m > 0$ satisfying (5.4.23) and (5.4.24).

Then the main results stated in Section 5.4 remain true provided that the condition $m < m_\infty$ is assumed in all the statements. For instance we have the following analog of Theorem 5.4.1.

THEOREM 5.4.8. *Assume the dead core condition* (5.4.2) *holds and let u be a solution of* (5.4.1), *with $0 \leqslant u(x) \leqslant m$ on $\partial\Omega$ for some positive constant $m < m_\infty$. Then the following properties are valid*:

(a) $0 \leqslant u < m$ *in* Ω.

(b) *Assume that*

$$R_0 = \int_0^{m_\infty} \frac{\mathrm{d}s}{H^{-1}(F(s)/n)} < \infty,$$

and let B_R be a ball with radius $R \geqslant R_0$, compactly contained in Ω. Then u has a dead core in Ω for all $m \in (0, m_\infty)$.

(c) *If B is any ball compactly contained in Ω, then $u \equiv 0$ in B provided that $m > 0$ is suitably small.*

It is not hard to show that if $\Phi(\infty) = \infty$ then necessarily $H(\infty) = \infty$, but it is possible to have $\Phi(\infty) < \infty$ and $H(\infty) = \infty$, as shown by $A(s) = 1\big/(1 + \sqrt{1+s^2})$, with correspondingly

$$H(s) = \frac{1}{2}\left[\frac{s^2}{1+\sqrt{1+s^2}} - \log\frac{1+\sqrt{1+s^2}}{2}\right].$$

In this example $\Phi(\infty) = 1$, while $H(\infty) = \infty$.

The case $H(\infty) < \infty$ for unrestricted $m > 0$ was treated by Siegel in [88].

A dead core with bursts

It is known that when (4.1.6) holds and when f appropriately changes sign for $z > \delta$, there are non-negative radially symmetric solutions v of (5.4.1) having compact support; see for example [36]. Let R_1 be the support radius of such a solution.

Next choose R and S in Theorem 5.4.7 so that $R_1 < S < R$, and let w denote a corresponding dead core solution with small m. This being done, we can now replace the solution w on the set B_{R_1}, where it vanishes, by the solution v, thus obtaining a new solution u which is then positive in B_{R_1} and $B_R \setminus \overline{B_S}$, and otherwise vanishes. This solution may be considered as a dead core with a symmetric *burst* centered at the origin.

Of course, the same procedure may be repeated at other suitably chosen origins in B_S, giving rise to multiple bursts. Naturally a given ball B_S can accommodate only a certain number of bursts, but the larger are R and S the more bursts which can be allowed. For details and further extensions the reader is referred to [72].

Notes

The results in Section 5.4 are taken from the paper [72]. In related work [12] Bandle and Vernier–Piro have studied the dead core problem for a weighted equation, see also other extensions given in [72]. A further related dead core theorem was given by Diaz and Véron [30].

Sperb [91] considers similar dead core problems for the special case of the Laplace operator, that is $A \equiv 1$. He estimates the critical value m_0 for more general domains than balls, but only for the homogeneous case $f(u) = \text{Const.}|u|^{q-1}u$, $0 < q < 1$. For balls B_R his estimate is weaker than the exact result (5.4.8). His estimates for the size of dead cores apply to more general domains than balls, but in the special case of balls are weaker than those given above.

Theorem 5.4.4 for the general equation (5.4.1) seems to capture and extend many of the ideas of these earlier papers (for further extensions see [72]).

5.5. The Harnack Inequality in $\mathbb{R}^2$

In this section we use maximum principle techniques to obtain a Harnack inequality in $\mathbb{R}^2$. Although the final result is restricted to plane domains, the class of equations covered is in many respects more extensive than in previous work. We consider the model equation

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} - f(u) = 0, \quad u > 0, \tag{5.5.1}$$

in $\mathbb{R}^2$, and more generally the equation

$$\operatorname{div}\big\{A\big(|Du|\big)Du\big\} + B(x, u, Du) = 0, \quad u > 0. \tag{5.5.2}$$

Throughout the section we assume the natural conditions (A1′) and (A2), and let $\Phi(s) = sA(s)$ for $s > 0$ with $\Phi(0) = 0$, Concerning the function $B(x, z, \boldsymbol{\xi})$ we suppose, in slight variation with the conditions of Section 4.7, that

There exist a constant $\kappa > 0$ and a nonlinearity f such that

(B1) $B(x, z, \boldsymbol{\xi}) \geqslant -\kappa\Phi(|\boldsymbol{\xi}|) - f(z)$,

(B2) $B(x, z, \boldsymbol{\xi}) \leqslant \kappa\Phi(|\boldsymbol{\xi}|)$,

for $x \in \Omega$, $z \geqslant 0$, and all $\xi \in \mathbb{R}^n$. Moreover f is assumed to satisfy

(F2) $f(0) = 0$ *and f is continuous and non-decreasing in $\mathbb{R}_0^+$.*

In addition to these assumptions, several further technical conditions will be placed on the function Φ to allow the derivations to proceed. It is convenient to defer their statements for the moment (see (A3)–(A4) below). At the same time we note that the full set of conditions (A1′)–(A4) are valid for the prime examples of the p-Laplace operator $A(s) = s^{p-2}$, $p > 1$, and the *mean curvature operator* $A(s) = 1/\sqrt{1+s^2}$, as shown below.

Let us denote by B_R the disk in $\mathbb{R}^2$ centered at $(0, 0)$ with radius $R > 0$. Then the following main results hold under the main conditions (A1′)–(A2), (B1)–(B2).

THEOREM 5.5.1. *Let Φ also satisfy* (A3)–(A4) *below. Suppose either $f \equiv 0$ on $[0, d]$, $d > 0$, or that $f > 0$ on $\mathbb{R}^+$ and that*

$$\int_{0^+} \frac{\mathrm{d}s}{H^{-1}(F(s))} = \infty. \tag{5.5.3}$$

Then for every $R > 0$ there is a strictly increasing continuous function Φ_R from $\mathbb{R}_0^+$ to $\mathbb{R}_0^+$, with $\Phi_R(0) = 0$, such that any C^1 distribution solution u of equation (5.5.2) *in the disk B_R satisfies the Harnack condition*

$$u(x, y) \geqslant \Phi_R\big(u(0, 0)\big) \quad \textit{for all } (x, y) \in B_{R/3}. \tag{5.5.4}$$

Our method of proof does not make obvious the dependence of the function Φ_R on R or on the parameters of (5.5.2). For a given function A, however, it is worth noting that the dependence of Φ_R on f arises from the divergence rate as $s \to 0$ of the integral (5.5.3).

Theorem 5.5.1 has the following important companion result.

THEOREM 5.5.2. *Under the assumptions of Theorem* 5.5.1, *for every $R > 0$ there exists a strictly increasing continuous function $\tilde{\Psi}_R$ from $[0, M_R)$ to $\mathbb{R}_0^+$, with $\Psi_R(0) = 0$, such that any C^1 distribution solution u of* (5.5.2) *in the disk B_R, with $u(0, 0) < M_R$, satisfies*

$$u(x, y) \leqslant \tilde{\Psi}_R\big(u(0, 0)\big) \quad \textit{for all } (x, y) \in B_{R/4}. \tag{5.5.5}$$

(*In fact, one can take $\tilde{\Psi} = \Psi^{-1}$.*)

In the next Theorem 5.5.3 we give a criterion for the value M_R to be infinite, in which case the restriction $u(0, 0) < M_R$ in Theorem 5.5.2 can be omitted.

THEOREM 5.5.3. *Let $\Phi(\infty) = \infty$. If either $f \equiv 0$, or $f(z) > 0$ for z suitably large, say $z > 1$, and*

$$\int^{\infty} \frac{\mathrm{d}s}{H^{-1}(F(s))} = \infty, \tag{5.5.6}$$

then the function Ψ_R in Theorem 5.5.2 *is onto $\mathbb{R}_0^+$ and $M_R = \infty$.*

Conversely, if $f(z) > 0$ *for all* $z > 0$ *and*

$$\int^{\infty} \frac{\mathrm{d}s}{H^{-1}(F(s))} < \infty, \tag{5.5.7}$$

then M_R *is finite.*

The standard extension of the Harnack inequality to arbitrary domains is also true in the present circumstances. In stating Theorem 5.5.3, observe that $\Phi(\infty) = \infty$ implies $H(\infty) = \infty$, so the integral in (5.5.6) is well-defined.

Conditions (5.5.3) and (5.5.6) place non-trivial restrictions on the behavior of the function f for z near 0 and z near ∞. For example, if $A(s) = s^{p-2}$ (p-Laplace operator), then $H(s) = s^p/p'$. Thus a function f with polynomial behavior $f(z) \approx z^m$ near $z = 0$ and $f(z) \approx z^q$ near $z = \infty$ satisfies (5.5.3) and (5.5.6) only when $m \geqslant p - 1$, $q \leqslant p - 1$.

The further conditions for the function Φ, whose statement has previously been deferred, can now be given.

(A3) *There is a positive constant* c *such that*

$$s\Phi'(s) \leqslant c\Phi(s) \quad \textit{for all } s > 0;$$

(A4) *For all* $\theta \geqslant 1$ *there is a number* $c_1 = c_1(\theta) \geqslant 1$ *such that*

$$\Phi'(\lambda s) \leqslant c_1 \Phi'(s) \quad \textit{for all } \lambda \in [1, \theta] \textit{ and } s > 0.$$

Conditions (A1′)–(A4) are comparatively weak. For example, they hold for the following examples:

1. The generalized mean curvature operator $A(s) = (1 + s^2)^{-p/2}$, $p \leqslant 1$. Here one can take $c = \max\{1, 1 - p\}$; similarly $c_1 = 1$ if $p = 1$ and $c_1 = \theta^q$, $q = \max\{2, -p\}$ if $p < 1$. Note also that $\Phi(\infty) = 1$ for $p = 1$, while $\Phi(\infty) = \infty$ for $p < 1$.
2. The p-Laplace operator $A(s) = s^{p-2}$, $p > 1$. Here one takes $c = p - 1$, $c_1 = \theta^{p-2}$.
3. $A(s) = s^{p-2} + s^{p_1-2}$, $1 < p < p_1$. Now $c = p_1 - 1$ and $c_1 = \theta^{p_1-2}$. See [9] for applications in quantum physics.

Construction of a comparison function

Let E denote the closed region bounded by an arbitrary ellipse in $\mathbb{R}^2$. It is convenient to introduce *local Cartesian coordinates* (ξ, η) on E, so that E is given by

$$\sigma^2 = \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} \leqslant 1, \quad a \geqslant b > 0$$

(note that the coordinates (ξ, η) on E are related by translation and rotation to the global coordinates (x, y) on $\mathbb{R}^2$). The subset

$$\frac{1}{2} \leqslant \sigma \leqslant 1, \qquad \xi \geqslant \frac{1}{2}a$$

of E will be called a *minorant region* and denoted by Σ. To each minorant region Σ we shall associate a C^1 function $v = v(\sigma) : \Sigma \to \mathbb{R}^+$ (where $\sigma \in [\frac{1}{2}, 1]$ is the local coordinate on Σ defined above), called a *minorant function* and constructed in such a way that

$$\begin{aligned} &\operatorname{div}\{A(|Dv|Dv)\} - \kappa\Phi(|Dv|) - f(v) \geqslant 0, \quad |Dv| > 0 \quad \text{in } \Sigma, \\ &v(1) = 0, \qquad v\left(\frac{1}{2}\right) = m > 0. \end{aligned} \tag{5.5.8}$$

It is easy to check that $v(\sigma)$ is necessarily a positive increasing function in $(\frac{1}{2}, 1)$.

To carry out the construction, we shall need the following lemma; it is convenient in stating and proving these to replace the variables (ξ, η) by (x, y) – the corresponding results for the variables (ξ, η) then follow from the translational and rotational invariance of (5.5.8).

LEMMA 5.5.4. *Let $w = v/b$. Then*

$$-\operatorname{div}\{A(|Dv|)Dv\} \leqslant \frac{1}{\ell}\{\Phi(|w'|)\}' + \frac{\gamma}{\ell\sigma}\Phi(|w'|),$$

where

$$\gamma = c_1(\theta)\left[\theta^3 + \frac{1}{4}(\theta^2 - 1)^2 c\right], \quad \ell = a\theta c_1(\theta), \quad \theta = \frac{a}{b}$$

(*here $'$ denotes differentiation with respect to the variable σ*).[14]

For the proof of this lemma the reader is referred to [70]. We can now complete the construction of the minorant function $v = bw$ by choosing w as a solution of the following initial value problem (see (5.5.8) and Lemma 5.5.4):

$$\{\Phi(|w'|)\}' + \frac{k-1}{\sigma}\Phi(|w'|) + \ell f(bw) = 0, \quad k = \gamma + \kappa\ell + 1, \tag{5.5.9}$$

$$w(1) = 0, \quad w\left(\frac{1}{2}\right) = \frac{m}{b} > 0; \quad w' < 0. \tag{5.5.10}$$

If w is a solution of the problem (5.5.9)–(5.5.10) in $[\frac{1}{2}, 1]$, then $v = bw$: $\Sigma \to \mathbb{R}^+$ is a minorant function on Σ. Here one uses the relation $|Dv| = (s/\sigma)|v'| = (bs/\sigma)|w'| \leqslant |w'|$ and $\Phi(|Dv|) \leqslant \Phi(|w'|)$, since Φ is increasing.

Since (5.5.9) can be written in the form

$$\sigma^{1-k}\{\sigma^{k-1}\Phi(|w'|)\}' + \ell f(bw) = 0,$$

[14] By standard abuse of notation, the letter v denotes either the mapping $v : \Sigma \to \mathbb{R}^+$ or the ordinary real function $v : [\frac{1}{2}, 1] \to \mathbb{R}^+$. The distinction will always be clear form the context.

the existence of a solution of (5.5.9)–(5.5.10) is a consequence of Lemma 4.3.1 in a space of k dimensions, together with (5.5.3). As a final note, the function $v(\sigma) = v(\sigma, m)$ is a non-decreasing function of m, this being a consequence of the comparison Theorem 3.5.1.

For the proof of Theorem 5.5.1 one further lemma is necessary.

LEMMA 5.5.5. *Let u be a solution of* (5.5.2) *in a bounded domain Ω and let $u_0 > 0$ be such that the set $U = \{x \in \Omega\colon\ u(x) > u_0\}$ is non-empty. Then for any component K of U we have $\partial K \cap \partial\Omega \neq \emptyset$.*

This is a direct consequence of Corollary 4.7.6.

PROOF OF THEOREM 5.5.1. This is exactly the same as the proof of the corresponding Harnack theorem for linear elliptic equations, [66, pages 111–117] or [40, pages 41–44], using the minorant function constructed above. The comparison Theorem 3.6.5 replaces Hopf's Theorem 2.1.2, while Lemma 5.5.5 guarantees the existence of a connected region joining $(0, 0)$ to ∂B_R along which $u(x, y) \geqslant u(0, 0)$. We refer the reader to [66] and [40] for the details, noting only that the function $\Psi_R(m)$ has the form

$$v_2\big(\sigma_2, v_1(\sigma_1, m)\big), \tag{5.5.11}$$

where v_1, v_2 are specially chosen minorant functions and σ_1, $\sigma_2 \in (0, 1)$ are appropriate small parameters. □

PROOF OF THEOREM 5.5.2. Define $M_R = \lim_{m\to\infty} \Psi_R(m)$, possibly infinite, and let

$$\tilde{\Psi}_R = \Psi_R^{-1} : [0, M_R) \to \mathbb{R}_0^+.$$

Let $z = (x, y) \in B_{R/4}$. The disk $B_{3R/4}(z)$ centered at z is then contained in B_R, so by Theorem 5.5.1 applied in $B_{3R/4}(z)$ we get $u(0, 0) \leqslant \Psi_R(u(x, y))$. □

The Harnack function

The proof of Theorem 5.5.3 requires a careful analysis of the solution $w = w(\sigma, m)$ of (5.5.9)–(5.5.10). For this purpose it is convenient to change variables according to $t = 1 - \sigma$ and to consider (without confusion) the function $w = w(t, m)$, where $t \in [0, 1/2]$ and

$$w(0) = 0, \quad w\left(\frac{1}{2}\right) = m, \quad w' > 0.$$

LEMMA 5.5.6. *With $' = \mathrm{d}/\mathrm{d}t$, we have*

(i) $$\big\{\Phi(w')\big\}' = \frac{k-1}{\sigma}\Phi(w') + \tilde{f}(w) \geqslant 0,$$
where $\tilde{f}(s) = \ell f(bs)$;

(ii) $$\Phi(w') \leqslant \sigma^{1-k}\Phi(\alpha) + \frac{\sigma}{k}\big(2^k - 1\big)\tilde{f}(w), \quad \alpha = w'(0).$$

PROOF. (i) is an immediate consequence of (5.5.9). It follows then that w' is non-decreasing and in turn that w is convex in the variable t. Integrating (5.5.9) from 0 to t yields

$$\sigma^{k-1}\Phi\big(w'(t)\big) - \Phi(\alpha) = \int_0^t \sigma^{k-1}\tilde{f}\big(w(s)\big)\,\mathrm{d}s \leqslant \tilde{f}\big(w(t)\big)\frac{1-\sigma^k}{k},$$

since $\tilde{f}(w)$ is non-decreasing in $[0, 1/2]$. Then (ii) follows at once since $\sigma^{-k} \leqslant 2^k$. □

LEMMA 5.5.7.

$$H(w') \leqslant H(\alpha) + (k-1)2^k\Phi(\alpha)w + 2^k\widetilde{F}(w). \tag{5.5.12}$$

PROOF. From (4.1.3) and (i)

$$\begin{aligned} H\big(w'(t)\big) &= \big\{\Phi\big(w'(t)\big)\big\}'w'(t) = \left\{\frac{k-1}{\sigma}\Phi(w') + \tilde{f}(w)\right\}w' \\ &\leqslant \big\{(k-1)\sigma^{-k}\Phi(\alpha) + 2^k\tilde{f}(w)\big\}w' \end{aligned}$$

using (ii) of Lemma 5.5.6. The inequality (5.5.12) then follows by integration from 0 to t. □

Rewriting (5.5.12) as

$$w'(t) \leqslant H^{-1}\big(2^k\widetilde{F}\big(w(t)\big) + (k-1)2^k\Phi(\alpha)w(t) + H(\alpha)\big)$$

and integrating over $(t, 1/2)$ gives

$$\int_{w(t)}^{m} \frac{\mathrm{d}s}{H^{-1}(Q(s))} \leqslant \frac{1}{2} - t < \frac{1}{2}, \tag{5.5.13}$$

with $Q(s) = 2^k\widetilde{F}(s) + (k-1)2^k\Phi(\alpha)s + H(\alpha)$.

PROOF OF THEOREM 5.5.3. *First part*. In view of the form of Ψ_R in (5.5.11), and the fact that $v(\sigma, m) = bw(t, m)$, it is enough to show that, for each $t \in (0, 1)$,

$$w(t, m) \to \infty \quad \text{as } m \to \infty.$$

Suppose for contradiction that at some fixed $t_0 > 0$ we have $w(t_0, m) \leqslant w_0$ for all m sufficiently large. By convexity of w it would then follow that

$$0 < \alpha = \alpha(m) \leqslant w_0/t_0.$$

In turn

$$Q(s) \leqslant 2^k\widetilde{F}(s) + Cs, \quad s > 1,$$

where C is a positive constant.

Case 1. $f \equiv 0$ for all $u > 0$. Then, writing m for m/b without confusion, and putting $Cs = H(\rho)$, we have from (5.5.13)

$$\frac{1}{2} > \int_{w_0}^{m} \frac{\mathrm{d}s}{H^{-1}(Cs)} = \frac{1}{C}\int_{H^{-1}(Cw_0)}^{H^{-1}(Cm)} \frac{H'(\rho)\,\mathrm{d}\rho}{\rho} = \frac{1}{C}\int_{H^{-1}(Cw_0)}^{H^{-1}(Cm)} \Phi'(\rho)\,\mathrm{d}\rho$$
$$= \frac{1}{C}\Phi(\rho)\Big|_{H^{-1}(Cw_0)}^{H^{-1}(Cm)}$$

by (4.4.3). But $H(\infty) = \infty$ so $H^{-1}(Cm) \to \infty$ as $m \to \infty$, which yields a contradiction since $\Phi(\infty) = \infty$.

Case 2. $f(u) > 0$ for, say, $u > 1$. Since f is non-decreasing and $\widetilde{F}(s) = (\ell/b)F(bs)$, it follows that for all sufficiently large s, say $s \geqslant s_0 > 1$, we would have

$$\widetilde{F}(s) \geqslant C_1 s, \quad C_1 > 0.$$

In turn, now letting $C > 0$ be a generic constant, we find

$$Q(s) \leqslant C\widetilde{F}(s) = CF(bs)$$

for $s \geqslant s_0$. Using (5.5.13) then gives, for $\max\{s_0,\ w_0\} < s_1 < m$,

$$\frac{1}{2} > \int_{w_0}^{m} \frac{\mathrm{d}s}{H^{-1}(Q(s))} > \int_{s_1}^{m} \frac{\mathrm{d}s}{H^{-1}(CF(bs))} = \frac{1}{b}\int_{bs_1}^{bm} \frac{\mathrm{d}t}{H^{-1}(CF(t))} \to \infty$$

as $m \to \infty$ by virtue of Lemma 4.4.2 and (5.5.6). This is an immediate contradiction, and the first part of the theorem is proved.

Second part. For simplicity we consider only the canonical case $B(x, z, \boldsymbol{\xi}) = -f(z)$. In this case the one-dimensional function $u(x, y) = w(x + a)$, where

$$t = \int_1^{w(t)} \frac{\mathrm{d}s}{H^{-1}(F(s))}, \quad t > 0,$$

is a positive solution of (5.5.1) for all $x > -a$. For $m > 1$ define

$$a = \int_1^m \frac{\mathrm{d}s}{H^{-1}(F(s))}, \qquad d = \int_1^\infty \frac{\mathrm{d}s}{H^{-1}(F(s))} < \infty.$$

Consider any fixed radius R such that $\frac{3}{4}d < R < d$. For suitably large m, say $m > m_0$, it follows that

$$\frac{3}{4}a < R < a.$$

Consequently u is a solution of (5.5.1) in the strip $-R < x < R$, and a fortiori in the disk B_R.

In turn,

$$\int_1^{u(-R/3,0)} \frac{\mathrm{d}s}{H^{-1}(F(s))} = \int_1^{w(a-R/3)} \frac{\mathrm{d}s}{H^{-1}(F(s))} = -\frac{R}{3} + a < \frac{3}{4}a$$
$$= \frac{3}{4}\int_1^m \frac{\mathrm{d}s}{H^{-1}(F(s))} < \frac{3}{4}\int_1^\infty \frac{\mathrm{d}s}{H^{-1}(F(s))} < \infty.$$

It follows that $u(-R/3, 0) \leqslant$ Constant for all suitably large m.[15] On the other hand $u(0, 0) = w(a) = m$ so that

$$\mu \geqslant \inf_{B_{R/3}} u(x, y) \geqslant \Psi_R(m), \quad m > m_0.$$

Thus $\Psi_R(\infty) \leqslant \mu$, that is $M_R < \infty$. □

The above argument can be refined to obtain asymptotic behavior of Ψ_R at $m = 0$ and at $m = \infty$. For example, for the equation

$$\Delta_p u + |u|^{q-2}u = 0,$$

one obtains

$$\Psi_1(m) \sim m^{[(q-1)/(p-1)]2} \quad \text{as } m \to 0,$$

while Ψ_R is then found by scaling.

Notes

The theorems of this section are due to Pucci and Serrin [70]. Previous results concerning the Harnack inequality for divergence structure elliptic equations are due to the second author [81] and to Lieberman [49], both of whom studied the general divergence structure equation (5.5.2) in n-dimensional space. In comparison with that work, the restriction here to $\mathbb{R}^2$ is a serious drawback, cf. [80]. On the other hand, for the inequality (5.5.2) itself, conditions (A1′)–(A4) are weaker than those earlier required; for example, in [81] the operator A is closely related to the degenerate Laplace case $\Phi(s) = s^{p-1}$.

For the special case of the mean curvature and other closely related operators, Trudinger [96] has also given a Harnack inequality in n dimensions, but only for bounded solutions, with the constant in the Harnack principle depending on the bound. Moreover, in the context of inequality (5.5.2) and the mean curvature operator $A(s) = 1/\sqrt{1+s^2}$, Trudinger's conditions are stronger than conditions (B1)–(B2) and (5.5.3).

[15] In fact for the constant we can take the value μ given by

$$\int_\mu^\infty \frac{\mathrm{d}s}{H^{-1}(F(s))} = \frac{d}{4}.$$

5.6. The Strong Maximum Principle for Riemannian manifolds

Let $\mathcal{M}$ be an n-dimensional Riemannian manifold of class C^1, with controvariant metric tensor $[g^{ij}]$ continuous in local coordinates $x = (x^1, \dots, x^n)$. Let u be a real-valued C^1 function defined on some open connected submanifold Ω of $\mathcal{M}$. The Riemannian norm of the gradient vector ∇u on Ω is then defined as the non-negative continuous function on Ω given in local coordinates by

$$|\nabla u|_g = \sqrt{g^{ij}\partial_{x_i}u\partial_{x_j}u}, \quad \partial_{x_i}u = \frac{\partial u}{\partial x^i}.$$

Consider the variational integral

$$I[u] = \int_\Omega \{\mathcal{G}(|\nabla u|_g) + F(u)\}\,\mathrm{d}\mathcal{M}.$$

The corresponding Euler–Lagrange equation is then

$$\operatorname{div}_g\{A(|\nabla u|_g)\nabla u\} - f(u) = 0, \tag{5.6.1}$$

where div_g is the Riemannian divergence operator and $A(s) = \mathcal{G}'(s)/s$, $s > 0$, as in Section 4.1, see (4.1.2). More explicitly, in local coordinates $x = (x^1, \dots, x^n)$ in Ω, one has $\mathrm{d}\mathcal{M} = \sqrt{g}\,\mathrm{d}x$, where $g = 1/\det[g^{ij}]$. Then a direct calculation of the Euler–Lagrange equation yields

$$\frac{1}{\sqrt{g(x)}}\partial_{x_i}\big(\sqrt{g(x)}g^{ij}(x)A(|\nabla u|_g)\partial_{x_j}u\big) - f(u) = 0, \tag{5.6.2}$$

that is, exactly (5.6.1). When $A \equiv 1$ the differential operator in (5.6.2) reduces just to the manifold Laplacian, see [100], page 232.

A specific example is given by the variational integral

$$\int_\Omega \left\{\frac{1}{p}|\nabla u|_g^p + F(u)\right\}\mathrm{d}\mathcal{M}, \quad p > 1, \text{ where } \mathrm{d}\mathcal{M} = \sqrt{g}\,\mathrm{d}x \text{ on } \Omega,$$

introduced by Mossino ([58], page 40), though without the volume factor $\sqrt{g}$. Here of course $A(s) = s^{p-2}$, $p > 1$. Other examples are given also in [4,63,67].

Obviously (5.6.2) is the special case of

$$\partial_{x_i}\{a_{ij}(x,u)A(|Du|_g)\partial_{x_j}u\} - B(x,u,Du) \leqslant 0, \tag{5.6.3}$$

where $|Du|_g = \sqrt{g^{ij}(x,u)\partial_{x_i}u\partial_{x_j}u}$ is a gradient norm of Riemannian type and

$$a_{ij}(x,u) = \sqrt{g(x)}\,g^{ij}(x), \qquad B(x,u,\xi) = \sqrt{g(x)}\,f(u).$$

With this motivation in hand in Section 9 of [71] we established a strong maximum principle for (5.6.3), but with a somewhat difficult proof. A strong maximum principle for the Riemannian equation (5.6.1)–(5.6.2), or for the corresponding inequality, can be treated more simply and under slightly lighter hypotheses. The result is as follows.

THEOREM 5.6.1. *Let conditions* (A1), (A2), (F1) *and* (F2) *hold, as in Section* 4.1. *Assume that the Riemannian manifold $\mathcal{M}$ is of class C^3. Then the strong maximum principle is valid for the inequality*

$$\operatorname{div}_g\big\{A\big(|\nabla u|_g\big)\nabla u\big\} - f(u) \leqslant 0 \quad \text{in } \Omega, \tag{5.6.4}$$

provided that $f(z) \equiv 0$ for $z \in [0, d]$, $d > 0$, or $f(z) > 0$ for $z \in (0, \delta)$ and (4.1.4) *is satisfied.*

PROOF. In essence, we follow the proof of Theorem 4.1.1 in Section 4.5, but in the Hopf construction we replace the ball B_R tangent to the support of u by a small *geodesic ball* $\{x \in \Omega\colon\ s(x) \leqslant S\}$ centered at y and tangent to the singular set where $u = 0$, $Du = \mathbf{0}$; here $s(x)$ denotes the geodesic distance (with respect to the metric induced by the matrix $[g^{ij}]$) from the given center y to nearby points $x \in \Omega$. The existence of such a tangent ball can be shown exactly as in Hopf's original proof, at least provided that $|Ds|$ is equally bounded above and bounded away from zero.

To show this fact, we observe by Gauss' lemma (see [100], page 235) that

$$\big|Ds(x)\big|_g^2 = g^{ij}(x)\partial_{x_i}s(x)\partial_{x_j}s(x) = 1, \quad x \neq x_0. \tag{5.6.5}$$

Thus, letting θ^2 and Θ^2 be the least and greatest eigenvalues of $[g^{ij}]$, we get

$$\Theta^{-1} \leqslant |Ds| \leqslant \theta^{-1},$$

as required.

Consider the geodesic annular set $G_S = \{x \in \Omega\colon\ S/2 \leqslant s(x) \leqslant S\}$ and let v the unique solution of (4.3.1) given by Lemma 4.3.1, in k-dimensional space, where $R = S$ and the constant k will be determined later. In view of (4.1.4) of course $|Dv| > 0$ and so $|Dv|_g = |Dv|/|Ds| \geqslant \theta|Dv| > 0$.

Also by restricting the boundary value $v = m$ at $\partial B_{R/2}$ to be sufficiently small, one can maintain $\sup|Dv|_g \leqslant \Theta|Dv| \leqslant 1$.

The principal calculation, for $x \in G_S$, is the following:

$$\begin{aligned}
&\frac{1}{\sqrt{g(x)}}\partial_{x_i}\big\{\sqrt{g(x)}\,g^{ij}(x)A\big(|Dv|_g\big)\partial_{x_j}v\big\} - f(v)\\
&\quad = -\frac{1}{\sqrt{g(x)}}\partial_{x_i}\big\{\sqrt{g(x)}\,g^{ij}(x)\partial_{x_j}sA(w')w'\big\} - f(w)\\
&\quad = \big[\Phi(w')\big]' - \Delta s\Phi(w') - f(w) \geqslant \big[\Phi(w')\big]' - \frac{k}{s}\Phi(w') - f(w),
\end{aligned}$$

where k is an appropriate constant. The remaining part of the proof involves application of the comparison Theorem 2.4.1. To this end, we have to check (2.4.3) when $\hat{\boldsymbol{A}}(x,\boldsymbol{\xi}) = \sqrt{g(x)}\, g^{ij}(x) A(|\boldsymbol{\xi}|_g)\boldsymbol{\xi}$, that is, in Riemannian notation,

$$\begin{aligned}&\sqrt{g(x)}\big\langle A\big(|\boldsymbol{\eta}|_g\big)\boldsymbol{\eta} - A\big(|\boldsymbol{\xi}|_g\big)\boldsymbol{\xi}, \boldsymbol{\eta}-\boldsymbol{\xi}\big\rangle_{\mathcal{M}}\\ &\quad\geqslant \sqrt{g(x)}\,\big(\Phi\big(|\boldsymbol{\eta}|_g\big) - \Phi\big(|\boldsymbol{\xi}|_g\big)\big)\cdot\big(|\boldsymbol{\eta}|_g - |\boldsymbol{\xi}|_g\big)\end{aligned}$$

since $\langle \boldsymbol{\xi}, \boldsymbol{\eta}\rangle_{\mathcal{M}} \leqslant |\boldsymbol{\xi}|_g |\boldsymbol{\eta}|_g$, and (2.4.3) now follows because Φ is strictly increasing by (A2). □

The strong maximum principle Theorem 5.6.1 was given in [71]. For the corresponding necessity of the conditions in Theorem 5.6.1 we refer to [67].

Acknowledgement

The first author was supported by the Italian MIUR project titled *"Metodi Variazionali ed Equazioni Differenziali non Lineari"*.

References

[1] A. Aftalion and J. Busca, *Radial symmetry of overdetermined boundary value problems in exterior domains*, Archive Ration. Mech. Anal. **143** (1998), 195–206.
[2] A.D. Alexandroff, *A characterization property of the spheres*, Ann. Mat. Pura Appl. **58** (1962), 303–354.
[3] F.J. Almgren, *A maximum principle for elliptic variational problems*, J. Funct. Anal. **4** (1969), 380–390.
[4] P. Antonini, M. Mugnai and P. Pucci, *Singular elliptic inequalities on complete manifolds* (2006), submitted for publication.
[5] C. Bandle and I. Stakgold, *The formation of the dead core in parabolic reaction–diffusion problems*, Trans. Amer. Math. Soc. **286** (1984), 275–293.
[6] C. Bandle, R.P. Sperb and I. Stakgold, *Diffusion and reaction with monotone kinetics*, Nonlinear Anal. TMA **8** (1984), 321–333.
[7] M. Bardi and F. Da Lio, *On the strong maximum principle for fully nonlinear degenerate elliptic equations*, Arch. Math. **73** (2000), 276–285.
[8] G. Barles, G. Diaz and J.I. Diaz, *Uniqueness and continuum of foliated solutions for a quasilinear elliptic equation with a nonlipschitz nonlinearity*, Comm. Partial Differential Equations **17** (1992), 1037–1050.
[9] V. Benci, D. Fortunato and L. Pisani, *Solitons like solutions of a Lorentz invariant equation in dimension 3*, Rev. Math. Phys. **10** (1998), 315–344.
[10] P. Benilan, H. Brezis and M. Crandall, *A semilinear equation in $L^1(\mathbb{R}^n)$*, Ann. Scuola Norm. Sup. Pisa **4** (1975), 523–555.
[11] H. Berestycki and L. Nirenberg, *On the method of moving planes and the sliding methods*, Bol. Soc. Brasil. Mat. **22** (1991), 1–37.
[12] M.-F. Bidaut-Veron, *Local and global behavior of solutions of quasilinear equations of Emden–Fowler type*, Arch. Ration. Mech. Anal. **107** (1989), 293–324.
[13] M.-F. Bidaut-Veron and S.I. Pohozaev, *Nonexistence results and estimates for some nonlinear elliptic problems*, J. Anal. Math. **84** (2001), 1–49.
[14] I. Birindelli and F. Demengel, *First eigenvalue and maximum principle for fully nonlinear singular operators*, Adv. Differential Equations **11** (2006), 91–119.
[15] H. Brezis, *Symmetry in nonlinear PDE's*, Proc. Sympos Pure Math., Amer. Math. Soc., vol. 65 (1999), 1–12.

[16] L.A. Caffarelli and X. Cabré, *Fully Nonlinear Elliptic Equations*, Amer. Math. Soc. Colloq. Publ., vol. 43 (1995).
[17] A. Castro and R. Shivaji, *Nonnegative solutions to a semilinear Dirichlet problem in a ball are positive and radially symmetric*, Comm. Partial Differential Equations **14** (1989), 1091–1100.
[18] A. Cellina, *On the strong maximum principle*, Proc. Amer. Math. Soc. **130** (2001), 413–418.
[19] C.H. Conley, P. Pucci and J. Serrin, *Elliptic equations and products of positive definite matrices*, Math. Nachr. **278** (2005), 1490–1508.
[20] C. Cortázar, M. Elgueta and P. Felmer, *On a semilinear elliptic problem in $\mathbb{R}^n$ with a non-Lipschitzian nonlinearity*, Adv. Differential Equations **1** (1996), 199–218.
[21] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, I, II, Springer-Verlag (1924), (1937), and *Methods of Mathematical Physik*, vols. 1, 2, Wiley-Interscience, New York (1962).
[22] M. Cuesta and P. Takáč, *A strong comparison principle for positive solutions of degenerate elliptic equations*, Differential Integral Equations **13** (2000), 721–746.
[23] L. Damascelli and F. Pacella, *Monotonicity and symmetry of solutions of p-Laplace equations, via the moving plane method*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **26** (1998), 689–707.
[24] L. Damascelli and B. Sciunzi, *Regularity monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Differential Equations **206** (2004), 483–515.
[25] E.N. Dancer, *Some notes on the method of moving plane*, Bull. Australian Math. Soc. **46** (1992), 425–434.
[26] J.I. Diaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Pitman Research Notes in Mathematics, vol. 106 (1985).
[27] J.I. Diaz, *Some properties of solutions of degenerate second order PDE in non divergence form*, Appl. Anal. **20** (1985), 309–336.
[28] J.I. Diaz and M.A. Herrero, *Estimates on the support of the solutions of some nonlinear elliptic and a parabolic problems*, Proc. Roy. Soc. Edinburgh Sect. A **89** (1981), 249–258.
[29] J.I. Diaz, J.E. Saa and U. Thiel, *Sobre la ecuación de curvatura media prescrita y otras ecuaciones cuasilineales elípticas con soluciones anulándose localmente*, Rev. Un. Mat. Argentina **35** (1990), 175–206.
[30] J.I. Diaz and L. Veron, *Local vanishing properties of solutions of elliptic and parabolic quasilinear equations*, Trans. Amer. Math. Soc. **290** (1985), 787–814.
[31] J. Dolbeault, P. Felmer and R. Monneau, *Symmetry and nonuniformly elliptic operators*, Differential Integral Equations **18** (2005), 141–154.
[32] L.C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, vol. 19, Amer. Math. Soc. (1998).
[33] P. Felmer and A. Quaas, *On the strong maximum principle for quasilinear elliptic equations and systems*, Adv. Differential Equations **7** (2002), 25–46.
[34] W.H. Fleming and R. Rishel, *An integral formula for total variation*, Arch. Math. **11** (1960), 218–222.
[35] L.E. Fraenkel, *An Introduction to Maximum Principles and Symmetry in Elliptic Problems*, Cambridge Tracts in Mathematics, Cambridge Univ. Press (2000).
[36] B. Franchi, E. Lanconelli and J. Serrin, *Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^n$*, Adv. Math. **118** (1996), 177–243.
[37] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.
[38] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525–598.
[39] D. Gilbarg, *Some hydrodynamic applications of function theoretic properties of elliptic equations*, Math. Z. **72** (1959), 165–174.
[40] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second ed., Springer-Verlag (1983).
[41] S. Granlund, *Strong maximum principle for a quasi-linear equation with applications*, Ann. Acad. Sci. Fenn. Ser. A, I. Mathematica Dissertationes **21** (1978).
[42] E. Hopf, *Elementäre Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitzungsberichte Preussiche Akademie Wissenschaften, Berlin (1927), 147–152.
[43] E. Hopf, *A remark on linear elliptic differential equations of the second order*, Proc. Amer. Math. Soc. **3** (1952), 791–793.

[44] B. Kawohl and N. Kutev, *Strong maximum principle for semicontinuous viscosity solutions to nonlinear partial differential equations*, Arch. Math. **70** (1998), 470–478.
[45] J. Keller, *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), 503–510.
[46] N.V. Krylov, *Nonlinear Elliptic and Parabolic Equations of Second Order*, Reidel, Dordrecht, The Netherlands (1987).
[47] N.V. Krylov, *On the general notion of fully nonlinear second order elliptic equations*, Trans. Amer. Math. Soc. **347** (1995), 857–895.
[48] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Quasilinear elliptic equations and variational problems with several independent variables*, Uspekhi Mat. Nauk **16** (1961), 19–90; English translation in Russian Math. Surveys **16** (1961), 17–92.
[49] G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), 311–361.
[50] W. Littman, *A strong maximum principle for weakly L-subharmonic functions*, J. Math. Mech. **8** (1959), 761–770.
[51] V.G. Maz'ya, *Some estimates for the solutions of second order elliptic equations*, Dokl. Akad. Nauk SSSR **137** (1961), 299–302; English translation in Soviet Math. Dokl. **2** (1961), 413–415.
[52] A. McNabb, *Strong comparison theorems for elliptic equations of second order*, J. Math. Mech. **10** (1961), 431–440.
[53] N.G. Meyers, *An example of non-uniqueness in the theory of quasilinear elliptic equations of second order*, Arch. Ration. Mech. Anal. **14** (1963), 177–179.
[54] E. Mitidieri and S.I. Pohozaev, *The absence of global positive solutions to quasilinear elliptic inequalities*, Dokl. Akad. Nauk **359** (1998), 456–460; English translation in Dokl. Math. **57** (1998), 250–253.
[55] D.D. Monticelli, *Maximum principles and applications for a class of degenerate elliptic linear operators*, PhD Thesis at the University of Milan, Department of Mathematics "F. Enriques", a.y. (2005–2006).
[56] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468.
[57] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.
[58] J. Mossino, *Inégalités isopeérimétriques et applications en physique*, Travaux en Cours, Hermann, Paris (1984).
[59] J.C.C. Nitsche, *Elementary proof of Bernstein's theorem on minimal surfaces*, Ann. Math. **66** (1957), 543–544.
[60] O.A. Oleinik, *On properties of some boundary problems for equations of elliptic type*, Math. Sb. (NS) **30** (72) (1952), 695–702.
[61] R. Osserman, *On the inequality* $\Delta u \geqslant f(u)$, Pacific J. Math. **7** (1957), 1641–1647.
[62] L.A. Peletier and J. Serrin, *Gradient bounds and Liouville theorems for quasilinear elliptic equations*, Ann. Scuola Norm Sup. Pisa Cl. Sci. **5** (1978), 65–104.
[63] S. Pigola, M. Rigoli and A.G. Setti, *Maximum principles and singular elliptic inequalities*, J. Funct. Anal. **193** (2002), 224–260.
[64] P. Polacik, P. Quittner and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems* (2006), in manuscript.
[65] A. Porretta and L. Véron, *Symmetry properties of solutions of semilinear elliptic equations in the plane* (2006), in manuscript.
[66] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Englewood Cliffs, NJ, Prentice-Hall (1967).
[67] P. Pucci, M. Rigoli and J. Serrin, *Qualitative properties for solutions of singular elliptic inequalities on complete manifolds*, J. Differential Equations (2007), to appear.
[68] P. Pucci, B. Sciunzi and J. Serrin, *Partial and full symmetry of solutions of quasilinear elliptic equations, via the Comparison Principle*, Contemp. Math. (2006), special volume dedicated to H. Brézis, to appear.
[69] P. Pucci and J. Serrin, *A note on the strong maximum principle for elliptic differential inequalities*, J. Math. Pures Appl. **79** (2000), 57–71.
[70] P. Pucci and J. Serrin, *The Harnack inequality in* $\mathbb{R}^2$ *for quasilinear elliptic equations*, J. Anal. Math. **85** (2001), 307–321.

[71] P. Pucci and J. Serrin, *The Strong Maximum Principle revisited*, J. Differential Equations **196** (2004), 1–66; Erratum, J. Differential Equations **207** (2004), 226–227.
[72] P. Pucci and J. Serrin, *Dead cores and bursts for quasilinear singular elliptic equations*, SIAM J. Math. Anal. **38** (2006), 259–278.
[73] P. Pucci and J. Serrin, *The Maximum Principle*, monograph in preparation.
[74] P. Pucci, J. Serrin and H. Zou, *A strong maximum principle and a compact support principle for singular elliptic inequalities*, J. Math. Pures Appl. **78** (1999), 769–789.
[75] R.M. Redheffer, *On the inequality* $\Delta u \geqslant f(u, |\operatorname{grad} u|)$, J. Math. Anal. Appl. **1** (1960), 277–299.
[76] R.M. Redheffer, *Nonlinear differential inequalities and functions of compact support*, Trans. Amer. Math. Soc. **220** (1976), 133–157.
[77] W. Reichel, *Radial symmetry for elliptic boundary value problems on exterior domains*, Arch. Ration. Mech. Anal. **137** (1997), 381–394.
[78] F. Rellich, *Zur ersten Randwertaufgabe bei Monge–Ampèreschen Differentialgleichungen vom elliptischen Typus; differentialgeometrische Anwendungen*, Math. Ann. **107** (1932), 505–513.
[79] M.V. Safonov, *Harnack's inequality for elliptic equations and Hőlder property of their solutions*, Zap. Nauchn. Sem. Leningrad Otdel Mat. Inst. Steklov **96** (1980), 272–287 (in Russian); J. Soviet Math. **21** (1983), 851–863.
[80] J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Anal. Math. **4** (1954–56), 292–308.
[81] J. Serrin, *Local behavior of solutions of quasilinear elliptic equations*, Acta Math. **111** (1964), 247–302.
[82] J. Serrin, *On the strong maximum principle for quasilinear second order differential inequalities*, J. Funct. Anal. **5** (1970), 184-193.
[83] J. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal. **43** (1971), 304–318.
[84] J. Serrin, *Entire solutions of nonlinear Poisson equations*, Proc. London Math. Soc. **24** (1972), 348–366.
[85] J. Serrin, *Commentary on the Hopf strong maximum principle*, Selected Works of Eberhard Hopf with Commentaries, C.S. Morawetz, J.B. Serrin and Y.G. Sinai, eds., Amer. Math. Soc., Providence, RI, (2002).
[86] J. Serrin and H. Zou, *Symmetry od ground states of quasilinear elliptic equations*, Arch. Ration. Mech. Anal. **148** (1999), 265–290.
[87] J. Serrin and H. Zou, *Cauchy–Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities*, Acta Math. **189** (2002), 79–142.
[88] D. Siegel, *Height estimates for capillary surfaces*, Pacific J. Math. **88** (1980), 471–515.
[89] J. Simon, *Minimal varieties in Riemannian manifolds*, Ann. Math. **88** (1968), 62–105.
[90] R. Sperb, *Maximum Principles and their Applications*, Academic Press, New York (1981).
[91] R. Sperb, *Some complementary estimates in the dead core problem*, Nonlinear Problems in Applied Mathematics. In honor of Ivar Stakgold on his 70th birthday, T.S. Angell et al., eds., Philadelphia (1996), 217–224.
[92] G. Stampacchia, *Problemi al contorno ellittici, con dati discontinui dotati di soluzioni Hőlderiane*, Ann. Mat. Pura Appl. **51** (1960), 1–38.
[93] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. **110** (1976), 353–372.
[94] P. Tolksdorf, *On the Dirichlet problem for quasilinear elliptic equations in domains with conical boundary points*, Comm. Partial Differential Equations **8** (1983), 773–817.
[95] N.S. Trudinger, *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.
[96] N.S. Trudinger, *Harnack inequalities for nonuniformly elliptic divergence structure equations*, Invent. Math. **64** (1981), 517–531.
[97] J.-L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.
[98] W. Walter, *Differential and Integral Inequalities*, Springer-Verlag, Berlin (1964) (in German), 1970 (in English).
[99] K. Widman, *A quantitative form of the maximum principle for elliptic partial differential equations with coefficients in* L^∞, Comm. Pure Appl. Math. **21** (1968), 507–513.
[100] T.J. Willmore, *An Introduction to Differential Geometry*, Oxford University Press, Oxford (1993).

This page intentionally left blank

CHAPTER 7

Singular Phenomena in Nonlinear Elliptic Problems
From Blow-Up Boundary Solutions to Equations with Singular Nonlinearities

Vicenţiu D. Rădulescu[1]

Department of Mathematics, University of Craiova, 200585 Craiova, Romania
and
Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania
E-mail: vicentiu.radulescu@math.cnrs.fr, url: http://inf.ucv.ro/∼radulescu

Contents

1. Motivation and previous results 487
2. Large solutions of elliptic equations with absorption and subquadratic convection term 489
3. Singular solutions with lack of the Keller–Osserman condition 495
4. Blow-up boundary solutions of the logistic equation 503
4.1. Uniqueness and asymptotic behavior of the large solution. A Karamata regular variation theory approach 515
5. Entire solutions blowing up at infinity of semilinear elliptic systems 527
6. Bifurcation problems for singular Lane–Emden–Fowler equations 541
7. Sublinear singular elliptic problems with two bifurcation parameters 552
8. Bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with a convection term . . . 564
References 590

Abstract

In this survey we report on some recent results related to various singular phenomena arising in the study of some classes of nonlinear elliptic equations. We establish qualitative results on the existence, nonexistence or the uniqueness of solutions and we focus on the following types of problems: (i) blow-up boundary solutions of logistic equations; (ii) Lane–Emden–Fowler equations with singular nonlinearities and subquadratic convection term. We study the combined effects of various terms involved in these problems: sublinear or superlinear nonlinearities, singular nonlinear terms, convection nonlinearities, as well as sign-changing potentials. We also take into account bifurcation nonlinear problems and we establish the precise

[1]The author is partially supported by Grants CEEX 05-D11-36 and 2-CEEX 06-11-18/2006.

HANDBOOK OF DIFFERENTIAL EQUATIONS
Stationary Partial Differential Equations, volume 4
Edited by M. Chipot
© 2007 Elsevier B.V. All rights reserved

rate decay of the solution in some concrete situations. Our approach combines standard techniques based on the maximum principle with nonstandard arguments, such as the Karamata regular variation theory.

Keywords: Nonlinear elliptic equation, Singularity, Boundary blow-up, Bifurcation, Asymptotic analysis, Maximum principle, Karamata regular variation theory

MSC: primary 35-02; secondary 35A20, 35B32, 35B40, 35B50, 35J60, 47J10, 58J55

1. Motivation and previous results

Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geqslant 2$. We are concerned in this paper with the following types of stationary singular problems:

I. *The logistic equation*

$$\begin{cases} \Delta u = \Phi(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

II. *The Lane–Emden–Fowler equation*

$$\begin{cases} -\Delta u = \Psi(x, u, \nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Φ is a smooth nonlinear function, while Ψ has one or more singularities. The solutions of (1.1) are called *large* (or *blow-up*) *solutions*.

In this work we focus on problems (1.1) and (1.2) and we establish several recent contributions in the study of these equations. In order to illustrate the link between these problems, consider the most natural case where $\Phi(u, \nabla u) = u^p$, where $p > 1$. Then the function $v = u^{-1}$ satisfies (1.2) for $\Psi(u, \nabla v) = v^{2-p} - 2v^{-1}|\nabla v|^2$.

The study of large solutions has been initiated in 1916 by Bieberbach [12] for the particular case $\Phi(x, u, \nabla u) = \exp(u)$ and $N = 2$. He showed that there exists a unique solution of (1.1) such that $u(x) - \log(d(x)^{-2})$ is bounded as $x \to \partial\Omega$, where $d(x) := \operatorname{dist}(x, \partial\Omega)$. Problems of this type arise in Riemannian geometry: if a Riemannian metric of the form $|\mathrm{d}s|^2 = \exp(2u(x))|\mathrm{d}x|^2$ has constant Gaussian curvature $-c^2$ then $\Delta u = c^2 \exp(2u)$. Motivated by a problem in mathematical physics, Rademacher [82] continued the study of Bieberbach on smooth bounded domains in $\mathbb{R}^3$. Lazer and McKenna [68] extended the results of Bieberbach and Rademacher for bounded domains in $\mathbb{R}^N$ satisfying a uniform external sphere condition and for nonlinearities $\Phi(x, u, \nabla u) = b(x)\exp(u)$, where b is continuous and strictly positive on $\overline{\Omega}$. Let $\Phi(x, u, \nabla u) = f(u)$ where $f \in C^1[0, \infty)$, $f'(s) \geqslant 0$ for $s \geqslant 0$, $f(0) = 0$ and $f(s) > 0$ for $s > 0$. In this case, Keller [62] and Osserman [78] proved that large solutions of (1.1) exist if and only if

$$\int_1^\infty \frac{\mathrm{d}t}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,\mathrm{d}s.$$

In a celebrated paper, Loewner and Nirenberg [72] linked the uniqueness of the blow-up solution to the growth rate at the boundary. Motivated by certain geometric problems, they established the uniqueness for the case $f(u) = u^{(N+2)/(N-2)}$, $N > 2$. Bandle and Marcus [8] give results on asymptotic behavior and uniqueness of the large solution for more general nonlinearities including $f(u) = u^p$ for any $p > 1$. We refer to Bandle [5], Bandle and M. Essèn [6], Bandle and Marcus [9], Du and Huang [39], García-Melián, Letelier-Albornoz, and Sabina de Lis [43], Lazer and McKenna [69], Le Gall [70], Marcus and

Véron [74,75], Ratto, Rigoli and Véron [83] and the references therein for several results on large solutions extended to N-dimensional domains and for other classes of nonlinearities.

Singular problems like (1.2) have been intensively studied in the last decades. Stationary problems involving singular nonlinearities, as well as the associated evolution equations, describe naturally several physical phenomena. At our best knowledge, the first study in this direction is due to Fulks and Maybee [41], who proved existence and uniqueness results by using a fixed point argument; moreover, they showed that solutions of the associated parabolic problem tend to the unique solution of the corresponding elliptic equation. A different approach (see Coclite and Palmieri [33], Crandall, Rabinowitz and Tartar [34], Stuart [88]) consists in approximating the singular equation with a regular problem, where the standard techniques (e.g., monotonicity methods) can be applied and then passing to the limit to obtain the solution of the original equation. Nonlinear singular boundary value problems arise in the context of chemical heterogeneous catalysts and chemical catalyst kinetics, in the theory of heat conduction in electrically conducting materials, singular minimal surfaces, as well as in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids (we refer for more details to Caffarelli, Hardt, and L. Simon [16], Callegari and Nachman [17,18], Chipot [20], Díaz [37], Díaz, Morel and Oswald [38] and the more recent papers by Haitao [57], Hernández, Mancebo and Vega [58,59], Meadows [76], Shi and Yao [86,87], Stuart and Zhou [89]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases. For instance, problems of this type characterize some reaction–diffusion processes where $u \geqslant 0$ is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, where no reaction takes place (see Aris [4] for the study of a single, irreversible steady-state reaction). Nonlinear singular elliptic equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents (see Callegari and Nachman [18] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence).

In [34], Crandall, Rabinowitz and Tartar established that the boundary value problem

$$\begin{cases} -\Delta u - u^{-\alpha} = -u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$. The importance of the linear and nonlinear terms is crucial for the existence of solutions. For instance, Coclite and Palmieri studied in [33] the problem

$$\begin{cases} -\Delta u - u^{-\alpha} = \lambda u^{p} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\lambda \geqslant 0$ and $\alpha, p \in (0, 1)$. In [33] it is proved that problem (1.3) has at least one solution for all $\lambda \geqslant 0$ and $0 < p < 1$. Moreover, if $p \geqslant 1$, then there exists λ^* such that

problem (1.3) has a solution for $\lambda \in [0, \lambda^*)$ and no solution for $\lambda > \lambda^*$. In [33] it is also proved a related nonexistence result. More exactly, the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution, provided that $0 < \alpha < 1$ and $\lambda_1 \geqslant 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

Problems related to multiplicity and uniqueness become difficult even in simple cases. Shi and Yao studied in [86] the existence of radial symmetric solutions of the problem

$$\begin{cases} \Delta u + \lambda(u^p - u^{-\alpha}) = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $\alpha > 0$, $0 < p < 1$, $\lambda > 0$, and B_1 is the unit ball in $\mathbb{R}^N$. Using a bifurcation theorem of Crandall and Rabinowitz, it has been shown in [86] that there exists $\lambda_1 > \lambda_0 > 0$ such that the above problem has no solutions for $\lambda < \lambda_0$, exactly one solution for $\lambda = \lambda_0$ or $\lambda > \lambda_1$, and two solutions for $\lambda_0 < \lambda \leqslant \lambda_1$.

A crucial role in our arguments in this work is played by the Maximum Principle. We refer to Pucci and and Serrin [79,80] for recent advances and applications to PDE's.

The author's interest for the study of singular problems is motivated by several stimulating discussions with Professor Haim Brezis in Spring 2001. I would like to use this opportunity to thank once again Professor Brezis for his constant scientific support during the years.

This work is organized as follows. Sections 2–5 are mainly devoted to the study of blow-up boundary solutions of logistic type equations with absorption. In the second part of this work (Sections 6–8), in connection with the previous results, we are concerned with the study of the Dirichlet boundary value problem for the singular Lane–Emden–Fowler equation. Our framework includes the presence of a convection term.

2. Large solutions of elliptic equations with absorption and subquadratic convection term

Consider the problem

$$\begin{cases} \Delta u + q(x)|\nabla u|^a = p(x) f(u) & \text{in } \Omega, \\ u \geqslant 0, \quad u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{2.4}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is a smooth domain (bounded or possibly unbounded) with compact (possibly empty) boundary. We assume that $a \leqslant 2$ is a positive real number, p, q are nonnegative function such that $p \not\equiv 0$, $p, q \in C^{0,\alpha}(\overline{\Omega})$ if Ω is bounded, and

$p, q \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, otherwise. Throughout this section we assume that the nonlinearity f fulfills the following conditions

(f1) $f \in C^1[0,\infty)$, $f' \geqslant 0$, $f(0) = 0$ and $f > 0$ on $(0,\infty)$.

(f2) $\int_1^\infty [F(t)]^{-1/2}\,\mathrm{d}t < \infty$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$.

(f3) $F(t)/f^{2/a}(t) \to 0$ as $t \to \infty$.

Cf. Véron [90], f is called an absorption term. The above conditions hold provided that $f(t) = t^k$, $k > 1$, and $0 < a < 2r/(r+1)(< 2)$, or $f(t) = \mathrm{e}^t - 1$, or $f(t) = \mathrm{e}^t - t$ and $a < 2$. We observe that by (f1) and (f3) it follows that $f/F^{a/2} \geqslant \beta > 0$ for t large enough, that is, $(F^{1-a/2})' \geqslant \beta > 0$ for t large enough which yields $0 < a \leqslant 2$. We also deduce that conditions (f2) and (f3) imply $\int_1^\infty f^{-1/a}(t)\,\mathrm{d}t < \infty$.

We are mainly interested in finding properties of *large* (*explosive*) *solutions* of (2.4), that is solutions u satisfying $u(x) \to \infty$ as $\mathrm{dist}(x, \partial\Omega) \to 0$ (if $\Omega \not\equiv \mathbb{R}^N$), or $u(x) \to \infty$ as $|x| \to \infty$ (if $\Omega = \mathbb{R}^N$). In the latter case the solution is called an *entire large* (*explosive*) solution.

Problems of this type appear in stochastic control theory and have been first study by Lasry and Lions [66]. The corresponding parabolic equation was considered in Quittner [81] and in Galaktionov and Vázquez [42]. In terms of the dynamic programming approach, an explosive solution of (2.4) corresponds to a value function (or Bellman function) associated to an infinite exit cost (see Lasry and Lions [66]).

Bandle and Giarrusso [7] studied the existence of a large solution of problem (2.4) in the case $p \equiv 1$, $q \equiv 1$ and Ω bounded. Lair and Wood [65] studied the sublinear case corresponding to $p \equiv 1$, while Cîrstea and Rădulescu [24] proved the existence of large solutions to (2.4) in the case $q \equiv 0$.

As observed by Bandle and Giarrusso [7], the simplest case is $a = 2$, which can be reduced to a problem without gradient term. Indeed, if u is a solution of (2.4) for $q \equiv 1$, then the function $v = \mathrm{e}^u$ (Gelfand transformation) satisfies

$$\begin{cases} \Delta v = p(x) v f(\ln v) & \text{in } \Omega, \\ v(x) \to +\infty & \text{if } \mathrm{dist}(x, \partial\Omega) \to 0. \end{cases}$$

We shall therefore mainly consider the case where $0 < a < 2$.

The main results in this Section are due to Ghergu, Niculescu and Rădulescu [44]. These results generalize those obtained by Cîrstea and Rădulescu [24] in the case of the presence of a convection (gradient) term.

Our first result concerns the existence of a large solution to problem (2.4) when Ω is bounded.

THEOREM 2.1. *Suppose that Ω is bounded and assume that p satisfies*

(p1) *for every $x_0 \in \Omega$ with $p(x_0) = 0$, there exists a domain $\Omega_0 \ni x_0$ such that $\overline{\Omega_0} \subset \Omega$ and $p > 0$ on $\partial\Omega_0$.*

Then problem (2.4) *has a positive large solution.*

A crucial role in the proof of the above result is played by the following auxiliary result (see Ghergu, Niculescu and Rădulescu [44]).

LEMMA 2.2. *Let Ω be a bounded domain. Assume that $p, q \in C^{0,\alpha}(\overline{\Omega})$ are nonnegative functions, $0 < a < 2$ is a real number, f satisfies* (f1) *and $g : \partial\Omega \to (0, \infty)$ is continuous. Then the boundary value problem*

$$\begin{cases} \Delta u + q(x)|\nabla u|^a = p(x) f(u) & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \\ u \geqslant 0, \quad u \not\equiv 0, & \text{in } \Omega \end{cases} \tag{2.5}$$

has a classical solution. If p is positive, then the solution is unique.

SKETCH OF THE PROOF OF THEOREM 2.1. By Lemma 2.2, the boundary value problem

$$\begin{cases} \Delta v_n + q(x)|\nabla v_n|^a = (p(x) + \frac{1}{n}) f(v_n) & \text{in } \Omega, \\ v_n = n & \text{on } \partial\Omega, \\ v_n \geqslant 0, \quad v_n \not\equiv 0 & \text{in } \Omega \end{cases}$$

has a unique positive solution, for any $n \geqslant 1$. Next, by the maximum principle, the sequence (v_n) is nondecreasing and is bounded from below in Ω by a positive function.

To conclude the proof, it is sufficient to show that

(a) for all $x_0 \in \Omega$ there exists an open set $\mathcal{O} \Subset \Omega$ which contains x_0 and $M_0 = M_0(x_0) > 0$ such that $v_n \leqslant M_0$ in $\mathcal{O}$ for all $n \geqslant 1$;

(b) $\lim_{x \to \partial\Omega} v(x) = \infty$, where $v(x) = \lim_{n\to\infty} v_n(x)$.

We observe that the statement (a) shows that the sequence (v_n) is uniformly bounded on every compact subset of Ω. Standard elliptic regularity arguments (see Gilbarg and Trudinger [54]) show that v is a solution of problem (2.4). Then, by (b), it follows that v is a large solution of problem (2.4).

To prove (a) we distinguish two cases:

Case $p(x_0) > 0$. By the continuity of p, there exists a ball $B = B(x_0, r) \Subset \Omega$ such that

$$m_0 := \min\{p(x);\ x \in \overline{B}\} > 0.$$

Let w be a positive solution of the problem

$$\begin{cases} \Delta w + q(x)|\nabla w|^a = m_0 f(w) & \text{in } B, \\ w(x) \to \infty & \text{as } x \to \partial B. \end{cases}$$

The existence of w follows by considering the problem

$$\begin{cases} \Delta w_n + q(x)|\nabla w_n|^a = m_0 f(w_n) & \text{in } B, \\ w_n = n & \text{on } \partial B. \end{cases}$$

The maximum principle implies $w_n \leqslant w_{n+1} \leqslant \theta$, where

$$\begin{cases} \Delta\theta + \|q\|_{L^\infty}|\nabla\theta|^a = m_0 f(\theta) & \text{in } B, \\ \theta(x) \to \infty & \text{as } x \to \partial B. \end{cases}$$

Standard arguments show that $v_n \leqslant w$ in B. Furthermore, w is bounded in $\overline{B(x_0, r/2)}$. Setting $M_0 = \sup_{\mathcal{O}} w$, where $\mathcal{O} = B(x_0, r/2)$, we obtain (a).

Case $p(x_0) = 0$. Our hypothesis (p1) and the boundedness of Ω imply the existence of a domain $\mathcal{O} \Subset \Omega$ which contains x_0 such that $p > 0$ on $\partial\mathcal{O}$. The above case shows that for any $x \in \partial\mathcal{O}$ there exist a ball $B(x, r_x)$ strictly contained in Ω and a constant $M_x > 0$ such that $v_n \leqslant M_x$ on $B(x, r_x/2)$, for any $n \geqslant 1$. Since $\partial\mathcal{O}$ is compact, it follows that it may be covered by a finite number of such balls, say $B(x_i, r_{x_i}/2)$, $i = 1, \ldots, k_0$. Setting $M_0 = \max\{M_{x_1}, \ldots, M_{x_{k_0}}\}$ we have $v_n \leqslant M_0$ on $\partial\mathcal{O}$, for any $n \geqslant 1$. Applying the maximum principle we obtain $v_n \leqslant M_0$ in $\mathcal{O}$ and (a) follows.

Let z be the unique function satisfying $-\Delta z = p(x)$ in Ω and $z = 0$, on $\partial\Omega$. Moreover, by the maximum principle, we have $z > 0$ in Ω. We first observe that for proving (b) it is sufficient to show that

$$\int_{v(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \leqslant z(x), \quad \text{for any } x \in \Omega. \tag{2.6}$$

By [24, Lemma 1], the left-hand side of (2.6) is well defined in Ω. We choose $R > 0$ so that $\overline{\Omega} \subset B(0, R)$ and fix $\varepsilon > 0$. Since $v_n = n$ on $\partial\Omega$, let $n_1 = n_1(\varepsilon)$ be such that

$$n_1 > \frac{1}{\varepsilon(N-3)(1+R^2)^{-1/2} + 3\varepsilon(1+R^2)^{-5/2}}, \tag{2.7}$$

and

$$\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \leqslant z(x) + \varepsilon\big(1+|x|^2\big)^{-1/2}, \quad \forall x \in \partial\Omega,\ \forall n \geqslant n_1. \tag{2.8}$$

In order to prove (2.6), it is enough to show that

$$\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} \leqslant z(x) + \varepsilon\big(1+|x|^2\big)^{-1/2}, \quad \forall x \in \Omega,\ \forall n \geqslant n_1. \tag{2.9}$$

Indeed, taking $n \to \infty$ in (2.9) we deduce (2.6), since $\varepsilon > 0$ is arbitrarily chosen. Assume now, by contradiction, that (2.9) fails. Then

$$\max_{x \in \overline{\Omega}} \left\{ \int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} - z(x) - \varepsilon\big(1+|x|^2\big)^{-1/2} \right\} > 0.$$

Using (2.8) we see that the point where the maximum is achieved must lie in Ω. A straightforward computation shows that at this point, say x_0, we have

$$0 \geqslant \Delta\left(\int_{v_n(x)}^{\infty} \frac{\mathrm{d}t}{f(t)} - z(x) - \varepsilon\big(1+|x|^2\big)^{-1/2} \right)_{|x=x_0} > 0.$$

This contradiction shows that inequality (2.8) holds and the proof of Theorem 2.1 is complete. □

Similar arguments based on the maximum principle and the approximation of large balls $B(0,n)$ imply the following existence result.

THEOREM 2.3. *Assume that $\Omega = \mathbb{R}^N$ and that problem* (2.4) *has at least one solution. Suppose that p satisfies the condition*

(p1′) *There exists a sequence of smooth bounded domains $(\Omega_n)_{n\geqslant 1}$ such that $\overline{\Omega_n} \subset \Omega_{n+1}$, $\mathbb{R}^N = \bigcup_{n=1}^{\infty} \Omega_n$, and* (p1) *holds in Ω_n, for any $n \geqslant 1$.*

Then there exists a classical solution U of (2.4) *which is a maximal solution if p is positive.*

Assume that p verifies the additional condition

(p2) $\int_0^{\infty} r\Phi(r)\,\mathrm{d}r < \infty$, *where* $\Phi(r) = \max\{p(x):\ |x| = r\}$.

Then U is an entire large solution of (2.4).

We now consider the case in which $\Omega \neq \mathbb{R}^N$ and Ω is unbounded. We say that a large solution u of (2.4) is *regular* if u tends to zero at infinity. In [73, Theorem 3.1] Marcus proved for this case (and if $q = 0$) the existence of regular large solutions to problem (2.4) by assuming that there exist $\gamma > 1$ and $\beta > 0$ such that

$$\liminf_{t\to 0} f(t)t^{-\gamma} > 0 \quad \text{and} \quad \liminf_{|x|\to\infty} p(x)|x|^{\beta} > 0.$$

The large solution constructed in Marcus [73] is the *smallest* large solution of problem (2.4). In the next result we show that problem (2.4) admits a *maximal* classical solution U and that U blows-up at infinity if $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$.

THEOREM 2.4. *Suppose that $\Omega \neq \mathbb{R}^N$ is unbounded and that problem* (2.4) *has at least a solution. Assume that p satisfies condition* (p1′) *in Ω. Then there exists a classical solution U of problem* (2.4) *which is maximal solution if p is positive.*

If $\Omega = \mathbb{R}^N \setminus \overline{B(0,R)}$ and p satisfies the additional condition (p2), *with $\Phi(r) = 0$ for $r \in [0,R]$, then the solution U of* (2.4) *is a large solution that blows-up at infinity.*

We refer to Ghergu, Niculescu and Rădulescu [44] for complete proofs of Theorems 2.3 and 2.4.

A useful observation is given in the following

REMARK 1. Assume that $p \in C(\mathbb{R}^N)$ is a nonnegative and nontrivial function which satisfies (p2). Let f be a function satisfying assumption (f1). Then condition

$$\int_1^{\infty} \frac{\mathrm{d}t}{f(t)} < \infty \tag{2.10}$$

is necessary for the existence of entire large solutions to (2.4).

Indeed, let u be an entire large solution of problem (2.4). Define

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S = \frac{1}{\omega_N} \int_{|\xi|=1} \left(\int_{a_0}^{u(r\xi)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S,$$

where ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$ and a_0 is chosen such that $a_0 \in (0, u_0)$, where $u_0 = \inf_{\mathbb{R}^N} u > 0$. By the divergence theorem we have

$$\bar{u}'(r) = \frac{1}{\omega_N r^{N-1}} \int_{B(0,r)} \Delta\left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}x.$$

Since u is a positive classical solution it follows that

$$\left|\bar{u}'(r)\right| \leqslant Cr \to 0 \quad \text{as } r \to 0.$$

On the other hand

$$\omega_N \left(R^{N-1}\bar{u}'(R) - r^{N-1}\bar{u}'(r) \right) = \int_r^R \left(\int_{|x|=z} \Delta\left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S \right) \mathrm{d}z.$$

Dividing by $R - r$ and taking $R \to r$ we find

$$\begin{aligned} \omega_N \left(r^{N-1}\bar{u}'(r) \right)' &= \int_{|x|=r} \Delta\left(\int_{a_0}^{u(x)} \frac{\mathrm{d}t}{f(t)} \right) \mathrm{d}S = \int_{|x|=r} \operatorname{div}\left(\frac{1}{f(u(x))} \nabla u(x) \right) \mathrm{d}S \\ &= \int_{|x|=r} \left[\left(\frac{1}{f} \right)' \left(u(x) \right) \cdot \left| \nabla u(x) \right|^2 + \frac{1}{f(u(x))} \Delta u(x) \right] \mathrm{d}S \\ &\leqslant \int_{|x|=r} \frac{p(x) f(u(x))}{f(u(x))} \, \mathrm{d}S \leqslant \omega_N r^{N-1} \Phi(r). \end{aligned}$$

The above inequality yields by integration

$$\bar{u}(r) \leqslant \bar{u}(0) + \int_0^r \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1} \Phi(\tau) \, \mathrm{d}\tau \right) \mathrm{d}\sigma \quad \forall r \geqslant 0. \tag{2.11}$$

On the other hand, according to (p2), for all $r > 0$ we have

$$\begin{aligned} &\int_0^r \sigma^{1-N} \left(\int_0^\sigma \tau^{N-1} \Phi(\tau) \, \mathrm{d}\tau \right) \mathrm{d}\sigma \\ &\quad = \frac{1}{2-N} r^{2-N} \int_0^r \tau^{N-1} \Phi(\tau) \, \mathrm{d}\tau - \frac{1}{2-N} \int_0^r \sigma \Phi(\sigma) \, \mathrm{d}\sigma \\ &\quad \leqslant \frac{1}{N-2} \int_0^\infty r \Phi(r) \, \mathrm{d}r < \infty. \end{aligned}$$

So, by (2.11), $\bar{u}(r) \leqslant \bar{u}(0) + K$, for all $r \geqslant 0$. The last inequality implies that $\bar{u}$ is bounded and assuming that (2.10) is not fulfilled it follows that u cannot be a large solution.

We point out that the hypothesis (p2) on p is essential in the statement of Remark 1. Indeed, let us consider $f(t) = t$, $p \equiv 1$, $\alpha \in (0, 1)$, $q(x) = 2^{\alpha-2} \cdot |x|^\alpha$, $a = 2 - \alpha \in (1, 2)$. Then the corresponding problem has the entire large solution $u(x) = |x|^2 + 2N$, but (2.10) is not fulfilled.

3. Singular solutions with lack of the Keller–Osserman condition

We have already seen that if f is smooth and increasing on $[0,\infty)$ such that $f(0)=0$ and $f>0$ in $(0,\infty)$, then the problem

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega \end{cases}$$

has a solution if and only if the Keller–Osserman condition $\int_1^\infty [F(t)]^{-1/2}\,\mathrm{d}t < \infty$ is fulfilled, where $F(t)=\int_0^t f(s)\,\mathrm{d}s$. In particular, this implies that f must have a superlinear growth. In this section we are concerned with the problem

$$\begin{cases} \Delta u + |\nabla u| = p(x) f(u) & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \end{cases} \tag{3.12}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 3)$ is either a smooth bounded domain or the whole space. Our main assumptions on f is that it has a *sublinear* growth, so we cannot expect that problem (3.12) admits a blow-up boundary solution. Our main purpose in this section is to establish a necessary and sufficient condition on the variable potential $p(x)$ for the existence of an entire large solution.

Throughout this section we assume that p is a nonnegative function such that $p \in C^{0,\alpha}(\overline{\Omega})$ $(0<\alpha<1)$ if Ω is bounded, and $p \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$, otherwise. The nondecreasing nonlinearity $f \in C^{0,\alpha}_{\mathrm{loc}}[0,\infty)$ fulfills $f(0)=0$ and $f>0$ on $(0,\infty)$. We also assume that f is sublinear at infinity, in the sense that $\Lambda := \sup_{s\geqslant 1}(f(s)/s) < \infty$.

The main results in this section have been established by Ghergu and Rădulescu [50].

If Ω is bounded we prove the following nonexistence result.

THEOREM 3.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Then problem* (3.12) *has no positive large solution in Ω.*

PROOF. Suppose by contradiction that problem (3.12) has a positive large solution u and define $v(x)=\ln(1+u(x))$, $x\in\Omega$. It follows that v is positive and $v(x)\to\infty$ as $\operatorname{dist}(x,\partial\Omega)\to 0$. We have

$$\Delta v = \frac{1}{1+u}\Delta u - \frac{1}{(1+u)^2}|\nabla u|^2 \quad \text{in } \Omega$$

and so

$$\Delta v \leqslant p(x)\frac{f(u)}{1+u} \leqslant \|p\|_\infty \frac{f(u)}{1+u} \leqslant A \quad \text{in } \Omega,$$

for some constant $A>0$. Therefore

$$\Delta\big(v(x) - A|x|^2\big) < 0, \quad \text{for all } x\in\Omega.$$

Let $w(x) = v(x) - A|x|^2,\ x \in \Omega$. Then $\Delta w < 0$ in Ω. Moreover, since Ω is bounded, it follows that $w(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$.

Let $M > 0$ be arbitrary. We claim that $w \geqslant M$ in Ω. For all $\delta > 0$, we set

$$\Omega_\delta = \big\{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \delta\big\}.$$

Since $w(x) \to \infty$ as $\operatorname{dist}(x, \partial\Omega) \to 0$, we can choose $\delta > 0$ such that

$$w(x) \geqslant M, \quad \text{for all } x \in \Omega \setminus \Omega_\delta. \tag{3.13}$$

On the other hand,

$$-\Delta\big(w(x) - M\big) > 0 \quad \text{in } \Omega_\delta,$$
$$w(x) - M \geqslant 0 \quad \text{on } \partial\Omega_\delta.$$

By the maximum principle we get $w(x) - M \geqslant 0$ in Ω_δ. So, by (3.13), $w \geqslant M$ in Ω. Since $M > 0$ is arbitrary, it follows that $w \geqslant n$ in Ω, for all $n \geqslant 1$. Obviously, this is a contradiction and the proof is now complete. □

Next, we consider the problem (3.12) when $\Omega = \mathbb{R}^N$. For all $r \geqslant 0$ we set

$$\phi(r) = \max_{|x|=r} p(x), \quad \psi(r) = \min_{|x|=r} p(x), \quad \text{and} \quad h(r) = \phi(r) - \psi(r).$$

We suppose that

$$\int_0^\infty r h(r) \Psi(r)\, \mathrm{d}r < \infty, \tag{3.14}$$

where

$$\Psi(r) = \exp\left(\Lambda_N \int_0^r s\psi(s)\, \mathrm{d}s\right), \quad \Lambda_N = \frac{\Lambda}{N-2}.$$

Obviously, if p is radial then $h \equiv 0$ and (3.14) occurs. Assumption (3.14) shows that the variable potential $p(x)$ has a slow variation. An example of nonradial potential for which (3.14) holds is

$$p(x) = \frac{1 + |x_1|^2}{(1 + |x_1|^2)(1 + |x|^2) + 1}.$$

In this case

$$\phi(r) = \frac{r^2 + 1}{(r^2 + 1)^2 + 1} \quad \text{and} \quad \psi(r) = \frac{1}{r^2 + 2}.$$

If $\Lambda_N = 1$, by direct computation we get $r h(r) \Psi(r) = \mathrm{O}(r^{-2})$ as $r \to \infty$ and so (3.14) holds.

THEOREM 3.2. *Assume $\Omega = \mathbb{R}^N$ and p satisfies* (3.14). *Then problem* (3.12) *has a positive entire large solution if and only if*

$$\int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s)\,\mathrm{d}s\,\mathrm{d}t = \infty. \tag{3.15}$$

PROOF. Several times in the proof of Theorem 3.2 we shall apply the following elementary inequality:

$$\int_0^r \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} g(s)\,\mathrm{d}s\,\mathrm{d}t \leqslant \frac{1}{N-2} \int_0^r t g(t)\,\mathrm{d}t, \quad \forall r > 0, \tag{3.16}$$

for any continuous function $g : [0, \infty) \to [0, \infty)$. The proof follows easily by integration by parts.

Necessary condition. Suppose that (3.14) fails and the equation (3.12) has a positive entire large solution u. We claim that

$$\int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \phi(s)\,\mathrm{d}s\,\mathrm{d}t < \infty. \tag{3.17}$$

We first recall that $\phi = h + \psi$. Thus

$$\begin{aligned}\int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \phi(s)\,\mathrm{d}s\,\mathrm{d}t &= \int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad + \int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} h(s)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

By virtue of (3.16) we find

$$\begin{aligned}&\int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \phi(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\quad \leqslant \int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s)\,\mathrm{d}s\,\mathrm{d}t + \frac{1}{N-2} \int_0^\infty t h(t)\,\mathrm{d}t \\ &\quad \leqslant \int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s)\,\mathrm{d}s\,\mathrm{d}t + \frac{1}{N-2} \int_0^\infty t h(t) \Psi(t)\,\mathrm{d}t.\end{aligned}$$

Since $\int_1^\infty \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s)\,\mathrm{d}s\,\mathrm{d}t < \infty$, by (3.14) we deduce that (3.17) follows.

Now, let $\bar{u}$ be the spherical average of u, i.e.,

$$\bar{u}(r) = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} u(x)\,\mathrm{d}\sigma_x, \quad r \geqslant 0,$$

where ω_N is the surface area of the unit sphere in $\mathbb{R}^N$. Since u is a positive entire large solution of (2.4) it follows that $\bar{u}$ is positive and $\bar{u}(r) \to \infty$ as $r \to \infty$. With the change of variable $x \to ry$, we have

$$\bar{u}(r) = \frac{1}{\omega_N} \int_{|y|=1} u(ry)\, d\sigma_y, \quad r \geqslant 0$$

and

$$\bar{u}'(r) = \frac{1}{\omega_N} \int_{|y|=1} \nabla u(ry) \cdot y\, d\sigma_y, \quad r \geqslant 0. \tag{3.18}$$

Hence

$$\bar{u}'(r) = \frac{1}{\omega_N} \int_{|y|=1} \frac{\partial u}{\partial r}(ry)\, d\sigma_y = \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \frac{\partial u}{\partial r}(x)\, d\sigma_x,$$

that is

$$\bar{u}'(r) = \frac{1}{\omega_N r^{N-1}} \int_{B(0,R)} \Delta u(x)\, dx, \quad \text{for all } r \geqslant 0. \tag{3.19}$$

Due to the gradient term $|\nabla u|$ in (2.4), we cannot infer that $\Delta u \geqslant 0$ in $\mathbb{R}^N$ and so we cannot expect that $\bar{u}' \geqslant 0$ in $[0, \infty)$. We define the auxiliary function

$$U(r) = \max_{0 \leqslant t \leqslant r} \bar{u}(t), \quad r \geqslant 0. \tag{3.20}$$

Then U is positive and nondecreasing. Moreover, $U \geqslant \bar{u}$ and $U(r) \to \infty$ as $r \to \infty$.

The assumptions (f1) and (f2) yield $f(t) \leqslant \Lambda(1 + t)$, for all $t \geqslant 0$. So, by (3.18) and (3.19),

$$\begin{aligned}
\bar{u}'' + \frac{N-1}{r}\bar{u}' + \bar{u}' &\leqslant \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \big[\Delta u(x) + |\nabla u|(x)\big]\, d\sigma_x \\
&= \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} p(r) f\big(u(x)\big)\, d\sigma_x \\
&\leqslant \Lambda\phi(r) \frac{1}{\omega_N r^{N-1}} \int_{|x|=r} \big(1 + u(x)\big)\, d\sigma_x \\
&= \Lambda\phi(r)\big(1 + \bar{u}(r)\big) \leqslant \Lambda\phi(r)\big(1 + U(r)\big),
\end{aligned}$$

for all $r \geqslant 0$. It follows that

$$\big(r^{N-1} e^r \bar{u}'\big)' \leqslant \Lambda e^r r^{N-1} \phi(r)\big(1 + U(r)\big), \quad \text{for all } r \geqslant 0.$$

So, for all $r \geqslant r_0 > 0$,

$$\bar{u}(r) \leqslant \bar{u}(r_0) + \Lambda \int_{r_0}^{r} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \phi(s)\big(1 + U(s)\big)\,\mathrm{d}s\,\mathrm{d}t.$$

The monotonicity of U implies

$$\bar{u}(r) \leqslant \bar{u}(r_0) + \Lambda\big(1 + U(r)\big) \int_{r_0}^{r} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \phi(s)\,\mathrm{d}s\,\mathrm{d}t, \tag{3.21}$$

for all $r \geqslant r_0 \geqslant 0$. By (3.17) we can choose $r_0 \geqslant 1$ such that

$$\int_{r_0}^{\infty} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \phi(s)\,\mathrm{d}s\,\mathrm{d}t < \frac{1}{2\Lambda}. \tag{3.22}$$

Thus (3.21) and (3.22) yield

$$\bar{u}(r) \leqslant \bar{u}(r_0) + \frac{1}{2}\big(1 + U(r)\big), \quad \text{for all } r \geqslant r_0. \tag{3.23}$$

By the definition of U and $\lim_{r\to\infty} \bar{u}(r) = \infty$, we find $r_1 \geqslant r_0$ such that

$$U(r) = \max_{r_0 \leqslant t \leqslant r} \bar{u}(r), \quad \text{for all } r \geqslant r_1. \tag{3.24}$$

Considering now (3.23) and (3.24) we obtain

$$U(r) \leqslant \bar{u}(r_0) + \frac{1}{2}\big(1 + U(r)\big), \quad \text{for all } r \geqslant r_1.$$

Hence

$$U(r) \leqslant 2\bar{u}(r_0) + 1, \quad \text{for all } r \geqslant r_1.$$

This means that U is bounded, so u is also bounded, a contradiction. It follows that (2.4) has no positive entire large solutions.

Sufficient condition. We need the following auxiliary comparison result.

LEMMA 3.3. *Assume that* (3.14) *and* (3.15) *hold. Then the equations*

$$\Delta v + |\nabla v| = \phi\big(|x|\big) f(v), \qquad \Delta w + |\nabla w| = \psi\big(|x|\big) f(w) \tag{3.25}$$

have positive entire large solution such that

$$v \leqslant w \quad \text{in } \mathbb{R}^N. \tag{3.26}$$

PROOF. Radial solutions of (3.25) satisfy

$$v'' + \frac{N-1}{r} v' + |v'| = \phi(r) f(v)$$

and

$$w'' + \frac{N-1}{r} w' + |w'| = \psi(r) f(w).$$

Assuming that v' and w' are nonnegative, we deduce

$$\left(\mathrm{e}^r r^{N-1} v'\right)' = \mathrm{e}^r r^{N-1} \phi(r) f(v)$$

and

$$\left(\mathrm{e}^r r^{N-1} w'\right)' = \mathrm{e}^r r^{N-1} \psi(r) f(w).$$

Thus any positive solutions v and w of the integral equations

$$v(r) = 1 + \int_0^r \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \phi(s) f\big(v(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0, \tag{3.27}$$

$$w(r) = b + \int_0^r \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s) f\big(w(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0, \tag{3.28}$$

provide a solution of (3.25), for any $b > 0$. Since $w \geqslant b$, it follows that $f(w) \geqslant f(b) > 0$ which yields

$$w(r) \geqslant b + f(b) \int_0^r \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0.$$

By (3.15), the right-hand side of this inequality goes to $+\infty$ as $r \to \infty$. Thus $w(r) \to \infty$ as $r \to \infty$. With a similar argument we find $v(r) \to \infty$ as $r \to \infty$.

Let $b > 1$ be fixed. We first show that (3.28) has a positive solution. Similarly, (3.27) has a positive solution.

Let $\{w_k\}$ be the sequence defined by $w_1 = b$ and

$$w_{k+1}(r) = b + \int_0^r \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s) f\big(w_k(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad k \geqslant 1. \tag{3.29}$$

We remark that $\{w_k\}$ is a nondecreasing sequence. To get the convergence of $\{w_k\}$ we will show that $\{w_k\}$ is bounded from above on bounded subsets. To this aim, we fix $R > 0$ and we prove that

$$w_k(r) \leqslant b\mathrm{e}^{Mr}, \quad \text{for any } 0 \leqslant r \leqslant R, \text{ and for all } k \geqslant 1, \tag{3.30}$$

where $M \equiv \Lambda_N \max_{t\in[0,R]} t\psi(t)$.

We achieve (3.30) by induction. We first notice that (3.30) is true for $k = 1$. Furthermore, the assumption (f2) and the fact that $w_k \geqslant 1$ lead us to $f(w_k) \leqslant \Lambda w_k$, for all $k \geqslant 1$. So, by (3.29),

$$w_{k+1}(r) \leqslant b + \Lambda \int_0^r \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s) w_k(s)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0.$$

Using now (3.16) (for $g(t) = \psi(t) w_k(t)$) we deduce

$$w_{k+1}(r) \leqslant b + \Lambda_N \int_0^r t\psi(t) w_k(t)\,\mathrm{d}t, \quad \forall r \in [0, R].$$

The induction hypothesis yields

$$w_{k+1}(r) \leqslant b + bM \int_0^r \mathrm{e}^{Mt}\,\mathrm{d}t = b\mathrm{e}^{Mr}, \quad \forall r \in [0, R].$$

Hence, by induction, the sequence $\{w_k\}$ is bounded in $[0, R]$, for any $R > 0$. It follows that $w(r) = \lim_{k\to\infty} w_k(r)$ is a positive solution of (3.28). In a similar way we conclude that (3.27) has a positive solution on $[0, \infty)$.

The next step is to show that the constant b may be chosen sufficiently large so that (3.26) holds. More exactly, if

$$b > 1 + K\Lambda_N \int_0^\infty sh(s)\Psi(s)\,\mathrm{d}s, \tag{3.31}$$

where $K = \exp(\Lambda_N \int_0^\infty th(t)\,\mathrm{d}t)$, then (3.26) occurs.

We first prove that the solution v of (3.27) satisfies

$$v(r) \leqslant K\Psi(r), \quad \forall r \geqslant 0. \tag{3.32}$$

Since $v \geqslant 1$, from (f2) we have $f(v) \leqslant \Lambda v$. We use this fact in (3.27) and then we apply the estimate (3.16) for $g = \phi$. It follows that

$$v(r) \leqslant 1 + \Lambda_N \int_0^r s\phi(s) v(s)\,\mathrm{d}s, \quad \forall r \geqslant 0. \tag{3.33}$$

By Gronwall's inequality we obtain

$$v(r) \leqslant \exp\left(\Lambda_N \int_0^r s\phi(s)\,\mathrm{d}s\right), \qquad \forall r \geqslant 0,$$

and, by (3.33),

$$v(r) \leqslant 1 + \Lambda_N \int_0^r s\phi(s) \exp\left(\Lambda_N \int_0^s t\phi(t)\,\mathrm{d}t\right)\mathrm{d}s, \quad \forall r \geqslant 0.$$

Hence

$$v(r) \leqslant 1 + \int_0^r \left(\exp\left(\Lambda_N \int_0^s t\phi(t)\,\mathrm{d}t \right) \right)' \mathrm{d}s, \quad \forall r \geqslant 0,$$

that is

$$v(r) \leqslant \exp\left(\Lambda_N \int_0^r t\phi(t)\,\mathrm{d}t \right), \quad \forall r \geqslant 0. \tag{3.34}$$

Inserting $\phi = h + \psi$ in (3.34) we have

$$v(r) \leqslant \mathrm{e}^{\Lambda_N \int_0^r th(t)\,\mathrm{d}t} \Psi(r) \leqslant K\Psi(r), \quad \forall r \geqslant 0,$$

so (3.32) follows.

Since $b > 1$ it follows that $v(0) < w(0)$. Then there exists $R > 0$ such that $v(r) < w(r)$, for any $0 \leqslant r \leqslant R$. Set

$$R_\infty = \sup\big\{ R > 0 \mid v(r) < w(r),\ \forall r \in [0, R] \big\}.$$

In order to conclude our proof, it remains to show that $R_\infty = \infty$. Suppose the contrary. Since $v \leqslant w$ on $[0, R_\infty]$ and $\phi = h + \psi$, from (3.27) we deduce

$$\begin{aligned} v(R_\infty) = 1 &+ \int_0^{R_\infty} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} h(s) f\big(v(s)\big)\,\mathrm{d}s\,\mathrm{d}t \\ &+ \int_0^{R_\infty} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s) f\big(v(s)\big)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

So, by (3.16),

$$\begin{aligned} v(R_\infty) \leqslant 1 &+ \frac{1}{N-2} \int_0^{R_\infty} th(t) f\big(v(t)\big)\,\mathrm{d}t \\ &+ \int_0^{R_\infty} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s) f\big(w(s)\big)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Taking into account that $v \geqslant 1$ and the assumption (f2), it follows that

$$\begin{aligned} v(R_\infty) \leqslant 1 &+ K\Lambda_N \int_0^{R_\infty} th(t)\Psi(t)\,\mathrm{d}t \\ &+ \int_0^{R_\infty} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s) f\big(w(s)\big)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned}$$

Now, using (3.31) we obtain

$$v(R_\infty) < b + \int_0^{R_\infty} \mathrm{e}^{-t} t^{1-N} \int_0^t \mathrm{e}^s s^{N-1} \psi(s) f\big(w(s)\big)\,\mathrm{d}s\,\mathrm{d}t = w(R_\infty).$$

Hence $v(R_\infty) < w(R_\infty)$. Therefore, there exists $R > R_\infty$ such that $v < w$ on $[0, R]$, which contradicts the maximality of R_∞. This contradiction shows that inequality (3.26) holds and the proof of Lemma 3.3 is now complete. □

PROOF OF THEOREM 3.2 COMPLETED. Suppose that (3.15) holds. For all $k \geqslant 1$ we consider the problem

$$\begin{cases} \Delta u_k + |\nabla u_k| = p(x) f(u_k) & \text{in } B(0,k), \\ u_k(x) = w(k) & \text{on } \partial B(0,k). \end{cases} \tag{3.35}$$

Then v and w defined by (3.27) and (3.28) are positive sub and super-solutions of (3.35). So this problem has at least a positive solution u_k and

$$v\big(|x|\big) \leqslant u_k(x) \leqslant w\big(|x|\big) \quad \text{in } B(0,k), \text{ for all } k \geqslant 1.$$

By Theorem 14.3 in Gilbarg and Trudinger [54], the sequence $\{\nabla u_k\}$ is bounded on every compact set in $\mathbb{R}^N$. Hence the sequence $\{u_k\}$ is bounded and equicontinuous on compact subsets of $\mathbb{R}^N$. So, by the Arzela–Ascoli Theorem, the sequence $\{u_k\}$ has a uniform convergent subsequence, $\{u_k^1\}$ on the ball $B(0,1)$. Let $u^1 = \lim_{k\to\infty} u_k^1$. Then $\{f(u_k^1)\}$ converges uniformly to $f(u^1)$ on $B(0,1)$ and, by (3.35), the sequence $\{\Delta u_k^1 + |\nabla u_k^1|\}$ converges uniformly to $pf(u^1)$. Since the sum of the Laplace and Gradient operators is a closed operator, we deduce that u^1 satisfies (2.4) on $B(0,1)$.

Now, the sequence $\{u_k^1\}$ is bounded and equicontinuous on the ball $B(0,2)$, so it has a convergent subsequence $\{u_k^2\}$. Let $u^2 = \lim_{k\to\infty} u_k^2$, on $B(0,2)$, and u^2 satisfies (2.4) on $B(0,2)$. Proceeding in the same way, we construct a sequence $\{u^n\}$ so that u^n satisfies (2.4) on $B(0,n)$ and $u^{n+1} = u^n$ on $B(0,n)$ for all n. Moreover, the sequence $\{u^n\}$ converges in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ to the function u defined by

$$u(x) = u^m(x), \quad \text{for } x \in B(0,m).$$

Since $v \leqslant u^n \leqslant w$ on $B(0,n)$ it follows that $v \leqslant u \leqslant w$ on $\mathbb{R}^N$, and u, satisfies (2.4). From $v \leqslant u$ we deduce that u is a positive entire large solution of (2.4). This completes the proof. □

4. Blow-up boundary solutions of the logistic equation

Consider the semilinear elliptic equation

$$\Delta u + au = b(x) f(u) \quad \text{in } \Omega, \tag{4.36}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$. Let a be a real parameter and $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, such that $b \geqslant 0$ and $b \not\equiv 0$ in Ω. Set

$$\Omega_0 = \operatorname{int}\big\{x \in \Omega\colon\ b(x) = 0\big\}$$

and suppose, throughout, that $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ on $\Omega \setminus \overline{\Omega}_0$. Assume that $f \in C^1[0,\infty)$ satisfies

(A_1) $f \geqslant 0$ and $f(u)/u$ is increasing on $(0,\infty)$.

Following Alama and Tarantello [2], define by H_∞ the Dirichlet Laplacian on Ω_0 as the unique self-adjoint operator associated to the quadratic form $\psi(u) = \int_\Omega |\nabla u|^2 \,\mathrm{d}x$ with form domain

$$H_D^1(\Omega_0) = \big\{u \in H_0^1(\Omega)\colon\ u(x) = 0 \text{ for a.e. } x \in \Omega \setminus \Omega_0\big\}.$$

If $\partial\Omega_0$ satisfies the exterior cone condition then, according to [2], $H_D^1(\Omega_0)$ coincides with $H_0^1(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. We understand $\lambda_{\infty,1} = \infty$ if $\Omega_0 = \emptyset$.

Set $\mu_0 := \lim_{u \searrow 0}(f(u)/u)$, $\mu_\infty := \lim_{u\to\infty}(f(u)/u)$, and denote by $\lambda_1(\mu_0)$ (resp., $\lambda_1(\mu_\infty)$) the first eigenvalue of the operator $H_{\mu_0} = -\Delta + \mu_0 b$ (resp., $H_{\mu_\infty} = -\Delta + \mu_\infty b$) in $H_0^1(\Omega)$. Recall that $\lambda_1(+\infty) = \lambda_{\infty,1}$.

Alama and Tarantello [2] proved that problem (4.36) subject to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \tag{4.37}$$

has a positive solution u_a if and only if $a \in (\lambda_1(\mu_0), \lambda_1(\mu_\infty))$. Moreover, u_a is the unique positive solution for (4.36) + (4.37) (see [2, Theorem A (bis)]). We shall refer to the combination of (4.36) + (4.37) as problem (E_a).

Our first aim in this section is to give a corresponding necessary and sufficient condition, but for the existence of *large* (or *explosive*) solutions of (4.36). An elementary argument based on the maximum principle shows that if such a solution exists, then it is *positive* even if f satisfies a weaker condition than (A_1), namely

(A_1') $f(0) = 0$, $f' \geqslant 0$ and $f > 0$ on $(0,\infty)$.

We recall that Keller [62] and Osserman [78] supplied a necessary and sufficient condition on f for the existence of large solutions to (1) when $a \equiv 0$, $b \equiv 1$ and f is assumed to fulfill (A_1'). More precisely, f must satisfy the Keller–Osserman condition (see [62,78]),

(A_2) $\int_1^\infty (1/F(t))\,\mathrm{d}t < \infty$, where $F(t) = \int_0^t f(s)\,\mathrm{d}s$.

Typical examples of nonlinearities satisfying (A_1) and (A_2) are:

$$\text{(i) } f(u) = \mathrm{e}^u - 1; \quad \text{(ii) } f(u) = u^p,\ p > 1;$$
$$\text{(iii) } f(u) = u\big[\ln(u+1)\big]^p,\ p > 2.$$

Our first result gives the maximal interval for the parameter a that ensures the existence of large solutions to problem (4.36). More precisely, we prove

THEOREM 4.1. *Assume that f satisfies conditions* (A_1) *and* (A_2). *Then problem* (4.36) *has a large solution if and only if $a \in (-\infty, \lambda_{\infty,1})$.*

We point out that our framework in the above result includes the case when b vanishes at some points on $\partial\Omega$, or even if $b \equiv 0$ on $\partial\Omega$. This later case includes the "competition" $0 \cdot \infty$ on $\partial\Omega$. We also point out that, under our hypotheses, $\mu_\infty := \lim_{u\to\infty} f(u)/u = \lim_{u\to\infty} f'(u) = \infty$. Indeed, by l'Hospital's rule, $\lim_{u\to\infty} F(u)/u^2 = \mu_\infty/2$. But, by (A_2), we deduce that $\mu_\infty = \infty$. Then, by (A_1) we find that $f'(u) \geqslant f(u)/u$ for any $u > 0$, which shows that $\lim_{u\to\infty} f'(u) = \infty$.

Before giving the proof of Theorem 4.1 we claim that assuming (A_1), then problem (4.36) can have large solutions only if f satisfies the Keller–Osserman condition (A_2). Indeed, suppose that problem (4.36) has a large solution u_∞. Set $\tilde{f}(u) = |a|u + \|b\|_\infty f(u)$ for $u \geqslant 0$. Notice that $\tilde{f} \in C^1[0,\infty)$ satisfies (A_1'). For any $n \geqslant 1$, consider the problem

$$\begin{cases} \Delta u = \tilde{f}(u) & \text{in } \Omega, \\ u = n & \text{on } \partial\Omega, \\ u \geqslant 0 & \text{in } \Omega. \end{cases}$$

A standard argument based on the maximum principle shows that this problem has a unique solution, say u_n, which, moreover, is positive in $\overline{\Omega}$. Applying again the maximum principle we deduce that $0 < u_n \leqslant u_{n+1} \leqslant u_\infty$, in Ω, for all $n \geqslant 1$. Thus, for every $x \in \Omega$, we can define $\bar{u}(x) = \lim_{n\to\infty} u_n(x)$. Moreover, since (u_n) is uniformly bounded on every compact subset of Ω, standard elliptic regularity arguments show that $\bar{u}$ is a positive large solution of the problem $\Delta u = \tilde{f}(u)$. It follows that $\tilde{f}$ satisfies the Keller–Osserman condition (A_2). Then, by (A_1), $\mu_\infty := \lim_{u\to\infty} f(u)/u > 0$ which yields $\lim_{u\to\infty} \tilde{f}(u)/f(u) = |a|/\mu_\infty + \|b\|_\infty < \infty$. Consequently, our claim follows.

PROOF OF THEOREM 4.1. A. *Necessary condition*. Let u_∞ be a large solution of problem (4.36). Then, by the maximum principle, u_∞ is positive. Suppose $\lambda_{\infty,1}$ is finite. Arguing by contradiction, let us assume $a \geqslant \lambda_{\infty,1}$. Set $\lambda \in (\lambda_1(\mu_0), \lambda_{\infty,1})$ and denote by u_λ the unique positive solution of problem (E_a) with $a = \lambda$. We have

$$\begin{cases} \Delta(Mu_\infty) + \lambda_{\infty,1}(Mu_\infty) \leqslant b(x) f(Mu_\infty) & \text{in } \Omega, \\ Mu_\infty = \infty & \text{on } \partial\Omega, \\ Mu_\infty \geqslant u_\lambda & \text{in } \Omega, \end{cases}$$

where $M := \max\{\max_{\overline{\Omega}} u_\lambda / \min_\Omega u_\infty; 1\}$. By the sub-super solution method we conclude that problem (E_a) with $a = \lambda_{\infty,1}$ has at least a positive solution (between u_λ and Mu_∞). But this is a contradiction. So, necessarily, $a \in (-\infty, \lambda_{\infty,1})$.

B. *Sufficient condition*. This will be proved with the aid of several results.

LEMMA 4.2. *Let ω be a smooth bounded domain in $\mathbb{R}^N$. Assume p, q, r are $C^{0,\mu}$-functions on $\overline{\omega}$ such that $r \geqslant 0$ and $p > 0$ in $\overline{\omega}$. Then for any nonnegative function $0 \not\equiv \Phi \in C^{0,\mu}(\partial\omega)$ the boundary value problem*

$$\begin{cases} \Delta u + q(x)u = p(x) f(u) - r(x) & \text{in } \omega, \\ u > 0 & \text{in } \omega, \\ u = \Phi & \text{on } \partial\omega, \end{cases} \tag{4.38}$$

has a unique solution.

We refer to Cîrstea and Rădulescu [27, Lemma 3.1] for the proof of the above result.

Under the assumptions of Lemma 4.2 we obtain the following result which generalizes [74, Lemma 1.3].

COROLLARY 4.3. *There exists a positive large solution of the problem*

$$\Delta u + q(x)u = p(x)f(u) - r(x) \quad \text{in } \omega. \tag{4.39}$$

PROOF. Set $\Phi = n$ and let u_n be the unique solution of (4.38). By the maximum principle, $u_n \leqslant u_{n+1} \leqslant \bar{u}$ in ω, where $\bar{u}$ denotes a large solution of

$$\Delta u + \|q\|_\infty u = p_0 f(u) - \bar{r} \quad \text{in } \omega.$$

Thus $\lim_{n\to\infty} u_n(x) = u_\infty(x)$ exists and is a positive large solution of (4.39). Furthermore, every positive large solution of (4.39) dominates u_∞, i.e., the solution u_∞ is the *minimal large solution*. This follows from the definition of u_∞ and the maximum principle. □

LEMMA 4.4. *If $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ is a nonnegative function and $b > 0$ on $\partial\Omega$, then the boundary value problem*

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \end{cases} \tag{4.40}$$

has a solution if and only if $a \in (-\infty, \lambda_{\infty,1})$. Moreover, in this case, the solution is unique.

PROOF. The first part follows exactly in the same way as the proof of Theorem 4.1 (necessary condition).

For the sufficient condition, fix $a < \lambda_{\infty,1}$ and let $\lambda_{\infty,1} > \lambda_* > \max\{a, \lambda_1(\mu_0)\}$. Let u_* be the unique positive solution of (E_a) with $a = \lambda_*$.

Let Ω_i $(i = 1, 2)$ be subdomains of Ω such that $\Omega_0 \Subset \Omega_1 \Subset \Omega_2 \Subset \Omega$ and $\Omega \setminus \overline{\Omega}_1$ is smooth. We define $u_+ \in C^2(\Omega)$ as a positive function in Ω such that $u_+ \equiv u_\infty$ on $\Omega \setminus \Omega_2$ and $u_+ \equiv u_*$ on Ω_1. Here u_∞ denotes a positive large solution of (4.39) for $p(x) = b(x)$, $r(x) = 0$, $q(x) = a$ and $\omega = \Omega \setminus \overline{\Omega}_1$. So, since $b_0 := \inf_{\Omega_2\setminus\Omega_1} b$ is positive, it is easy to check that if $C > 0$ is large enough then $\bar{v}_\Phi = Cu_+$ satisfies

$$\begin{cases} \Delta\bar{v}_\Phi + a\bar{v}_\Phi \leqslant b(x)f(\bar{v}_\Phi) & \text{in } \Omega, \\ \bar{v}_\Phi = \infty & \text{on } \partial\Omega, \\ \bar{v}_\Phi \geqslant \max_{\partial\Omega} \Phi & \text{in } \Omega. \end{cases}$$

Let $\underline{v}_\Phi$ be the unique classical solution of the problem

$$\begin{cases} \Delta\underline{v}_\Phi = |a|\underline{v}_\Phi + \|b\|_\infty f(\underline{v}_\Phi) & \text{in } \Omega, \\ \underline{v}_\Phi > 0 & \text{in } \Omega, \\ \underline{v}_\Phi = \Phi & \text{on } \partial\Omega. \end{cases}$$

It is clear that $\underline{v}_\Phi$ is a positive sub-solution of (4.40) and $\underline{v}_\Phi \leqslant \max_{\partial\Omega} \Phi \leqslant \bar{v}_\Phi$ in Ω. Therefore, by the sub-super solution method, problem (4.40) has at least a solution v_Φ between $\underline{v}_\Phi$ and $\bar{v}_\Phi$. Next, the uniqueness of solution to (4.40) can be obtained by using essentially the same technique as in [15, Theorem 1] or [14, Appendix II]. □

PROOF OF THEOREM 4.1 COMPLETED. Fix $a \in (-\infty, \lambda_{\infty,1})$. Two cases may occur:

Case 1: $b > 0$ on $\partial\Omega$. Denote by v_n the unique solution of (4.40) with $\Phi \equiv n$. For $\Phi \equiv 1$, set $v := \underline{v}_\Phi$ and $V := \bar{v}_\Phi$, where $\underline{v}_\Phi$ and $\bar{v}_\Phi$ are defined in the proof of Lemma 4.4. The sub and super-solutions method combined with the uniqueness of solution of (4.40) shows that $v \leqslant v_n \leqslant v_{n+1} \leqslant V$ in Ω. Hence $v_\infty(x) := \lim_{n\to\infty} v_n(x)$ exists and is a positive large solution of (4.36).

Case 2: $b \geqslant 0$ on $\partial\Omega$. Let z_n ($n \geqslant 1$) be the unique solution of (4.38) for $p \equiv b + 1/n$, $r \equiv 0$, $q \equiv a$, $\Phi \equiv n$ and $\omega = \Omega$. By the maximum principle, (z_n) is nondecreasing. Moreover, (z_n) is uniformly bounded on every compact subdomain of Ω. Indeed, if $K \subset \Omega$ is an arbitrary compact set, then $d := \operatorname{dist}(K, \partial\Omega) > 0$. Choose $\delta \in (0, d)$ small enough so that $\overline{\Omega}_0 \subset C_\delta$, where $C_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$. Since $b > 0$ on ∂C_δ, Case 1 allows us to define z_+ as a positive large solution of (4.36) for $\Omega = C_\delta$. A standard argument based on the maximum principle implies that $z_n \leqslant z_+$ in C_δ, for all $n \geqslant 1$. So, (z_n) is uniformly bounded on K. By the monotonicity of (z_n), we conclude that $z_n \to \underline{z}$ in $L^\infty_{\mathrm{loc}}(\Omega)$. Finally, standard elliptic regularity arguments lead to $z_n \to \underline{z}$ in $C^{2,\mu}(\Omega)$. This completes the proof of Theorem 4.1. □

Denote by $\mathcal{D}$ and $\mathcal{R}$ the boundary operators

$$\mathcal{D}u := u \quad \text{and} \quad \mathcal{R}u := \partial_\nu u + \beta(x)u,$$

where ν is the unit outward normal to $\partial\Omega$, and $\beta \in C^{1,\mu}(\partial\Omega)$ is nonnegative. Hence, $\mathcal{D}$ is the *Dirichlet* boundary operator and $\mathcal{R}$ is either the *Neumann* boundary operator, if $\beta \equiv 0$, or the *Robin* boundary operator, if $\beta \not\equiv 0$. Throughout this work, $\mathcal{B}$ can define any of these boundary operators.

Note that the Robin condition $\mathcal{R} = 0$ relies essentially to heat flow problems in a body with constant temperature in the surrounding medium. More generally, if α and β are smooth functions on $\partial\Omega$ such that $\alpha, \beta \geqslant 0$, $\alpha + \beta > 0$, then the boundary condition $Bu = \alpha\partial_\nu u + \beta u = 0$ represents the exchange of heat at the surface of the reactant by Newtonian cooling. Moreover, the boundary condition $Bu = 0$ is called isothermal (Dirichlet) condition if $\alpha \equiv 0$, and it becomes an adiabatic (Neumann) condition if $\beta \equiv 0$. An intuitive meaning of the condition $\alpha + \beta > 0$ on $\partial\Omega$ is that, for the diffusion process described by problem (4.36), either the reflection phenomenon or the absorption phenomenon may occur at each point of the boundary.

We are now concerned with the following boundary blow-up problem

$$\begin{cases} \Delta u + au = b(x) f(u) & \text{in } \Omega \setminus \overline{\Omega}_0, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \\ u = \infty & \text{on } \partial\Omega_0, \end{cases} \tag{4.41}$$

where $b > 0$ on $\partial\Omega$, while $\overline{\Omega}_0$ is nonempty, connected and with smooth boundary. Here, $u = \infty$ on $\partial\Omega_0$ means that $u(x) \to \infty$ as $x \in \Omega \setminus \overline{\Omega}_0$ and $d(x) := \operatorname{dist}(x, \Omega_0) \to 0$.

The question of existence and uniqueness of positive solutions for problem (4.41) in the case of pure superlinear power in the nonlinearity is treated by Du and Huang [39]. Our next results extend their previous paper to the case of much more general nonlinearities of Keller–Osserman type.

In the following, by $(\tilde{A}_1)$ we mean that (A_1) is fulfilled and there exists $\lim_{u\to\infty}(F/f)'(u) := \gamma$. Then, $\gamma \geqslant 0$.

We prove

THEOREM 4.5. *Let* $(\tilde{A}_1)$ *and* (A_2) *hold. Then, for any* $a \in \mathbb{R}$, *problem* (4.41) *has a minimal* (*resp.*, *maximal*) *positive solution* $\underline{U}_a$ (*resp.*, $\overline{U}_a$).

PROOF. In proving Theorem 4.5 we rely on an appropriate comparison principle which allows us to show that $(u_n)_{n\geqslant 1}$ is nondecreasing, where u_n is the unique positive solution of problem (4.43) with $\Phi \equiv n$. The minimal positive solution of (4.41) will be obtained as the limit of the sequence $(u_n)_{n\geqslant 1}$. Note that, since $b = 0$ on $\partial\Omega_0$, the main difficulty is related to the construction of an upper bound of this sequence which must fit to our general framework. Next, we deduce the maximal positive solution of (4.41) as the limit of the nonincreasing sequence $(v_m)_{m\geqslant m_1}$ provided m_1 is large so that $\Omega_{m_1} \Subset \Omega$. We denoted by v_m the minimal positive solution of (4.41) with Ω_0 replaced by

$$\Omega_m := \left\{ x \in \Omega \colon d(x) < \frac{1}{m} \right\}, \quad m \geqslant m_1. \tag{4.42}$$

We start with the following auxiliary result (see Cîrstea and Rădulescu [27]).

LEMMA 4.6. *Assume* $b > 0$ *on* $\partial\Omega$. *If* (A_1) *and* (A_2) *hold, then for any positive function* $\Phi \in C^{2,\mu}(\partial\Omega_0)$ *and* $a \in \mathbb{R}$ *the problem*

$$\begin{cases} \Delta u + au = b(x) f(u) & \text{in } \Omega \setminus \overline{\Omega}_0, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega, \\ u = \Phi & \text{on } \partial\Omega_0, \end{cases} \tag{4.43}$$

has a unique positive solution.

We now come back to the proof of Theorem 4.5, that will be divided into two steps:

Step 1. *Existence of the minimal positive solution for problem* (4.41).

For any $n \geqslant 1$, let u_n be the unique positive solution of problem (4.43) with $\Phi \equiv n$. By the maximum principle, $u_n(x)$ increases with n for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. Moreover, we prove

LEMMA 4.7. *The sequence* $(u_n(x))_n$ *is bounded from above by some function* $V(x)$ *which is uniformly bounded on all compact subsets of* $\overline{\Omega} \setminus \overline{\Omega}_0$.

PROOF. Let b^* be a C^2-function on $\overline{\Omega} \setminus \Omega_0$ such that

$$0 < b^*(x) \leqslant b(x) \quad \forall x \in \overline{\Omega} \setminus \overline{\Omega}_0.$$

For x bounded away from $\partial\Omega_0$ is not a problem to find such a function b^*. For x satisfying $0 < d(x) < \delta$ with $\delta > 0$ small such that $x \to d(x)$ is a C^2-function, we can take

$$b^*(x) = \int_0^{d(x)} \int_0^t \Big[\min_{d(z) \geqslant s} b(z) \Big] \, \mathrm{d}s \, \mathrm{d}t.$$

Let $g \in \mathcal{G}$ be a function such that (A_g) holds. Since $b^*(x) \to 0$ as $d(x) \searrow 0$, we deduce, by (A_1), the existence of some $\delta > 0$ such that for all $x \in \Omega$ with $0 < d(x) < \delta$ and $\xi > 1$

$$\frac{b^*(x) f(g(b^*(x))\xi)}{g''(b^*(x))\xi} > \sup_{\overline{\Omega}\setminus\Omega_0} |\nabla b^*|^2 + \frac{g'(b^*(x))}{g''(b^*(x))} \inf_{\overline{\Omega}\setminus\Omega_0} (\Delta b^*) + a \frac{g(b^*(x))}{g''(b^*(x))}.$$

Here, $\delta > 0$ is taken sufficiently small so that $g'(b^*(x)) < 0$ and $g''(b^*(x)) > 0$ for all x with $0 < d(x) < \delta$.

For $n_0 \geqslant 1$ fixed, define V^* as follows

(i) $V^*(x) = u_{n_0}(x) + 1$ for $x \in \overline{\Omega}$ and near $\partial\Omega$;

(ii) $V^*(x) = g(b^*(x))$ for x satisfying $0 < d(x) < \delta$;

(iii) $V^* \in C^2(\overline{\Omega} \setminus \overline{\Omega}_0)$ is positive on $\overline{\Omega} \setminus \overline{\Omega}_0$.

We show that for $\xi > 1$ large enough the upper bound of the sequence $(u_n(x))_n$ can be taken as $V(x) = \xi V^*(x)$. Since

$$\mathcal{B}V(x) = \xi \mathcal{B}V^*(x) \geqslant \xi \min\big\{1, \beta(x)\big\} \geqslant 0, \quad \forall x \in \partial\Omega \quad \text{and}$$
$$\lim_{d(x)\searrow 0} \big[u_n(x) - V(x)\big] = -\infty < 0,$$

to conclude that $u_n(x) \leqslant V(x)$ for all $x \in \overline{\Omega} \setminus \overline{\Omega}_0$ it is sufficient to show that

$$-\Delta V(x) \geqslant aV(x) - b(x) f\big(V(x)\big), \quad \forall x \in \Omega \setminus \overline{\Omega}_0. \tag{4.44}$$

For $x \in \Omega$ satisfying $0 < d(x) < \delta$ and $\xi > 1$ we have

$$\begin{aligned}
&-\Delta V(x) - aV(x) + b(x) f\big(V(x)\big) \\
&\quad = -\xi \Delta g\big(b^*(x)\big) - a\xi g\big(b^*(x)\big) + b(x) f\big(g\big(b^*(x)\big)\xi\big) \\
&\quad \geqslant \xi g''\big(b^*(x)\big) \bigg(-\frac{g'(b^*(x))}{g''(b^*(x))} \Delta b^*(x) - \big|\nabla b^*(x)\big|^2 - a \frac{g(b^*(x))}{g''(b^*(x))} \\
&\qquad + b^*(x) \frac{f(g(b^*(x))\xi)}{g''(b^*(x))\xi} \bigg) > 0.
\end{aligned}$$

For $x \in \Omega$ satisfying $d(x) \geqslant \delta$,

$$-\Delta V(x) - aV(x) + b(x) f\big(V(x)\big) = \xi\left(-\Delta V^*(x) - aV^*(x) + b(x)\frac{f(\xi V^*(x))}{\xi}\right) \geqslant 0$$

for ξ sufficiently large. It follows that (4.44) is fulfilled provided ξ is large enough. This finishes the proof of the lemma. □

By Lemma 4.7, $\underline{U}_a(x) \equiv \lim_{n\to\infty} u_n(x)$ exists, for any $x \in \overline{\Omega} \setminus \overline{\Omega}_0$. Moreover, $\underline{U}_a$ is a positive solution of (4.41). Using the maximum principle once more, we find that any positive solution u of (4.41) satisfies $u \geqslant u_n$ on $\overline{\Omega} \setminus \overline{\Omega}_0$, for all $n \geqslant 1$. Hence $\underline{U}_a$ is the minimal positive solution of (4.41).

PROOF OF THEOREM 4.5 COMPLETED.

Step 2. *Existence of the maximal positive solution for problem* (4.41).

LEMMA 4.8. *If Ω_0 is replaced by Ω_m defined in* (4.42), *then problem* (4.41) *has a minimal positive solution provided that* (A_1) *and* (A_2) *are fulfilled.*

PROOF. The argument used here (since $b > 0$ on $\overline{\Omega} \setminus \Omega_m$) is similar to that in Step 1. The only difference which appears in the proof (except the replacement of Ω_0 by Ω_m) is related to the construction of $V^*(x)$ for x near $\partial\Omega_m$. Here, we use our Theorem 4.1 which says that, for any $a \in \mathbb{R}$, there exists a positive large solution $u_{a,\infty}$ of problem (4.36) in the domain $\Omega \setminus \overline{\Omega}_m$. We define $V^*(x) = u_{a,\infty}(x)$ for $x \in \Omega \setminus \overline{\Omega}_m$ and near $\partial\Omega_m$. For $\xi > 1$ and $x \in \Omega \setminus \overline{\Omega}_m$ near $\partial\Omega_m$ we have

$$\begin{aligned}-\Delta V(x) - aV(x) + b(x) f\big(V(x)\big) &= -\xi\Delta V^*(x) - a\xi V^*(x) + b(x) f\big(\xi V^*(x)\big) \\ &= b(x)\big[f\big(\xi V^*(x)\big) - \xi f\big(V^*(x)\big)\big] \geqslant 0.\end{aligned}$$

This completes the proof. □

Let v_m be the minimal positive solution for the problem considered in the statement of Lemma 4.8. By the maximum principle, $v_m \geqslant v_{m+1} \geqslant u$ on $\overline{\Omega} \setminus \overline{\Omega}_m$, where u is any positive solution of (4.41). Hence $\overline{U}_a(x) := \lim_{m\to\infty} v_m(x) \geqslant u(x)$. A regularity and compactness argument shows that $\overline{U}_a$ is a positive solution of (4.41). Consequently, $\overline{U}_a$ is the maximal positive solution. This concludes the proof of Theorem 4.5. □

The next question is whether one can conclude the uniqueness of positive solutions of problem (4.41). We recall first what is already known in this direction. When $f(u) = u^p$, $p > 1$, Du and Huang [39] proved the uniqueness of solution to problem (4.41) and established its behavior near $\partial\Omega_0$, under the assumption

$$\lim_{d(x)\searrow 0} \frac{b(x)}{[d(x)]^\tau} = c \quad \text{for some positive constants } \tau, c > 0. \tag{4.45}$$

We shall give a general uniqueness result provided that b and f satisfy the following assumptions:

(B_1) $\lim_{d(x)\searrow 0} b(x)/k(d(x)) = c$ for some constant $c > 0$, where $0 < k \in C^1(0, \delta_0)$ is increasing and satisfies

(B_2) $K(t) = (\int_0^t \sqrt{k(s)}\,ds)/\sqrt{k(t)} \in C^1[0, \delta_0)$, for some $\delta_0 > 0$.

Assume there exist $\zeta > 0$ and $t_0 \geqslant 1$ such that

(A_3) $f(\xi t) \leqslant \xi^{1+\zeta} f(t),\ \forall \xi \in (0,1),\ \forall t \geqslant t_0/\xi$

(A_4) the mapping $(0, 1] \ni \xi \mapsto A(\xi) = \lim_{u\to\infty}(f(\xi u)/\xi f(u))$ is a continuous positive function.

Our uniqueness result is

THEOREM 4.9. *Assume the conditions* ($\tilde{A}_1$) *with* $\gamma \neq 0$, (A_3), (A_4), (B_1) *and* (B_2) *hold. Then, for any* $a \in \mathbb{R}$, *problem* (4.41) *has a unique positive solution* U_a. *Moreover,*

$$\lim_{d(x)\searrow 0} \frac{U_a(x)}{h(d(x))} = \xi_0,$$

where h *is defined by*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t \sqrt{k(s)}\,ds, \quad \forall t \in (0, \delta_0) \tag{4.46}$$

and ξ_0 *is the unique positive solution of* $A(\xi) = (K'(0)(1-2\gamma) + 2\gamma)/c$.

REMARK 2.

(a) (A_1) + (A_3) $\Rightarrow$ (A_2). Indeed, $\lim_{u\to\infty} f(u)/u^{1+\zeta} > 0$ since $f(t)/t^{1+\zeta}$ is nondecreasing for $t \geqslant t_0$.

(b) $K'(0)(1-2\gamma) + 2\gamma \in (0, 1]$ when ($\tilde{A}_1$) with $\gamma \neq 0$, (A_2), (B_1) and (B_2) hold.

(c) The function $(0, \infty) \ni \xi \mapsto A(\xi) \in (0, \infty)$ is bijective when (A_3) and (A_4) hold (see Lemma 4.10).

Among the nonlinearities f that satisfy the assumptions of Theorem 4.9 we note: (i) $f(u) = u^p$, $p > 1$; (ii) $f(u) = u^p \ln(u+1)$, $p > 1$; (iii) $f(u) = u^p \arctan u$, $p > 1$.

PROOF OF THEOREM 4.9. By (A_4) we deduce that the mapping $(0, \infty) \ni \xi \mapsto A(\xi) = \lim_{u\to\infty} f(\xi u)/(\xi f(u))$ is a continuous positive function, since $A(1/\xi) = 1/A(\xi)$ for any $\xi \in (0, 1)$. Moreover, we claim

LEMMA 4.10. *The function* $A : (0, \infty) \to (0, \infty)$ *is bijective, provided that* (A_3) *and* (A_4) *are fulfilled.*

PROOF. By the continuity of A, we see that the surjectivity of A follows if we prove that $\lim_{\xi\searrow 0} A(\xi) = 0$. To this aim, let $\xi \in (0, 1)$ be fixed. Using (A_3) we find

$$\frac{f(\xi u)}{\xi f(u)} \leqslant \xi^{\zeta}, \quad \forall u \geqslant \frac{t_0}{\xi}$$

which yields $A(\xi) \leqslant \xi^{\zeta}$. Since $\xi \in (0,1)$ is arbitrary, it follows that $\lim_{\xi \searrow 0} A(\xi) = 0$.

We now prove that the function $\xi \mapsto A(\xi)$ is increasing on $(0,\infty)$ which concludes our lemma. Let $0 < \xi_1 < \xi_2 < \infty$ be chosen arbitrarily. Using assumption (A_3) once more, we obtain

$$f(\xi_1 u) = f\left(\frac{\xi_1}{\xi_2}\xi_2 u\right) \leqslant \left(\frac{\xi_1}{\xi_2}\right)^{1+\zeta} f(\xi_2 u), \quad \forall u \geqslant t_0 \frac{\xi_2}{\xi_1}.$$

It follows that

$$\frac{f(\xi_1 u)}{\xi_1 f(u)} \leqslant \left(\frac{\xi_1}{\xi_2}\right)^{\zeta} \frac{f(\xi_2 u)}{\xi_2 f(u)}, \quad \forall u \geqslant t_0 \frac{\xi_2}{\xi_1}.$$

Passing to the limit as $u \to \infty$ we find

$$A(\xi_1) \leqslant \left(\frac{\xi_1}{\xi_2}\right)^{\zeta} A(\xi_2) < A(\xi_2),$$

which finishes the proof. □

PROOF OF THEOREM 4.9 COMPLETED. Set

$$\Pi(\xi) = \lim_{d(x)\searrow 0} b(x) \frac{f(h(d(x))\xi)}{h''(d(x))\xi},$$

for any $\xi > 0$. Using (B_1) we find

$$\begin{aligned}\Pi(\xi) &= \lim_{d(x)\searrow 0} \frac{b(x)}{k(d(x))} \frac{k(d(x))f(h(d(x)))}{h''(d(x))} \frac{f(h(d(x))\xi)}{\xi f(h(d(x)))} \\ &= c \lim_{t\searrow 0} \frac{k(t)f(h(t))}{h''(t)} \lim_{u\to\infty} \frac{f(\xi u)}{\xi f(u)} = \frac{c}{K'(0)(1-2\gamma)+2\gamma} A(\xi).\end{aligned}$$

This and Lemma 4.10 imply that the function $\Pi : (0,\infty) \to (0,\infty)$ is bijective. Let ξ_0 be the unique positive solution of $\Pi(\xi) = 1$, that is $A(\xi_0) = (K'(0)(1-2\gamma)+2\gamma)/c$.

For $\varepsilon \in (0, 1/4)$ arbitrary, we denote $\xi_1 = \Pi^{-1}(1-4\varepsilon)$, respectively $\xi_2 = \Pi^{-1}(1+4\varepsilon)$. We choose $\delta > 0$ small enough such that

(i) $\operatorname{dist}(x, \partial\Omega_0)$ is a C^2 function on the set $\{x \in \Omega\colon \operatorname{dist}(x,\partial\Omega_0) \leqslant 2\delta\}$;

(ii) $$\left|\frac{h'(s)}{h''(s)}\Delta d(x) + a\frac{h(s)}{h''(s)}\right| < \varepsilon \quad \text{and} \quad h''(s) > 0$$
for all $s \in (0, 2\delta)$ and x satisfying $0 < d(x) < 2\delta$;

(iii) $$\big(\Pi(\xi_2) - \varepsilon\big)\frac{h''(d(x))\xi_2}{f(h(d(x))\xi_2)} \leqslant b(x) \leqslant \big(\Pi(\xi_1) + \varepsilon\big)\frac{h''(d(x))\xi_1}{f(h(d(x))\xi_1)},$$
for every x with $0 < d(x) < 2\delta$.

(iv) $b(y) < (1+\varepsilon)b(x)$, for every x, y with $0 < d(y) < d(x) < 2\delta$.

Let $\sigma \in (0, \delta)$ be arbitrary. We define $\underline{v}_\sigma(x) = h(d(x)+\sigma)\xi_1$, for any x with $d(x)+\sigma < 2\delta$, respectively $\bar{v}_\sigma(x) = h(d(x)-\sigma)\xi_2$ for any x with $\sigma < d(x) < 2\delta$.

Using (ii), (iv) and the first inequality in (iii), when $\sigma < d(x) < 2\delta$, we obtain (since $|\nabla d(x)| \equiv 1$)

$$
\begin{aligned}
&-\Delta \bar{v}_\sigma(x) - a\bar{v}_\sigma(x) + b(x) f\big(\bar{v}_\sigma(x)\big) \\
&\quad = \xi_2 \bigg(-h'\big(d(x)-\sigma\big)\Delta d(x) - h''\big(d(x)-\sigma\big) - ah\big(d(x)-\sigma\big) \\
&\qquad + \frac{b(x) f(h(d(x)-\sigma)\xi_2)}{\xi_2} \bigg) \\
&\quad = \xi_2 h''\big(d(x)-\sigma\big) \bigg(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\Delta d(x) - a\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 \\
&\qquad + \frac{b(x) f(h(d(x)-\sigma)\xi_2)}{h''(d(x)-\sigma)\xi_2} \bigg) \\
&\quad \geqslant \xi_2 h''\big(d(x)-\sigma\big) \bigg(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\Delta d(x) - a\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 \\
&\qquad + \frac{\Pi(\xi_2) - \varepsilon}{1+\varepsilon} \bigg) \geqslant 0
\end{aligned}
$$

for all x satisfying $\sigma < d(x) < 2\delta$.

Similarly, using (ii), (iv) and the second inequality in (iii), when $d(x) + \sigma < 2\delta$ we find

$$
\begin{aligned}
&-\Delta \underline{v}_\sigma(x) - a\underline{v}_\sigma(x) + b(x) f\big(\underline{v}_\sigma(x)\big) \\
&\quad = \xi_1 h''\big(d(x)+\sigma\big) \bigg(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\Delta d(x) \\
&\qquad - a\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} - 1 + \frac{b(x) f(h(d(x)+\sigma)\xi_1)}{h''(d(x)+\sigma)\xi_1} \bigg) \\
&\quad \leqslant \xi_1 h''\big(d(x)+\sigma\big) \bigg(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\Delta d(x) \\
&\qquad - a\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} - 1 + (1+\varepsilon)\big(\Pi(\xi_1) + \varepsilon\big) \bigg) \leqslant 0,
\end{aligned}
$$

for all x satisfying $d(x) + \sigma < 2\delta$.

Define $\Omega_\delta \equiv \{x \in \Omega\colon\ d(x) < \delta\}$. Let $\omega \Subset \Omega_0$ be such that the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\Omega_0 \setminus \overline{\omega}$ is strictly greater than a. Denote by w a positive large solution to the following problem

$$
-\Delta w = aw - p(x) f(w) \quad \text{in } \Omega_\delta,
$$

where $p \in C^{0,\mu}(\overline{\Omega}_\delta)$ satisfies $0 < p(x) \leqslant b(x)$ for $x \in \overline{\Omega}_\delta \setminus \overline{\Omega}_0$, $p(x) = 0$ on $\overline{\Omega}_0 \setminus \omega$ and $p(x) > 0$ for $x \in \omega$. The existence of w is guaranteed by our Theorem 4.1.

Suppose that u is an arbitrary solution of (4.41) and let $v := u + w$. Then v satisfies

$$-\Delta v \geqslant av - b(x) f(v) \quad \text{in } \Omega_\delta \setminus \overline{\Omega}_0.$$

Since

$$v|_{\partial\Omega_0} = \infty > \underline{v}_\sigma|_{\partial\Omega_0} \quad \text{and} \quad v|_{\partial\Omega_\delta} = \infty > \underline{v}_\sigma|_{\partial\Omega_\delta},$$

we find

$$u + w \geqslant \underline{v}_\sigma \quad \text{on } \Omega_\delta \setminus \overline{\Omega}_0. \tag{4.47}$$

Similarly

$$\bar{v}_\sigma + w \geqslant u \quad \text{on } \Omega_\delta \setminus \overline{\Omega}_\sigma. \tag{4.48}$$

Letting $\sigma \to 0$ in (4.47) and (4.48), we deduce

$$h\big(d(x)\big)\xi_2 + 2w \geqslant u + w \geqslant h\big(d(x)\big)\xi_1, \quad \forall x \in \Omega_\delta \setminus \overline{\Omega}_0.$$

Since w is uniformly bounded on $\partial\Omega_0$, it follows that

$$\xi_1 \leqslant \liminf_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} \leqslant \limsup_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} \leqslant \xi_2. \tag{4.49}$$

Letting $\varepsilon \to 0$ in (4.49) and looking at the definition of ξ_1 respectively ξ_2 we find

$$\lim_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} = \xi_0. \tag{4.50}$$

This behavior of the solution will be speculated in order to prove that problem (4.41) has a unique solution. Indeed, let u_1, u_2 be two positive solutions of (4.41). For any $\varepsilon > 0$, denote $\tilde{u}_i = (1+\varepsilon)u_i$, $i = 1, 2$. By virtue of (4.50) we get

$$\lim_{d(x)\searrow 0} \frac{u_1(x) - \tilde{u}_2(x)}{h(d(x))} = \lim_{d(x)\searrow 0} \frac{u_2(x) - \tilde{u}_1(x)}{h(d(x))} = -\varepsilon\xi_0 < 0$$

which implies

$$\lim_{d(x)\searrow 0} \big[u_1(x) - \tilde{u}_2(x)\big] = \lim_{d(x)\searrow 0} \big[u_2(x) - \tilde{u}_1(x)\big] = -\infty.$$

On the other hand, since $f(u)/u$ is increasing for $u > 0$, we obtain

$$\begin{aligned} -\Delta\tilde{u}_i &= -(1+\varepsilon)\Delta u_i = (1+\varepsilon)\big(au_i - b(x)f(u_i)\big) \\ &\geqslant a\tilde{u}_i - b(x)f(\tilde{u}_i) \quad \text{in } \Omega \setminus \overline{\Omega}_0, \\ \mathcal{B}\tilde{u}_i &= \mathcal{B}u_i = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

So, by the maximum principle,

$$u_1(x) \leqslant \tilde{u}_2(x), \quad u_2(x) \leqslant \tilde{u}_1(x), \quad \forall x \in \Omega \setminus \overline{\Omega}_0.$$

Letting $\varepsilon \to 0$, we obtain $u_1 \equiv u_2$. The proof of Theorem 4.9 is complete. □

The above results have been established by Cîrstea and Rădulescu [27,29].

4.1. *Uniqueness and asymptotic behavior of the large solution. A Karamata regular variation theory approach*

The major purpose in this section is to advance innovative methods to study the uniqueness and asymptotic behavior of large solutions of (4.36). This approach is due to Cîrstea and Rădulescu [25,28,30–32] and it relies essentially on the *regular variation theory* introduced by Karamata (see Bingham, Goldie and Teugels [13], Karamata [71]), not only in the statement but in the proof as well. This enables us to obtain significant information about the qualitative behavior of the large solution to (4.36) in a general framework that removes previous restrictions in the literature.

DEFINITION 4.11. A positive measurable function R defined on $[D, \infty)$, for some $D > 0$, is called regularly varying (at infinity) with index $q \in \mathbb{R}$ (written $R \in RV_q$) if for all $\xi > 0$

$$\lim_{u\to\infty} R(\xi u)/R(u) = \xi^q.$$

When the index of regular variation q is zero, we say that the function is slowly varying.

We remark that any function $R \in RV_q$ can be written in terms of a slowly varying function. Indeed, set $R(u) = u^q L(u)$. From the above definition we easily deduce that L varies slowly.

The canonical q-varying function is u^q. The functions $\ln(1+u)$, $\ln\ln(\mathrm{e}+u)$, $\exp\{(\ln u)^\alpha\}$, $\alpha \in (0,1)$ vary slowly, as well as any measurable function on $[D,\infty)$ with positive limit at infinity.

In what follows L denotes an arbitrary slowly varying function and $D > 0$ a positive number. For details on the below properties, we refer to Seneta [85].

PROPOSITION 4.12.

(i) *For any* $m > 0$, $u^m L(u) \to \infty$, $u^{-m} L(u) \to 0$ *as* $u \to \infty$.

(ii) *Any positive C^1-function on $[D,\infty)$ satisfying $uL_1'(u)/L_1(u)\to 0$ as $u\to\infty$ is slowly varying. Moreover, if the above limit is $q\in\mathbb{R}$, then $L_1\in RV_q$.*

(iii) *Assume $R:[D,\infty)\to(0,\infty)$ is measurable and Lebesgue integrable on each finite subinterval of $[D,\infty)$. Then R varies regularly iff there exists $j\in\mathbb{R}$ such that*

$$\lim_{u\to\infty}\frac{u^{j+1}R(u)}{\int_D^u x^j R(x)\,dx} \tag{4.51}$$

exists and is a positive number, say a_j+1. In this case, $R\in RV_q$ with $q=a_j-j$.

(iv) (*Karamata Theorem, 1933*) *If $R\in RV_q$ is Lebesgue integrable on each finite subinterval of $[D,\infty)$, then the limit defined by* (4.51) *is $q+j+1$, for every $j>-q-1$.*

LEMMA 4.13. *Assume* (A_1) *holds. Then we have the equivalence*

$$\text{(a) } f'\in RV_\rho \Longleftrightarrow \text{(b) } \lim_{u\to\infty} uf'(u)/f(u):=\vartheta<\infty$$
$$\Longleftrightarrow \text{(c) } \lim_{u\to\infty}(F/f)'(u):=\gamma>0.$$

REMARK 3. Let (a) of Lemma 4.13 be fulfilled. Then the following assertions hold

(i) ρ is nonnegative;

(ii) $\gamma=1/(\rho+2)=1/(\vartheta+1)$;

(iii) If $\rho\neq 0$, then (A_2) holds (use $\lim_{u\to\infty} f(u)/u^p=\infty$, $\forall p\in(1,1+\rho)$). The converse implication is not necessarily true (take $f(u)=u\ln^4(u+1)$). However, there are cases when $\rho=0$ and (A_2) fails so that (4.36) has **no** large solutions. This is illustrated by $f(u)=u$ or $f(u)=u\ln(u+1)$.

Inspired by the definition of γ, we denote by $\mathcal{K}$ the set of all positive, increasing C^1-functions k defined on $(0,\nu)$, for some $\nu>0$, which satisfy

$$\lim_{t\to 0^+}\left(\left(\int_0^t k(s)\,ds\right)\Big/k(t)\right)^{(i)}:=\ell_i,\quad i=\overline{0,1}.$$

It is easy to see that $\ell_0=0$ and $\ell_1\in[0,1]$, for every $k\in\mathcal{K}$. Our next result gives examples of functions $k\in\mathcal{K}$ with $\lim_{t\to 0^+}k(t)=0$, for *every* $\ell_1\in[0,1]$.

LEMMA 4.14. *Let $S\in C^1[D,\infty)$ be such that $S'\in RV_q$ with $q>-1$. Hence the following hold*:

(a) *If $k(t)=\exp\{-S(1/t)\}$ $\forall t\leqslant 1/D$, then $k\in\mathcal{K}$ with $\ell_1=0$.*

(b) *If $k(t)=1/S(1/t)$ $\forall t\leqslant 1/D$, then $k\in\mathcal{K}$ with $\ell_1=1/(q+2)\in(0,1)$.*

(c) *If $k(t)=1/\ln S(1/t)$ $\forall t\leqslant 1/D$, then $k\in\mathcal{K}$ with $\ell_1=1$.*

REMARK 4. If $S\in C^1[D,\infty)$, then $S'\in RV_q$ with $q>-1$ iff for some $m>0$, $C>0$ and $B>D$ we have $S(u)=Cu^m\exp\{\int_B^u(y(t)/t)\,dt\}$, $\forall u\geqslant B$, where $y\in C[B,\infty)$ satisfies $\lim_{u\to\infty}y(u)=0$. In this case, $S'\in RV_q$ with $q=m-1$. (This is a consequence of Proposition 4.12 (iii) and (iv).)

Our main result is

THEOREM 4.15. *Let* (A_1) *hold and* $f' \in RV_\rho$ *with* $\rho > 0$. *Assume* $b \equiv 0$ *on* $\partial\Omega$ *satisfies*
(B) $b(x) = ck^2(d(x)) + o(k^2(d(x)))$ *as* $d(x) \to 0$, *for some constant* $c > 0$ *and* $k \in \mathcal{K}$.
Then, for any $a \in (-\infty, \lambda_{\infty,1})$, *equation* (4.36) *admits a unique large solution* u_a. *Moreover,*

$$\lim_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} = \xi_0, \tag{4.52}$$

where $\xi_0 = ((2 + \ell_1\rho)/(c(2+\rho)))^{1/\rho}$ *and* h *is defined by*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t k(s)\,ds, \quad \forall t \in (0, \nu). \tag{4.53}$$

By Remark 4, the assumption $f' \in RV_\rho$ with $\rho > 0$ holds if and only if there exist $p > 1$ and $B > 0$ such that $f(u) = Cu^p \exp\{\int_B^u (y(t))/t\,dt\}$, for all $u \geqslant B$ (y as before and $p = \rho + 1$). If B is large enough ($y > -\rho$ on $[B, \infty)$), then $f(u)/u$ is increasing on $[B, \infty)$. Thus, to get the whole range of functions f for which our Theorem 4.15 applies we have only to "paste" a suitable smooth function on $[0, B]$ in accordance with (A_1). A simple way to do this is to define $f(u) = u^p \exp\{\int_0^u (z(t)/t)\,dt\}$, for all $u \geqslant 0$, where $z \in C[0, \infty)$ is nonnegative such that $\lim_{t\to 0^+} z(t)/t \in [0, \infty)$ and $\lim_{u\to\infty} z(u) = 0$. Clearly, $f(u) = u^p$, $f(u) = u^p \ln(u+1)$, and $f(u) = u^p \arctan u$ ($p > 1$) fall into this category.

Lemma 4.14 provides a practical method to find functions k which can be considered in the statement of Theorem 4.15. Here are some examples:

$$k(t) = -\frac{1}{\ln t}, \quad k(t) = t^\alpha, \quad k(t) = \exp\left\{-\frac{1}{t^\alpha}\right\},$$
$$k(t) = \exp\left\{-\frac{\ln(1+1/t)}{t^\alpha}\right\}, \quad k(t) = \exp\left\{-\frac{\arctan(1/t)}{t^\alpha}\right\},$$
$$k(t) = \frac{t^\alpha}{\ln(1+1/t)},$$

for some $\alpha > 0$.

As we shall see, the uniqueness lies upon the crucial observation (4.52), which shows that all explosive solutions have the same boundary behavior. Note that the only case of Theorem 4.15 studied so far is $f(u) = u^p$ ($p > 1$) and $k(t) = t^\alpha$ ($\alpha > 0$) (see García-Melián, Letelier-Albornoz and Sabina de Lis [43]). For related results on the uniqueness of explosive solutions (mainly in the cases $b \equiv 1$ and $a = 0$) we refer to Bandle and Marcus [8], Loewner and Nirenberg [72], Marcus and Véron [74].

PROOF OF LEMMA 4.13. From Property 4.12(iv) and Remark 3(i) we deduce (a) $\Rightarrow$ (b) and $\vartheta = \rho + 1$. Conversely, (b) $\Rightarrow$ (a) follows by 4.12(iii) since $\vartheta \geqslant 1$ cf. (A_1).

(b) $\Longrightarrow$ (c). Indeed, $\lim_{u\to\infty} uf(u)/F(u) = 1+\vartheta$, which yields $\vartheta/(1+\vartheta) = \lim_{u\to\infty}[1-(F/f)'(u)] = 1-\gamma$.

(c) $\Longrightarrow$ (b). Choose $s_1 > 0$ such that $(F/f)'(u) \geqslant \gamma/2$, $\forall u \geqslant s_1$. So, $(F/f)(u) \geqslant (u-s_1)\gamma/2 + (F/f)(s_1)$, $\forall u \geqslant s_1$. Passing to the limit $u\to\infty$, we find

$$\lim_{u\to\infty} \frac{F(u)}{f(u)} = \infty.$$

Thus, $\lim_{u\to\infty} uf(u)/F(u) = \frac{1}{\gamma}$. Since $1-\gamma := \lim_{u\to\infty} F(u)f'(u)/f^2(u)$, we obtain $\lim_{u\to\infty} uf'(u)/f(u) = 1-\gamma/\gamma$. □

PROOF OF LEMMA 4.14. Since $\lim_{u\to\infty} uS'(u) = \infty$ (cf. Property 4.12(i)), from Karamata Theorem we deduce $\lim_{u\to\infty} uS'(u)/S(u) = q+1 > 0$. Therefore, in any of the cases (a)–(c), $\lim_{t\to 0^+} k(t) = 0$ and k is an increasing C^1-function on $(0,\nu)$, for $\nu > 0$ sufficiently small.

(a) It is clear that

$$\lim_{t\to 0^+} \frac{tk'(t)}{k(t)\ln k(t)} = \lim_{t\to 0^+} \frac{-S'(1/t)}{tS(1/t)} = -(q+1).$$

By l'Hospital's rule,

$$\ell_0 = \lim_{t\to 0^+} \frac{k(t)}{k'(t)} = 0 \quad \text{and} \quad \lim_{t\to 0^+} \frac{(\int_0^t k(s)\,\mathrm{d}s)\ln k(t)}{tk(t)} = -\frac{1}{q+1}.$$

So,

$$1-\ell_1 := \lim_{t\to 0^+} \frac{(\int_0^t k(s)\,\mathrm{d}s)k'(t)}{k^2(t)} = 1.$$

(b) We see that

$$\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \lim_{t\to 0^+} \frac{S'(1/t)}{tS(1/t)} = q+1.$$

By l'Hospital's rule, $\ell_0 = 0$ and

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)} = \frac{1}{q+2}.$$

So,

$$\ell_1 = 1 - \lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)} \frac{tk'(t)}{k(t)} = \frac{1}{q+2}.$$

(c) We have

$$\lim_{t\to 0^+} \frac{tk'(t)}{k^2(t)} = \lim_{t\to 0^+} \frac{S'(1/t)}{tS(1/t)} = q+1.$$

By l'Hospital's rule,

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)} = 1.$$

Thus, $\ell_0 = 0$ and

$$\ell_1 = 1 - \lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{t}\frac{tk'(t)}{k^2(t)} = 1. \qquad \square$$

PROOF OF THEOREM 4.15. Fix $a \in (-\infty, \lambda_{\infty,1})$. By Theorem 4.1, problem (4.36) has at least a large solution.

If we prove that (4.52) holds for an *arbitrary* large solution u_a of (4.36), then the uniqueness follows easily. Indeed, if u_1 and u_2 are two arbitrary large solutions of (4.36), then (4.52) yields $\lim_{d(x)\to 0^+}(u_1(x)/u_2(x)) = 1$. Hence, for any $\varepsilon \in (0,1)$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$(1-\varepsilon)u_2(x) \leqslant u_1(x) \leqslant (1+\varepsilon)u_2(x), \quad \forall x \in \Omega \text{ with } 0 < d(x) \leqslant \delta. \tag{4.54}$$

Choosing eventually a smaller $\delta > 0$, we can assume that $\overline{\Omega}_0 \subset C_\delta$, where $C_\delta := \{x \in \Omega: d(x) > \delta\}$.

It is clear that u_1 is a positive solution of the boundary value problem

$$\Delta\phi + a\phi = b(x)f(\phi) \quad \text{in } C_\delta, \qquad \phi = u_1 \quad \text{on } \partial C_\delta. \tag{4.55}$$

By (A_1) and (4.54), we see that $\phi^- = (1-\varepsilon)u_2$ (resp., $\phi^+ = (1+\varepsilon)u_2$) is a positive sub-solution (resp., super-solution) of (4.55). By the sub and super-solutions method, (4.55) has a positive solution ϕ_1 satisfying $\phi^- \leqslant \phi_1 \leqslant \phi^+$ in C_δ. Since $b > 0$ on $\overline{C}_\delta \setminus \overline{\Omega}_0$, we deduce that (4.55) has a *unique* positive solution, that is, $u_1 \equiv \phi_1$ in C_δ. This yields $(1-\varepsilon)u_2(x) \leqslant u_1(x) \leqslant (1+\varepsilon)u_2(x)$ in C_δ, so that (4.54) holds in Ω. Passing to the limit $\varepsilon \to 0^+$, we conclude that $u_1 \equiv u_2$.

In order to prove (4.52) we state some useful properties about h:

(h_1) $h \in C^2(0,\nu)$, $\lim_{t\to 0^+} h(t) = \infty$ (straightforward from (4.53)).

(h_2) $$\lim_{t\to 0^+} \frac{h''(t)}{k^2(t)f(h(t)\xi)} = \frac{1}{\xi^{\rho+1}}\frac{2+\rho\ell_1}{2+\rho},$$
$\forall \xi > 0$ (so, $h'' > 0$ on $(0, 2\delta)$, for $\delta > 0$ small enough).

(h_3) $\lim_{t\to 0^+} h(t)/h''(t) = \lim_{t\to 0^+} h'(t)/h''(t) = 0$.

We check (h_2) for $\xi = 1$ only, since $f \in RV_{\rho+1}$. Clearly, $h'(t) = -k(t)\sqrt{2F(h(t))}$ and

$$h''(t) = k^2(t) f(h(t)) \left(1 - 2 \frac{k'(t) (\int_0^t k(s)\,\mathrm{d}s)}{k^2(t)} \frac{\sqrt{F(h(t))}}{f(h(t)) \int_{h(t)}^{\infty} [F(s)]^{-1/2}\,\mathrm{d}s} \right)$$

$$\forall t \in (0, \nu). \tag{4.56}$$

We see that $\lim_{u\to\infty} \sqrt{F(u)}/f(u) = 0$. Thus, from l'Hospital's rule and Lemma 4.13 we infer that

$$\lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u) \int_u^{\infty} [F(s)]^{-1/2}\,\mathrm{d}s} = \frac{1}{2} - \gamma = \frac{\rho}{2(\rho+2)}. \tag{4.57}$$

Using (4.56) and (4.57) we derive (h_2) and also

$$\begin{aligned} \lim_{t\to 0^+} \frac{h'(t)}{h''(t)} &= \frac{-2(2+\rho)}{2+\ell_1\rho} \lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)} \lim_{u\to\infty} \frac{\sqrt{F(u)}}{f(u) \int_u^{\infty} [F(s)]^{-1/2}\,\mathrm{d}s} \\ &= \frac{-\rho\ell_0}{2+\ell_1\rho} = 0. \end{aligned} \tag{4.58}$$

From (h_1) and (h_2), $\lim_{t\to 0^+} h'(t) = -\infty$. So, l'Hospital's rule and (4.58) yield $\lim_{t\to 0^+} h(t)/h'(t) = 0$. This and (4.58) lead to $\lim_{t\to 0^+} h(t)/h''(t) = 0$ which proves (h_3).

PROOF OF (4.52). Fix $\varepsilon \in (0, c/2)$. Since $b \equiv 0$ on $\partial\Omega$ and (B) holds, we take $\delta > 0$ so that

(i) $d(x)$ is a C^2-function on the set $\{x \in \mathbb{R}^N\colon\ d(x) < 2\delta\}$;
(ii) k^2 is increasing on $(0, 2\delta)$;
(iii) $(c-\varepsilon)k^2(d(x)) < b(x) < (c+\varepsilon)k^2(d(x))$, $\forall x \in \Omega$ with $0 < d(x) < 2\delta$;
(iv) $h''(t) > 0$ $\forall t \in (0, 2\delta)$ (from (h_2)).

Let $\sigma \in (0, \delta)$ be arbitrary. We define $\xi^{\pm} = [(2+\ell_1\rho)/((c \mp 2\varepsilon)(2+\rho))]^{1/\rho}$ and $v_\sigma^-(x) = h(d(x) + \sigma)\xi^-$, for all x with $d(x) + \sigma < 2\delta$ resp., $v_\sigma^+(x) = h(d(x) - \sigma)\xi^+$, for all x with $\sigma < d(x) < 2\delta$.

Using (i)–(iv), when $\sigma < d(x) < 2\delta$ we obtain (since $|\nabla d(x)| \equiv 1$)

$$\begin{aligned} \Delta v_\sigma^+ + a v_\sigma^+ - b(x) f\big(v_\sigma^+\big) \leqslant \xi^+ h''\big(d(x) - \sigma\big) \Bigg(& \frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)} \Delta d(x) \\ & + a \frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} + 1 - (c-\varepsilon) \frac{k^2(d(x)-\sigma) f(h(d(x)-\sigma)\xi^+)}{h''(d(x)-\sigma)\xi^+} \Bigg). \end{aligned}$$

Similarly, when $d(x) + \sigma < 2\delta$ we find

$$\begin{aligned} \Delta v_\sigma^- + a v_\sigma^- - b(x) f\big(v_\sigma^-\big) \geqslant \xi^- h''\big(d(x) + \sigma\big) \Bigg(& \frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)} \Delta d(x) \\ & + a \frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} + 1 - (c+\varepsilon) \frac{k^2(d(x)+\sigma) f(h(d(x)+\sigma)\xi^-)}{h''(d(x)+\sigma)\xi^-} \Bigg). \end{aligned}$$

Using (h_2) and (h_3) we see that, by diminishing δ, we can assume

$$\Delta v_\sigma^+(x) + a v_\sigma^+(x) - b(x) f\left(v_\sigma^+(x)\right) \leqslant 0 \quad \forall x \text{ with } \sigma < d(x) < 2\delta;$$
$$\Delta v_\sigma^-(x) + a v_\sigma^-(x) - b(x) f\left(v_\sigma^-(x)\right) \geqslant 0 \quad \forall x \text{ with } d(x) + \sigma < 2\delta.$$

Let Ω_1 and Ω_2 be smooth bounded domains such that $\Omega \Subset \Omega_1 \subset\subset \Omega_2$ and the first Dirichlet eigenvalue of $(-\Delta)$ in the domain $\Omega_1 \setminus \overline{\Omega}$ is greater than a. Let $p \in C^{0,\mu}(\overline{\Omega}_2)$ satisfy $0 < p(x) \leqslant b(x)$ for $x \in \Omega \setminus C_{2\delta}$, $p = 0$ on $\overline{\Omega}_1 \setminus \Omega$ and $p > 0$ on $\Omega_2 \setminus \overline{\Omega}_1$. Denote by w a positive large solution of

$$\Delta w + a w = p(x) f(w) \quad \text{in } \Omega_2 \setminus \overline{C}_{2\delta}.$$

The existence of w is ensured by Theorem 4.1.

Suppose that u_a is an arbitrary large solution of (4.36) and let $v := u_a + w$. Then v satisfies

$$\Delta v + a v - b(x) f(v) \leqslant 0 \quad \text{in } \Omega \setminus \overline{C}_{2\delta}.$$

Since $v_{|\partial\Omega} = \infty > v_{\sigma|\partial\Omega}^-$ and $v_{|\partial C_{2\delta}} = \infty > v_{\sigma|\partial C_{2\delta}}^-$, the maximum principle implies

$$u_a + w \geqslant v_\sigma^- \quad \text{on } \Omega \setminus \overline{C}_{2\delta}. \tag{4.59}$$

Similarly,

$$v_\sigma^+ + w \geqslant u_a \quad \text{on } C_\sigma \setminus \overline{C}_{2\delta}. \tag{4.60}$$

Letting $\sigma \to 0$ in (4.59) and (4.60), we deduce $h(d(x))\xi^+ + 2w \geqslant u_a + w \geqslant h(d(x))\xi^-$, for all $x \in \Omega \setminus \overline{C}_{2\delta}$. Since w is uniformly bounded on $\partial\Omega$, we have

$$\xi^- \leqslant \liminf_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leqslant \limsup_{d(x)\to 0} \frac{u_a(x)}{h(d(x))} \leqslant \xi^+.$$

Letting $\varepsilon \to 0^+$ we obtain (4.52). This concludes the proof of Theorem 4.15. □

Bandle and Marcus proved in [9] that the blow-up rate of the unique large solution of (4.36) depends on the curvature of the boundary of Ω. Our purpose in what follows is to refine the blow-up rate of u_a near $\partial\Omega$ by giving the second term in its expansion near the boundary. This is a more subtle question which represents the goal of more recent literature (see García-Melián, Letelier-Albornoz and Sabina de Lis [43] and the references therein). The following is very general and, as a novelty, it relies on the Karamata regular variation theory.

Recall that $\mathcal{K}$ denotes the set of all positive increasing C^1-functions k defined on $(0, \nu)$, for some $\nu > 0$, which satisfy $\lim_{t\searrow 0}(\int_0^t k(s)\,ds/k(t))^{(i)} := \ell_i$, $i \in \overline{0,1}$. We also recall that RV_q ($q \in \mathbb{R}$) is the set of all positive measurable functions $Z : [A, \infty) \to \mathbb{R}$ (for some

$A > 0$) satisfying $\lim_{u\to\infty} Z(\xi u)/Z(u) = \xi^q$, $\forall \xi > 0$. Define by NRV_q the class of functions f in the form $f(u) = Cu^q \exp\{\int_B^u \phi(t)/t\,\mathrm{d}t\}$, $\forall u \geqslant B > 0$, where $C > 0$ is a constant and $\phi \in C[B, \infty)$ satisfies $\lim_{t\to\infty} \phi(t) = 0$. The Karamata Representation Theorem shows that $NRV_q \subset RV_q$.

For any $\zeta > 0$, set $\mathcal{K}_{0,\zeta}$ the subset of $\mathcal{K}$ with $\ell_1 = 0$ and $\lim_{t\searrow 0} t^{-\zeta}(\int_0^t k(s)\,\mathrm{d}s/k(t))' := L_\star \in \mathbb{R}$. It can be proven that $\mathcal{K}_{0,\zeta} \equiv \mathcal{R}_{0,\zeta}$, where

$$\mathcal{R}_{0,\zeta} = \left\{ \begin{array}{l} k\colon\ k(u^{-1}) = d_0 u[\Lambda(u)]^{-1} \exp[-\int_{d_1}^u (s\Lambda(s))^{-1}\,\mathrm{d}s]\ (u \geqslant d_1), \\ \quad 0 < \Lambda \in C^1[d_1, \infty), \\ \lim_{u\to\infty} \Lambda(u) = \lim_{u\to\infty} u\Lambda'(u) = 0, \\ \lim_{u\to\infty} u^{\zeta+1}\Lambda'(u) = \ell_\star \in \mathbb{R}, \quad d_0, d_1 > 0. \end{array} \right\}$$

Define

$$\mathcal{F}_{\rho\eta} = \left\{ f \in NRV_{\rho+1}\ (\rho > 0)\colon\ \phi \in RV_\eta \text{ or } -\phi \in RV_\eta \right\}, \quad \eta \in (-\rho - 2, 0];$$

$$\mathcal{F}_{\rho 0,\tau} = \left\{ f \in \mathcal{F}_{\rho 0}\colon\ \lim_{u\to\infty} (\ln u)^\tau \phi(u) = \ell^\star \in \mathbb{R} \right\}, \quad \tau \in (0, \infty).$$

The following result establishes a precise asymptotic estimate in the neighborhood of the boundary.

THEOREM 4.16. *Assume that*

$$b(x) = k^2(d)\big(1 + \tilde{c}d^\theta + \mathrm{o}\big(d^\theta\big)\big) \quad \text{if } d(x) \to 0,$$
$$\text{where } k \in \mathcal{R}_{0,\zeta},\ \theta > 0,\ \tilde{c} \in \mathbb{R}. \tag{4.61}$$

Suppose that f fulfills (A_1) *and one of the following growth conditions at infinity*:
(i) $f(u) = Cu^{\rho+1}$ *in a neighborhood of infinity*;
(ii) $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta \neq 0$;
(iii) $f \in \mathcal{F}_{\rho 0,\tau_1}$ *with* $\tau_1 = \varpi/\zeta$, *where* $\varpi = \min\{\theta, \zeta\}$.
Then, for any $a \in (-\infty, \lambda_{\infty,1})$, *the unique positive solution* u_a *of* (4.36) *satisfies*

$$u_a(x) = \xi_0 h(d)\big(1 + \chi d^\varpi + \mathrm{o}\big(d^\varpi\big)\big) \quad \text{if } d(x) \to 0,$$
$$\text{where } \xi_0 = \big[2(2+\rho)^{-1}\big]^{1/\rho} \tag{4.62}$$

and h is defined by $\int_{h(t)}^\infty [2F(s)]^{-1/2}\,\mathrm{d}s = \int_0^t k(s)\,\mathrm{d}s$, *for $t > 0$ small enough. The expression of χ is*

$$\chi = \left\{ \begin{array}{l} -(1+\zeta)\ell_\star(2\zeta)^{-1}\ \mathrm{Heaviside}(\theta - \zeta) - \tilde{c}\rho^{-1}\ \mathrm{Heaviside}(\zeta - \theta) := \chi_1 \\ \quad \textit{if} \text{ (i) } \textit{or} \text{ (ii) } \textit{holds} \\ \chi_1 - \ell^\star \rho^{-1}(-\rho\ell_\star/2)^{\tau_1}[1/(\rho+2) + \ln\xi_0] \quad \textit{if } f \textit{ obeys} \text{ (iii)}. \end{array} \right.$$

Note that the only case related, in same way, to our Theorem 4.16 corresponds to $\Omega_0 = \emptyset$, $f(u) = u^{\rho+1}$ on $[0,\infty)$, $k(t) = ct^\alpha \in \mathcal{K}$ (where $c, \alpha > 0$), $\theta = 1$ in (4.61), being studied in [43]. There, the two-term asymptotic expansion of u_a near $\partial\Omega$ ($a \in \mathbb{R}$ since $\lambda_{\infty,1} = \infty$) involves both the distance function $d(x)$ and the mean curvature H of $\partial\Omega$. However, the blow-up rate of u_a we present in Theorem 4.16 is of a different nature since the class $\mathcal{R}_{0,\zeta}$ does not include $k(t) = ct^\alpha$.

Our main result contributes to the knowledge in some new directions. More precisely, the blow-up rate of the unique positive solution u_a of (4.36) is refined as follows in the above result:

(a) on the maximal interval $(-\infty, \lambda_{\infty,1})$ for the parameter a, which is in connection with an appropriate semilinear eigenvalue problem; thus, the condition $b > 0$ in Ω is removed by defining the set Ω_0, but we maintain $b \equiv 0$ on $\partial\Omega$ since this is a *natural* restriction inherited from the logistic problem.

(b) When b satisfies (4.61), where θ is *any* positive number and k belongs to a very rich class of functions, namely $\mathcal{R}_{0,\zeta}$. The equivalence $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$ shows the connection to the larger class $\mathcal{K}$ for which the uniqueness of u_a holds. In addition, the explicit form of $k \in \mathcal{R}_{0,\zeta}$ shows us how to built $k \in \mathcal{K}_{0,\zeta}$.

(c) For a wide class of functions $f \in NRV_{\rho+1}$ where either $\phi \equiv 0$ (case (i)) or ϕ (resp., $-\phi$) belongs to RV_η with $\eta \in (-\rho-2, 0]$ (cases (ii) and (iii)). Therefore, the theory of regular variation plays a key role in understanding the general framework and the approach as well.

PROOF OF THEOREM 4.16. We first state two auxiliary results. Their proofs are straightforward and we shall omit them.

LEMMA 4.17. *Assume* (4.61) *and* $f \in NRV_{\rho+1}$ *satisfies* (A_1). *Then* h *has the following properties*:

(i) $h \in C^2(0,\nu)$, $\lim_{t\searrow 0} h(t) = \infty$ *and* $\lim_{t\searrow 0} h'(t) = -\infty$;
(ii) $\lim_{t\searrow 0} h''(t)/[k^2(t) f(h(t)\xi)] = (2+\rho\ell_1)/[\xi^{\rho+1}(2+\rho)]$, $\forall \xi > 0$;
(iii) $\lim_{t\searrow 0} h(t)/h''(t) = \lim_{t\searrow 0} h'(t)/h''(t) = \lim_{t\searrow 0} h(t)/h'(t) = 0$;
(iv) $\lim_{t\searrow 0} h'(t)/[th''(t)] = -\rho\ell_1/(2+\rho\ell_1)$ *and* $\lim_{t\searrow 0} h(t)/[t^2 h''(t)] = \rho^2\ell_1^2/[2(2+\rho\ell_1)]$;
(v) $\lim_{t\searrow 0} h(t)/[th'(t)] = \lim_{t\searrow 0}[\ln t]/[\ln h(t)] = -\rho\ell_1/2$;
(vi) *If* $\ell_1 = 0$, *then* $\lim_{t\searrow 0} t^j h(t) = \infty$, *for all* $j > 0$;
(vii) $\lim_{t\searrow 0} 1/[t^\zeta \ln h(t)] = -\rho\ell_\star/2$ *and* $\lim_{t\searrow 0} h'(t)/[t^{\zeta+1}h''(t)] = \rho\ell_\star/(2\zeta)$, $\forall k \in \mathcal{R}_{0,\zeta}$.

Let $\tau > 0$ be arbitrary. For any $u > 0$, define $T_{1,\tau}(u) = \{\rho/[2(\rho+2)] - \Xi(u)\}(\ln u)^\tau$ and $T_{2,\tau}(u) = \{f(\xi_0 u)/[\xi_0 f(u)] - \xi_0^\rho\}(\ln u)^\tau$. Note that if $f(u) = Cu^{\rho+1}$, for u in a neighborhood V_∞ of infinity, then $T_{1,\tau}(u) = T_{2,\tau}(u) = 0$ for each $u \in V_\infty$.

LEMMA 4.18. *Assume* (A_1) *and* $f \in \mathcal{F}_{\rho\eta}$. *The following hold*:

(i) *If* $f \in \mathcal{F}_{\rho 0,\tau}$, *then*

$$\lim_{u\to\infty} T_{1,\tau}(u) = \frac{-\ell^\star}{(\rho+2)^2} \quad \textit{and} \quad \lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \ell^\star \ln \xi_0.$$

(ii) *If* $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta \neq 0$, *then* $\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} T_{2,\tau}(u) = 0$.

Fix $\varepsilon \in (0, 1/2)$. We can find $\delta > 0$ such that $d(x)$ is of class C^2 on $\{x \in \mathbb{R}^N\colon d(x) < \delta\}$, k is nondecreasing on $(0, \delta)$, and $h'(t) < 0 < h''(t)$ for all $t \in (0, \delta)$. A straightforward computation shows that $\lim_{t\searrow 0} t^{1-\theta}k'(t)/k(t) = \infty$, for every $\theta > 0$. Using now (4.61), it follows that we can diminish $\delta > 0$ such that $k^2(t)[1 + (\tilde{c} - \varepsilon)t^\theta]$ is increasing on $(0, \delta)$ and

$$1 + (\tilde{c} - \varepsilon)d^\theta < b(x)/k^2(d) < 1 + (\tilde{c} + \varepsilon)d^\theta,$$
$$\forall x \in \Omega \text{ with } d \in (0, \delta). \tag{4.63}$$

We define

$$u^\pm(x) = \xi_0 h(d)\big(1 + \chi_\varepsilon^\pm d^\varpi\big),$$

with $d \in (0, \delta)$, where $\chi_\varepsilon^\pm = \chi \pm \varepsilon\,[1 + \text{Heaviside}(\zeta - \theta)]/\rho$. Take $\delta > 0$ small enough such that $u^\pm(x) > 0$, for each $x \in \Omega$ with $d \in (0, \delta)$. By the Lagrange mean value theorem, we obtain $f(u^\pm(x)) = f(\xi_0 h(d)) + \xi_0 \chi_\varepsilon^\pm d^\varpi h(d) f'(\Upsilon^\pm(d))$, where $\Upsilon^\pm(d) = \xi_0 h(d)(1 + \lambda^\pm(d)\chi_\varepsilon^\pm d^\varpi)$, for some $\lambda^\pm(d) \in [0, 1]$. We claim that

$$\lim_{d\searrow 0} f\big(\Upsilon^\pm(d)\big)/f\big(\xi_0 h(d)\big) = 1. \tag{4.64}$$

Fix $\sigma \in (0, 1)$ and $M > 0$ such that $|\chi_\varepsilon^\pm| < M$. Choose $\mu^\star > 0$ so that $|(1 \pm Mt)^{\rho+1} - 1| < \sigma/2$, for all $t \in (0, 2\mu^\star)$. Let $\mu_\star \in (0, (\mu^\star)^{1/\varpi})$ be such that, for every $x \in \Omega$ with $d \in (0, \mu_\star)$

$$\big|f\big(\xi_0 h(d)(1 \pm M\mu^\star)\big)/f\big(\xi_0 h(d)\big) - (1 \pm M\mu^\star)^{\rho+1}\big| < \sigma/2.$$

Hence, $1 - \sigma < (1 - M\mu^\star)^{\rho+1} - \sigma/2 < f(\Upsilon^\pm(d))/f(\xi_0 h(d)) < (1 + M\mu^\star)^{\rho+1} + \sigma/2 < 1 + \sigma$, for every $x \in \Omega$ with $d \in (0, \mu_\star)$. This proves (4.64).

Step 1. There exists $\delta_1 \in (0, \delta)$ so that $\Delta u^+ + au^+ - k^2(d)[1 + (\tilde{c} - \varepsilon)d^\theta]f(u^+) \leqslant 0$, $\forall x \in \Omega$ with $d \in (0, \delta_1)$ and $\Delta u^- + au^- - k^2(d)[1 + (\tilde{c} + \varepsilon)d^\theta]f(u^-) \geqslant 0$, $\forall x \in \Omega$ with $d \in (0, \delta_1)$.

Indeed, for every $x \in \Omega$ with $d \in (0, \delta)$, we have

$$\begin{aligned}
&\Delta u^\pm + au^\pm - k^2(d)\big[1 + (\tilde{c} \mp \varepsilon)d^\theta\big]f(u^\pm)\\
&\quad= \xi_0 d^\varpi h''(d)\Bigg[a\chi_\varepsilon^\pm \frac{h(d)}{h''(d)} + \chi_\varepsilon^\pm \Delta d \frac{h'(d)}{h''(d)} + 2\varpi\chi_\varepsilon^\pm \frac{h'(d)}{dh''(d)}\\
&\qquad+ \varpi\chi_\varepsilon^\pm \Delta d \frac{h(d)}{dh''(d)} + \varpi(\varpi - 1)\chi_\varepsilon^\pm \frac{h(d)}{d^2 h''(d)} + \Delta d \frac{h'(d)}{d^\varpi h''(d)}\\
&\qquad+ \frac{ah(d)}{d^\varpi h''(d)} + \sum_{j=1}^4 \mathcal{S}_j^\pm(d)\Bigg]
\end{aligned}$$

where, for any $t \in (0, \delta)$, we denote

$$\mathcal{S}_1^{\pm}(t) = (-\tilde{c} \pm \varepsilon) t^{\theta-\varpi} k^2(t) f\big(\xi_0 h(t)\big) / \big[\xi_0 h''(t)\big],$$
$$\mathcal{S}_2^{\pm}(t) = \chi_\varepsilon^{\pm}\big(1 - k^2(t) h(t) f'\big(\Upsilon^{\pm}(t)\big) / h''(t)\big),$$
$$\mathcal{S}_3^{\pm}(t) = (-\tilde{c} \pm \varepsilon)\chi_\varepsilon^{\pm} t^{\theta} k^2(t) h(t) f'\big(\Upsilon^{\pm}(t)\big) / h''(t),$$
$$\mathcal{S}_4^{\pm}(t) = t^{-\varpi}\big(1 - k^2(t) f\big(\xi_0 h(t)\big) / \big[\xi_0 h''(t)\big]\big).$$

By Lemma 4.17(ii), we find $\lim_{t \searrow 0} k^2(t) f(\xi_0 h(t))[\xi_0 h''(t)]^{-1} = 1$, which yields $\lim_{t \searrow 0} \mathcal{S}_1^{\pm}(t) = (-\tilde{c} \pm \varepsilon)\text{Heaviside}(\zeta - \theta)$. Using (4.64), we obtain

$$\lim_{t \searrow 0} \frac{k^2(t) h(t) f'(\Upsilon^{\pm}(t))}{h''(t)} = \rho + 1.$$

Hence, $\lim_{t \searrow 0} \mathcal{S}_2^{\pm}(t) = -\rho \chi_\varepsilon^{\pm}$ and $\lim_{t \searrow 0} \mathcal{S}_3^{\pm}(t) = 0$.

Using the expression of h'', we derive

$$\mathcal{S}_4^{\pm}(t) = \frac{k^2(t) f(h(t))}{h''(t)} \sum_{i=1}^{3} \mathcal{S}_{4,i}(t), \quad \forall t \in (0, \delta),$$

where we denote

$$\mathcal{S}_{4,1}(t) = 2\frac{\Xi(h(t))}{t^{\varpi}} \left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)} \right)', \quad \mathcal{S}_{4,2}(t) = 2\frac{T_{1,\tau_1}(h(t))}{[t^{\zeta} \ln h(t)]^{\tau_1}}$$

and

$$\mathcal{S}_{4,3}(t) = -\frac{T_{2,\tau_1}(h(t))}{[t^{\zeta} \ln h(t)]^{\tau_1}}.$$

Since $\mathcal{R}_{0,\zeta} \equiv \mathcal{K}_{0,\zeta}$, we find

$$\lim_{t \searrow 0} \mathcal{S}_{4,1}(t) = -(1 + \zeta)\rho \ell_\star \zeta^{-1} (\rho + 2)^{-1} \text{Heaviside}(\theta - \zeta).$$

Cases (i), (ii). By Lemma 4.17(vii) and Lemma 4.18(ii), we find $\lim_{t \searrow 0} S_{4,2}(t) = \lim_{t \searrow 0} S_{4,3}(t) = 0$. In view of Lemma 4.17(ii), we derive that $\lim_{t \searrow 0} S_4^{\pm}(t) = -(1 + \zeta)\rho \ell_\star (2\zeta)^{-1} \text{Heaviside}(\theta - \zeta)$.

Case (iii). By Lemmas 4.17(vii) and 4.18(i), $\lim_{t \searrow 0} S_{4,2}(t) = -2\ell^\star (\rho+2)^{-2} (-\rho \ell_\star / 2)^{\tau_1}$ and $\lim_{t \searrow 0} S_{4,3}(t) = -2\ell^\star (\rho + 2)^{-1} (-\rho \ell_\star / 2)^{\tau_1} \ln \xi_0$. Using Lemma 4.17(ii) once more, we arrive at

$$\lim_{t \searrow 0} S_4^{\pm}(t)$$
$$= -(1 + \zeta)\rho \ell_\star (2\zeta)^{-1} \text{Heaviside}\,(\theta - \zeta) - \ell^\star \left(-\frac{\rho \ell_\star}{2} \right)^{\tau_1} \left[\frac{1}{\rho + 2} + \ln \xi_0 \right].$$

Note that in each of the cases (i)–(iii), the definition of $\chi_\varepsilon^\pm$ yields $\lim_{t\searrow 0}\sum_{j=1}^4 \mathcal{S}_j^+(t) = -\varepsilon < 0$ and $\lim_{t\searrow 0}\sum_{j=1}^4 \mathcal{S}_j^-(t) = \varepsilon > 0$. By Lemma 4.17(vii),

$$\lim_{t\searrow 0}\frac{h'(t)}{(t^\varpi h''(t))} = 0.$$

But $\lim_{t\searrow 0} h(t)/h'(t) = 0$, so $\lim_{t\searrow 0} h(t)/(t^\varpi h''(t)) = 0$. Thus, using Lemma 4.17 [(iii), (iv)], relation (4.65) concludes our Step 1.

Step 2. There exists M^+, $\delta^+ > 0$ such that $u_a(x) \leqslant u^+(x) + M^+$, for all $x \in \Omega$ with $0 < d < \delta^+$.

Define $(0,\infty) \ni u \mapsto \Psi_x(u) = au - b(x)f(u)$, $\forall x$ with $d \in (0,\delta_1)$. Clearly, $\Psi_x(u)$ is decreasing when $a \leqslant 0$. Suppose $a \in (0, \lambda_{\infty,1})$. Obviously, $f(t)/t : (0,\infty) o(f'(0),\infty)$ is bijective. Let $\delta_2 \in (0,\delta_1)$ be such that $b(x) < 1$, $\forall x$ with $d \in (0,\delta_2)$. Let u_x define the unique positive solution of $b(x)f(u)/u = a + f'(0)$, $\forall x$ with $d \in (0,\delta_2)$. Hence, for any x with $d \in (0,\delta_2)$, $u \to \Psi_x(u)$ is decreasing on (u_x,∞). But $\lim_{d(x)\searrow 0} b(x)f(u^+(x))/u^+(x) = +\infty$ (use $\lim_{d(x)\searrow 0} u^+(x)/h(d) = \xi_0$, (A_1) and Lemma 4.17 [(ii) and (iii)]). So, for δ_2 small enough, $u^+(x) > u_x$, $\forall x$ with $d \in (0,\delta_2)$.

Fix $\sigma \in (0,\delta_2/4)$ and set $\mathcal{N}_\sigma := \{x \in \Omega\colon\ \sigma < d(x) < \delta_2/2\}$. We define $u_\sigma^*(x) = u^+(d-\sigma,s) + M^+$, where (d,s) are the local coordinates of $x \in \mathcal{N}_\sigma$. We choose $M^+ > 0$ large enough to have $u_\sigma^*(\delta_2/2,s) \geqslant u_a(\delta_2/2,s)$, $\forall\sigma \in (0,\delta_2/4)$ and $\forall s \in \partial\Omega$. Using (4.63) and Step 1, we find

$$\begin{aligned}
-\Delta u_\sigma^*(x) &\geqslant au^+(d-\sigma,s) - \big[1 + (\tilde{c}-\varepsilon)(d-\sigma)^\theta\big]k^2(d-\sigma)f\big(u^+(d-\sigma,s)\big)\\
&\geqslant au^+(d-\sigma,s) - \big[1 + (\tilde{c}-\varepsilon)d^\theta\big]k^2(d)f\big(u^+(d-\sigma,s)\big)\\
&\geqslant \Psi_x\big(u^+(d-\sigma,s)\big)\\
&\geqslant \Psi_x\big(u_\sigma^*\big) = au_\sigma^*(x) - b(x)f\big(u_\sigma^*(x)\big) \quad \text{in } \mathcal{N}_\sigma.
\end{aligned}$$

Thus, by the maximum principle, $u_a \leqslant u_\sigma^*$ in $\mathcal{N}_\sigma$, $\forall\sigma \in (0,\delta_2/4)$. Letting $\sigma \to 0$, we have proved Step 2.

Step 3. There exists M^-, $\delta^- > 0$ such that $u_a(x) \geqslant u^-(x) - M^-$, for all $x \in \Omega$ with $0 < d < \delta^-$.

For every $r \in (0,\delta)$, define $\Omega_r = \{x \in \Omega\colon\ 0 < d(x) < r\}$. We will prove that for $\lambda > 0$ sufficiently small, $\lambda u^-(x) \leqslant u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$. Indeed, fix arbitrarily $\sigma \in (0,\delta_2/4)$. Define $v_\sigma^*(x) = \lambda u^-(d+\sigma,s)$, for $x = (d,s) \in \Omega_{\delta_2/2}$. We choose $\lambda \in (0,1)$ small enough such that $v_\sigma^*(\delta_2/4,s) \leqslant u_a(\delta_2/4,s)$, $\forall\sigma \in (0,\delta_2/4)$, $\forall s \in \partial\Omega$. Using (4.63), Step 1 and (A_1), we find

$$\begin{aligned}
\Delta v_\sigma^*(x) + av_\sigma^*(x) &\geqslant \lambda k^2(d+\sigma)\big[1 + (\tilde{c}+\varepsilon)(d+\sigma)^\theta\big]f\big(u^-(d+\sigma,s)\big)\\
&\geqslant k^2(d)\big[1 + (\tilde{c}+\varepsilon)d^\theta\big]f\big(\lambda u^-(d+\sigma,s)\big) \geqslant bf(v_\sigma^*),
\end{aligned}$$

for all $x = (d,s) \in \Omega_{\delta_2/4}$, that is v_σ^* is a sub-solution of $\Delta u + au = b(x)f(u)$ in $\Omega_{\delta_2/4}$. By the maximum principle, we conclude that $v_\sigma^* \leqslant u_a$ in $\Omega_{\delta_2/4}$. Letting $\sigma \to 0$, we find $\lambda u^-(x) \leqslant u_a(x)$, $\forall x \in \Omega_{\delta_2/4}$.

Since $\lim_{d\searrow 0} u^-(x)/h(d) = \xi_0$, by using (A_1) and Lemma 4.17 [(ii), (iii)], we can easily obtain $\lim_{d\searrow 0} k^2(d) f(\lambda^2 u^-(x))/u^-(x) = \infty$. So, there exists $\tilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d)\big[1 + (\tilde{c} + \varepsilon)d^\theta\big] f\big(\lambda^2 u^-\big)/u^- \geqslant \lambda^2 |a|, \quad \forall x \in \Omega \text{ with } 0 < d \leqslant \tilde{\delta}. \tag{4.65}$$

By Lemma 4.17 [(i) and (v)], we deduce that $u^-(x)$ decreases with d when $d \in (0, \tilde{\delta})$ (if necessary, $\tilde{\delta} > 0$ is diminished). Choose $\delta_* \in (0, \tilde{\delta})$, close enough to $\tilde{\delta}$, such that

$$h(\delta_*)\big(1 + \chi_\varepsilon^- \delta_*^\varpi\big)/\big[h(\tilde{\delta})\big(1 + \chi_\varepsilon^- \tilde{\delta}^\varpi\big)\big] < 1 + \lambda. \tag{4.66}$$

For each $\sigma \in (0, \tilde{\delta} - \delta_*)$, we define $z_\sigma(x) = u^-(d + \sigma, s) - (1 - \lambda)u^-(\delta_*, s)$. We prove that z_σ is a sub-solution of $\Delta u + au = b(x) f(u)$ in Ω_{δ_*}. Using (4.66), $z_\sigma(x) \geqslant u^-(\tilde{\delta}, s) - (1 - \lambda)u^-(\delta_*, s) > 0$ $\forall x = (d, s) \in \Omega_{\delta_*}$. By (4.63) and Step 1, z_σ is a sub-solution of $\Delta u + au = b(x) f(u)$ in Ω_{δ_*} if

$$\begin{aligned} &k^2(d + \sigma)\big[1 + (\tilde{c} + \varepsilon)(d + \sigma)^\theta\big]\big[f\big(u^-(d + \sigma, s)\big) - f\big(z_\sigma(d, s)\big)\big] \\ &\quad \geqslant a(1 - \lambda)u^-(\delta_*, s), \end{aligned} \tag{4.67}$$

for all $(d, s) \in \Omega_{\delta_*}$. Applying the Lagrange mean value theorem and (A_1), we infer that (4.67) is a consequence of $k^2(d + \sigma)[1 + (\tilde{c} + \varepsilon)(d + \sigma)^\theta] f(z_\sigma(d, s))/z_\sigma(d, s) \geqslant |a|$, $\forall (d, s) \in \Omega_{\delta_*}$. This inequality holds by virtue of (4.65), (4.66) and the decreasing character of u^- with d.

On the other hand, $z_\sigma(\delta_*, s) \leqslant \lambda u^-(\delta_*, s) \leqslant u_a(x)$, $\forall x = (\delta_*, s) \in \Omega$. Clearly, $\limsup_{d\to 0}(z_\sigma - u_a)(x) = -\infty$ and $b > 0$ in Ω_{δ_*}. Thus, by the maximum principle, $z_\sigma \leqslant u_a$ in Ω_{δ_*}, $\forall \sigma \in (0, \tilde{\delta} - \delta_*)$. Letting $\sigma \to 0$, we conclude the assertion of Step 3.

By Steps 2 and 3, $\chi_\varepsilon^+ \geqslant \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} - M^+/[\xi_0 d^\varpi h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^+)$ and $\chi_\varepsilon^- \leqslant \{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} + M^-/[\xi_0 d^\varpi h(d)]$ $\forall x \in \Omega$ with $d \in (0, \delta^-)$. Passing to the limit as $d \to 0$ and using Lemma 4.17(vi), we obtain $\chi_\varepsilon^- \leqslant \liminf_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi}$ and $\limsup_{d\to 0}\{-1 + u_a(x)/[\xi_0 h(d)]\}d^{-\varpi} \leqslant \chi_\varepsilon^+$. Letting $\varepsilon \to 0$, we conclude our proof. □

5. Entire solutions blowing up at infinity of semilinear elliptic systems

In this section we are concerned with the existence of solutions that blow up at infinity for a class of semilinear elliptic systems defined on the whole space.

Consider the following semilinear elliptic system

$$\begin{cases} \Delta u = p(x)g(v) & \text{in } \mathbb{R}^N, \\ \Delta v = q(x)f(u) & \text{in } \mathbb{R}^N, \end{cases} \tag{5.68}$$

where $N \geqslant 3$ and $p, q \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ $(0 < \alpha < 1)$ are nonnegative and radially symmetric functions. Throughout this paper we assume that $f, g \in C^{0,\beta}_{\text{loc}}[0, \infty)$ $(0 < \beta < 1)$ are positive and nondecreasing on $(0, \infty)$.

We are concerned here with the existence of positive *entire large solutions* of (5.68), that is positive classical solutions which satisfy $u(x) \to \infty$ and $v(x) \to \infty$ as $|x| \to \infty$. Set $\mathbb{R}^+ = (0, \infty)$ and define

$$\mathcal{G} = \big\{(a,b) \in \mathbb{R}^+ \times \mathbb{R}^+;\ (\exists) \text{ an entire radial solution of (5.68) so that } (u(0), v(0)) = (a,b)\big\}.$$

The case of pure powers in the nonlinearities was treated by Lair and Shaker in [64]. They proved that $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$ if $f(t) = t^\gamma$ and $g(t) = t^\theta$ for $t \geqslant 0$ with $0 < \gamma, \theta \leqslant 1$. Moreover, they established that all positive entire radial solutions of (5.68) are *large* provided that

$$\int_0^\infty tp(t)\,\mathrm{d}t = \infty, \qquad \int_0^\infty tq(t)\,\mathrm{d}t = \infty. \tag{5.69}$$

If, in turn

$$\int_0^\infty tp(t)\,\mathrm{d}t < \infty, \qquad \int_0^\infty tq(t)\,\mathrm{d}t < \infty \tag{5.70}$$

then all positive entire radial solutions of (5.68) are *bounded*.

In what follows we generalize the above results to a larger class of systems. Theorems 5.1 and 5.4 are due to Cîrstea and Rădulescu [26].

THEOREM 5.1. *Assume that*

$$\lim_{t\to\infty} \frac{g(cf(t))}{t} = 0 \quad \textit{for all } c > 0. \tag{5.71}$$

Then $\mathcal{G} = \mathbb{R}^+ \times \mathbb{R}^+$. *Moreover, the following hold*:

(i) *If p and q satisfy* (5.69), *then all positive entire radial solutions of* (5.68) *are large*.

(ii) *If p and q satisfy* (5.70), *then all positive entire radial solutions of* (5.68) *are bounded*.

Furthermore, if f, g are locally Lipschitz continuous on $(0, \infty)$ *and* (u, v), $(\tilde{u}, \tilde{v})$ *denote two positive entire radial solutions of* (5.68), *then there exists a positive constant C such that for all* $r \in [0, \infty)$

$$\max\big\{\big|u(r) - \tilde{u}(r)\big|, \big|v(r) - \tilde{v}(r)\big|\big\} \leqslant C \max\big\{\big|u(0) - \tilde{u}(0)\big|, \big|v(0) - \tilde{v}(0)\big|\big\}.$$

PROOF. We start with the following auxiliary results.

LEMMA 5.2. *Condition* (5.69) *holds if and only if* $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$ *where*

$$A(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} p(s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$B(r) \equiv \int_0^r t^{1-N} \int_0^t s^{N-1} q(s)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r > 0.$$

PROOF. Indeed, for any $r > 0$

$$A(r) = \frac{1}{N-2}\left[\int_0^r tp(t)\,\mathrm{d}t - \frac{1}{r^{N-2}}\int_0^r t^{N-1}p(t)\,\mathrm{d}t\right]$$
$$\leqslant \frac{1}{N-2}\int_0^r tp(t)\,\mathrm{d}t. \tag{5.72}$$

On the other hand,

$$\int_0^r tp(t)\,\mathrm{d}t - \frac{1}{r^{N-2}}\int_0^r t^{N-1}p(t)\,\mathrm{d}t = \frac{1}{r^{N-2}}\int_0^r \big(r^{N-2} - t^{N-2}\big)tp(t)\,\mathrm{d}t$$
$$\geqslant \frac{1}{r^{N-2}}\left[r^{N-2} - \left(\frac{r}{2}\right)^{N-2}\right]\int_0^{r/2} tp(t)\,\mathrm{d}t.$$

This combined with (5.72) yields

$$\frac{1}{N-2}\int_0^r tp(t)\,\mathrm{d}t \geqslant A(r) \geqslant \frac{1}{N-2}\left[1 - \left(\frac{1}{2}\right)^{N-2}\right]\int_0^{r/2} tp(t)\,\mathrm{d}t.$$

Our conclusion follows now by letting $r \to \infty$. □

LEMMA 5.3. *Assume that condition* (5.70) *holds. Let f and g be locally Lipschitz continuous functions on* $(0,\infty)$. *If (u,v) and $(\tilde u,\tilde v)$ denote two bounded positive entire radial solutions of* (5.68), *then there exists a positive constant C such that for all $r \in [0,\infty)$*

$$\max\big\{\big|u(r) - \tilde u(r)\big|, \big|v(r) - \tilde v(r)\big|\big\} \leqslant C\max\big\{\big|u(0) - \tilde u(0)\big|, \big|v(0) - \tilde v(0)\big|\big\}.$$

PROOF. We first see that radial solutions of (5.68) are solutions of the ordinary differential equations system

$$\begin{cases} u''(r) + \frac{N-1}{r}u'(r) = p(r)g(v(r)), & r > 0, \\ v''(r) + \frac{N-1}{r}v'(r) = q(r)f(u(r)), & r > 0. \end{cases} \tag{5.73}$$

Define $K = \max\{|u(0) - \tilde u(0)|, |v(0) - \tilde v(0)|\}$. Integrating the first equation of (5.73), we get

$$u'(r) - \tilde u'(r) = r^{1-N}\int_0^r s^{N-1}p(s)\big(g\big(v(s)\big) - g\big(\tilde v(s)\big)\big)\,\mathrm{d}s.$$

Hence

$$\big|u(r)-\tilde{u}(r)\big| \leqslant K + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) \big|g\big(v(s)\big) - g\big(\tilde{v}(s)\big)\big| \,\mathrm{d}s\,\mathrm{d}t. \tag{5.74}$$

Since (u, v) and $(\tilde{u}, \tilde{v})$ are bounded entire radial solutions of (5.68) we have

$$\begin{aligned}
&\big|g\big(v(r)\big) - g\big(\tilde{v}(r)\big)\big| \leqslant m\big|v(r)-\tilde{v}(r)\big| \quad \text{for any } r \in [0,\infty),\\
&\big|f\big(u(r)\big) - f\big(\tilde{u}(r)\big)\big| \leqslant m\big|u(r)-\tilde{u}(r)\big| \quad \text{for any } r \in [0,\infty),
\end{aligned}$$

where m denotes a Lipschitz constant for both functions f and g. Therefore, using (5.74) we find

$$\big|u(r)-\tilde{u}(r)\big| \leqslant K + m\int_0^r t^{1-N} \int_0^t s^{N-1} p(s) \big|v(s)-\tilde{v}(s)\big| \,\mathrm{d}s\,\mathrm{d}t. \tag{5.75}$$

Arguing as above, but now with the second equation of (5.73), we obtain

$$\big|v(r)-\tilde{v}(r)\big| \leqslant K + m\int_0^r t^{1-N} \int_0^t s^{N-1} q(s) \big|u(s)-\tilde{u}(s)\big| \,\mathrm{d}s\,\mathrm{d}t. \tag{5.76}$$

Define

$$\begin{aligned}
X(r) &= K + m\int_0^r t^{1-N} \int_0^t s^{N-1} p(s) \big|v(s)-\tilde{v}(s)\big| \,\mathrm{d}s\,\mathrm{d}t,\\
Y(r) &= K + m\int_0^r t^{1-N} \int_0^t s^{N-1} q(s) \big|u(s)-\tilde{u}(s)\big| \,\mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

It is clear that X and Y are nondecreasing functions with $X(0) = Y(0) = K$. By a simple calculation together with (5.75) and (5.76) we obtain

$$\begin{aligned}
\big(r^{N-1}X'\big)'(r) &= mr^{N-1}p(r)\big|v(r)-\tilde{v}(r)\big| \leqslant mr^{N-1}p(r)Y(r),\\
\big(r^{N-1}Y'\big)'(r) &= mr^{N-1}q(r)\big|u(r)-\tilde{u}(r)\big| \leqslant mr^{N-1}q(r)X(r).
\end{aligned} \tag{5.77}$$

Since Y is nondecreasing, we have

$$\begin{aligned}
X(r) &\leqslant K + mY(r)A(r) \leqslant K + \frac{m}{N-2}Y(r)\int_0^r tp(t)\,\mathrm{d}t\\
&\leqslant K + mC_pY(r)
\end{aligned} \tag{5.78}$$

where $C_p = (1/(N-2))\int_0^\infty tp(t)\,\mathrm{d}t$. Using (5.78) in the second inequality of (5.77) we find

$$\big(r^{N-1}Y'\big)'(r) \leqslant mr^{N-1}q(r)\big(K + mC_pY(r)\big).$$

Integrating twice this inequality from 0 to r, we obtain

$$Y(r) \leqslant K(1+mC_q) + \frac{m^2}{N-2} C_p \int_0^r tq(t)Y(t)\,\mathrm{d}t,$$

where $C_q = (1/(N-2)) \int_0^\infty tq(t)\,\mathrm{d}t$. From Gronwall's inequality, we deduce

$$\begin{aligned} Y(r) &\leqslant K(1+mC_q)\exp\left(\frac{m^2}{N-2} C_p \int_0^r tq(t)\right)\mathrm{d}t \\ &\leqslant K(1+mC_q)\exp\bigl(m^2 C_p C_q\bigr) \end{aligned}$$

and similarly for X. The conclusion follows now from the above inequality, (5.75) and (5.76). □

PROOF OF THEOREM 5.1 COMPLETED. Since the radial solutions of (5.68) are solutions of the ordinary differential equations system (5.73) it follows that the radial solutions of (5.68) with $u(0) = a > 0$, $v(0) = b > 0$ satisfy

$$u(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\bigl(v(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0, \tag{5.79}$$

$$v(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\bigl(u(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0. \tag{5.80}$$

Define $v_0(r) = b$ for all $r \geqslant 0$. Let $(u_k)_{k\geqslant 1}$ and $(v_k)_{k\geqslant 1}$ be two sequences of functions given by

$$u_k(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\bigl(v_{k-1}(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0,$$

$$v_k(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\bigl(u_k(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0.$$

Since $v_1(r) \geqslant b$, we find $u_2(r) \geqslant u_1(r)$ for all $r \geqslant 0$. This implies $v_2(r) \geqslant v_1(r)$ which further produces $u_3(r) \geqslant u_2(r)$ for all $r \geqslant 0$. Proceeding at the same manner we conclude that

$$u_k(r) \leqslant u_{k+1}(r) \quad \text{and} \quad v_k(r) \leqslant v_{k+1}(r), \qquad \forall r \geqslant 0 \quad \text{and } k \geqslant 1.$$

We now prove that the nondecreasing sequences $(u_k(r))_{k\geqslant 1}$ and $(v_k(r))_{k\geqslant 1}$ are bounded from above on bounded sets. Indeed, we have

$$u_k(r) \leqslant u_{k+1}(r) \leqslant a + g\bigl(v_k(r)\bigr)A(r), \quad \forall r \geqslant 0 \tag{5.81}$$

and

$$v_k(r) \leqslant b + f\bigl(u_k(r)\bigr)B(r), \quad \forall r \geqslant 0. \tag{5.82}$$

Let $R > 0$ be arbitrary. By (5.81) and (5.82) we find

$$u_k(R) \leqslant a + g\big(b + f\big(u_k(R)\big)B(R)\big)A(R), \quad \forall k \geqslant 1$$

or, equivalently,

$$1 \leqslant \frac{a}{u_k(R)} + \frac{g(b + f(u_k(R))B(R))}{u_k(R)}A(R), \quad \forall k \geqslant 1. \tag{5.83}$$

By the monotonicity of $(u_k(R))_{k\geqslant 1}$, there exists $\lim_{k\to\infty} u_k(R) := L(R)$. We claim that $L(R)$ is finite. Assume the contrary. Then, by taking $k \to \infty$ in (5.83) and using (5.71) we obtain a contradiction. Since $u_k'(r), v_k'(r) \geqslant 0$ we get that the map $(0, \infty) \ni R \to L(R)$ is nondecreasing on $(0, \infty)$ and

$$u_k(r) \leqslant u_k(R) \leqslant L(R), \quad \forall r \in [0, R],\ \forall k \geqslant 1, \tag{5.84}$$

$$v_k(r) \leqslant b + f\big(L(R)\big)B(R), \quad \forall r \in [0, R],\ \forall k \geqslant 1. \tag{5.85}$$

It follows that there exists $\lim_{R\to\infty} L(R) = \bar{L} \in (0, \infty]$ and the sequences $(u_k(r))_{k\geqslant 1}$, $(v_k(r))_{k\geqslant 1}$ are bounded above on bounded sets. Therefore, we can define $u(r) := \lim_{k\to\infty} u_k(r)$ and $v(r) := \lim_{k\to\infty} v_k(r)$ for all $r \geqslant 0$. By standard elliptic regularity theory we obtain that (u, v) is a positive entire solution of (5.68) with $u(0) = a$ and $v(0) = b$.

We now assume that, in addition, condition (5.70) is fulfilled. According to Lemma 5.2 we have that $\lim_{r\to\infty} A(r) = \bar{A} < \infty$ and $\lim_{r\to\infty} B(r) = \overline{B} < \infty$. Passing to the limit as $k \to \infty$ in (5.83) we find

$$1 \leqslant \frac{a}{L(R)} + \frac{g(b + f(L(R))B(R))}{L(R)}A(R) \leqslant \frac{a}{L(R)} + \frac{g(b + f(L(R))\overline{B})}{L(R)}\bar{A}.$$

Letting $R \to \infty$ and using (5.71) we deduce $\bar{L} < \infty$. Thus, taking into account (5.84) and (5.85), we obtain

$$u_k(r) \leqslant \bar{L} \quad \text{and} \quad v_k(r) \leqslant b + f(\bar{L})\overline{B}, \quad \forall r \geqslant 0,\ \forall k \geqslant 1.$$

So, we have found upper bounds for $(u_k(r))_{k\geqslant 1}$ and $(v_k(r))_{k\geqslant 1}$ which are independent of r. Thus, the solution (u, v) is bounded from above. This shows that any solution of (5.79) and (5.80) will be bounded from above provided (5.70) holds. Thus, we can apply Lemma 5.3 to achieve the second assertion of (ii).

Let us now drop the condition (5.70) and assume that (5.69) is fulfilled. In this case, Lemma 5.2 tells us that $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = \infty$. Let (u, v) be an entire positive radial solution of (5.68). Using (5.79) and (5.80) we obtain

$$u(r) \geqslant a + g(b)A(r), \quad \forall r \geqslant 0,$$

$$v(r) \geqslant b + f(a)B(r), \quad \forall r \geqslant 0.$$

Taking $r \to \infty$ we get that (u, v) is an entire large solution. This concludes the proof of Theorem 5.1. □

If f and g satisfy the stronger regularity $f, g \in C^1[0, \infty)$, then we drop the assumption (5.71) and require, in turn,

(H_1) $f(0) = g(0) = 0$, $\liminf_{u\to\infty}(f(u)/g(u)) =: \sigma > 0$

and the Keller–Osserman condition

(H_2) $\int_1^\infty (1/G(t))\,dt < \infty$, where $G(t) = \int_0^t g(s)\,ds$.

Observe that assumptions (H_1) and (H_2) imply that f satisfies condition (H_2), too.

Set $\eta = \min\{p, q\}$.

Our main result in this case is

THEOREM 5.4. *Let $f, g \in C^1[0, \infty)$ satisfy* (H_1) *and* (H_2). *Assume that* (5.70) *holds, η is not identically zero at infinity and $\nu := \max\{p(0), q(0)\} > 0$.*

Then any entire radial solution (u, v) of (5.68) *with $(u(0), v(0)) \in F(\mathcal{G})$ is large.*

PROOF. Under the assumptions of Theorem 5.4 we prove the following auxiliary results.

LEMMA 5.5. $\mathcal{G} \neq \emptyset$.

PROOF. Cf. Cîrstea and Rădulescu [26], the problem

$$\Delta\psi = (p+q)(x)(f+g)(\psi) \quad \text{in } \mathbb{R}^N,$$

has a positive radial entire large solution. Since ψ is radial, we have

$$\psi(r) = \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p+q)(s)(f+g)(\psi(s))\,ds\,dt, \quad \forall r \geqslant 0.$$

We claim that $(0, \psi(0)] \times (0, \psi(0)] \subseteq \mathcal{G}$. To prove this, fix $0 < a, b \leqslant \psi(0)$ and let $v_0(r) \equiv b$ for all $r \geqslant 0$. Define the sequences $(u_k)_{k\geqslant 1}$ and $(v_k)_{k\geqslant 1}$ by

$$u_k(r) = a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(v_{k-1}(s)\big)\,ds\,dt,$$
$$\forall r \in [0, \infty),\ \forall k \geqslant 1, \tag{5.86}$$
$$v_k(r) = b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(u_k(s)\big)\,ds\,dt,$$
$$\forall r \in [0, \infty),\ \forall k \geqslant 1. \tag{5.87}$$

We first see that $v_0 \leqslant v_1$ which produces $u_1 \leqslant u_2$. Consequently, $v_1 \leqslant v_2$ which further yields $u_2 \leqslant u_3$. With the same arguments, we obtain that (u_k) and (v_k) are nondecreasing sequences. Since $\psi'(r) \geqslant 0$ and $b = v_0 \leqslant \psi(0) \leqslant \psi(r)$ for all $r \geqslant 0$ we find

$$u_1(r) \leqslant a + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(\psi(s)\big)\,\mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} (p+q)(s)(f+g)\big(\psi(s)\big)\,\mathrm{d}s\,\mathrm{d}t = \psi(r).$$

Thus $u_1 \leqslant \psi$. It follows that

$$v_1(r) \leqslant b + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(\psi(s)\big)\,\mathrm{d}s\,\mathrm{d}t$$
$$\leqslant \psi(0) + \int_0^r t^{1-N} \int_0^t s^{N-1} (p+q)(s)(f+g)\big(\psi(s)\big)\,\mathrm{d}s\,\mathrm{d}t = \psi(r).$$

Similar arguments show that

$$u_k(r) \leqslant \psi(r) \quad \text{and} \quad v_k(r) \leqslant \psi(r) \quad \forall r \in [0,\infty),\ \forall k \geqslant 1.$$

Thus, (u_k) and (v_k) converge and $(u, v) = \lim_{k\to\infty}(u_k, v_k)$ is an entire radial solution of (5.68) such that $(u(0), v(0)) = (a, b)$. This completes the proof. □

An easy consequence of the above result is

COROLLARY 5.6. *If* $(a, b) \in \mathcal{G}$, *then* $(0, a] \times (0, b] \subseteq \mathcal{G}$.

PROOF. Indeed, the process used before can be repeated by taking

$$u_k(r) = a_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(v_{k-1}(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0,\infty),\ \forall k \geqslant 1,$$
$$v_k(r) = b_0 + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(u_k(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0,\infty),\ \forall k \geqslant 1,$$

where $0 < a_0 \leqslant a$, $0 < b_0 \leqslant b$ and $v_0(r) \equiv b_0$ for all $r \geqslant 0$.

Letting (U, V) be the entire radial solution of (5.68) with central values (a, b) we obtain as in Lemma 5.5,

$$u_k(r) \leqslant u_{k+1}(r) \leqslant U(r), \quad \forall r \in [0,\infty),\ \forall k \geqslant 1,$$
$$v_k(r) \leqslant v_{k+1}(r) \leqslant V(r), \quad \forall r \in [0,\infty),\ \forall k \geqslant 1.$$

Set $(u, v) = \lim_{k\to\infty}(u_k, v_k)$. We see that $u \leqslant U$, $v \leqslant V$ on $[0,\infty)$ and (u, v) is an entire radial solution of (5.68) with central values (a_0, b_0). This shows that $(a_0, b_0) \in \mathcal{G}$, so that our assertion is proved. □

LEMMA 5.7. $\mathcal{G}$ *is bounded.*

PROOF. Set $0 < \lambda < \min\{\sigma, 1\}$ and let $\delta = \delta(\lambda)$ be large enough so that

$$f(t) \geqslant \lambda g(t), \quad \forall t \geqslant \delta. \tag{5.88}$$

Since η is radially symmetric and not identically zero at infinity, we can assume $\eta > 0$ on $\partial B(0, R)$ for some $R > 0$. Let ζ be a positive large solution ζ of the problem

$$\Delta\zeta = \lambda\eta(x) g\left(\frac{\zeta}{2}\right) \quad \text{in } B(0, R).$$

Arguing by contradiction, we assume that $\mathcal{G}$ is not bounded. Then, there exists $(a, b) \in \mathcal{G}$ such that $a + b > \max\{2\delta, \zeta(0)\}$. Let (u, v) be the entire radial solution of (5.68) such that $(u(0), v(0)) = (a, b)$. Since $u(x) + v(x) \geqslant a + b > 2\delta$ for all $x \in \mathbb{R}^N$, by (5.88), we find

$$f(u(x)) \geqslant f\left(\frac{u(x) + v(x)}{2}\right) \geqslant \lambda g\left(\frac{u(x) + v(x)}{2}\right) \quad \text{if } u(x) \geqslant v(x)$$

and

$$g(v(x)) \geqslant g\left(\frac{u(x) + v(x)}{2}\right) \geqslant \lambda g\left(\frac{u(x) + v(x)}{2}\right) \quad \text{if } v(x) \geqslant u(x).$$

It follows that

$$\begin{aligned}\Delta(u + v) &= p(x)g(v) + q(x)f(u) \geqslant \eta(x)\big(g(v) + f(u)\big) \\ &\geqslant \lambda\eta(x) g\left(\frac{u + v}{2}\right) \quad \text{in } \mathbb{R}^N.\end{aligned}$$

On the other hand, $\zeta(x) \to \infty$ as $|x| \to R$ and $u, v \in C^2(\overline{B(0, R)})$. Thus, by the maximum principle, we conclude that $u + v \leqslant \zeta$ in $B(0, R)$. But this is impossible since $u(0) + v(0) = a + b > \zeta(0)$. □

LEMMA 5.8. $F(\mathcal{G}) \subset \mathcal{G}$.

PROOF. Let $(a, b) \in F(\mathcal{G})$. We claim that $(a - 1/n_0, b - 1/n_0) \in \mathcal{G}$ provided $n_0 \geqslant 1$ is large enough so that $\min\{a, b\} > 1/n_0$. Indeed, if this is not true, by Corollary 5.6

$$D := \left[a - \frac{1}{n_0}, \infty\right) \times \left[b - \frac{1}{n_0}, \infty\right) \subseteq (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}.$$

So, we can find a small ball B centered in (a, b) such that $B \subset\subset D$, i.e., $B \cap \mathcal{G} = \emptyset$. But this will contradict the choice of (a, b). Consequently, there exists (u_{n_0}, v_{n_0}) an entire radial solution of (5.68) such that $(u_{n_0}(0), v_{n_0}(0)) = (a - 1/n_0, b - 1/n_0)$. Thus, for any $n \geqslant n_0$,

we can define

$$u_n(r) = a - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(v_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0,$$

$$v_n(r) = b - \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(u_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad r \geqslant 0.$$

Using Corollary 5.6 once more, we conclude that $(u_n)_{n \geqslant n_0}$ and $(v_n)_{n \geqslant n_0}$ are non-decreasing sequences. We now prove that (u_n) and (v_n) converge on $\mathbb{R}^N$. To this aim, let $x_0 \in \mathbb{R}^N$ be arbitrary. But η is not identically zero at infinity so that, for some $R_0 > 0$, we have $\eta > 0$ on $\partial B(0, R_0)$ and $x_0 \in B(0, R_0)$.

Since $\sigma = \liminf_{u \to \infty} f(u)/g(u) > 0$, we find $\tau \in (0, 1)$ such that

$$f(t) \geqslant \tau g(t), \quad \forall t \geqslant \frac{a+b}{2} - \frac{1}{n_0}.$$

Therefore, on the set where $u_n \geqslant v_n$, we have

$$f(u_n) \geqslant f\left(\frac{u_n + v_n}{2}\right) \geqslant \tau g\left(\frac{u_n + v_n}{2}\right).$$

Similarly, on the set where $u_n \leqslant v_n$, we have

$$g(v_n) \geqslant g\left(\frac{u_n + v_n}{2}\right) \geqslant \tau g\left(\frac{u_n + v_n}{2}\right).$$

It follows that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned} \Delta(u_n + v_n) &= p(x) g(v_n) + q(x) f(u_n) \geqslant \eta(x)\big[g(v_n) + f(u_n)\big] \\ &\geqslant \tau \eta(x) g\left(\frac{u_n + v_n}{2}\right). \end{aligned}$$

On the other hand, there exists a positive large solution of

$$\Delta \zeta = \tau \eta(x) g\left(\frac{\zeta}{2}\right) \quad \text{in } B(0, R_0).$$

The maximum principle yields $u_n + v_n \leqslant \zeta$ in $B(0, R_0)$. So, it makes sense to define $(u(x_0), v(x_0)) = \lim_{n \to \infty} (u_n(x_0), v_n(x_0))$. Since x_0 is arbitrary, the functions u, v exist on $\mathbb{R}^N$. Hence (u, v) is an entire radial solution of (5.68) with central values (a, b), i.e., $(a, b) \in \mathcal{G}$. □

For $(c, d) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$, define

$$\begin{aligned} R_{c,d} = \sup\big\{ r > 0 \mid &\text{there exists a radial solution of (5.68) in } B(0, r) \\ &\text{so that } (u(0), v(0)) = (c, d) \big\}. \end{aligned} \tag{5.89}$$

LEMMA 5.9. *If, in addition,* $\nu = \max\{p(0), q(0)\} > 0$, *then* $0 < R_{c,d} < \infty$ *where* $R_{c,d}$ *is defined by* (5.89).

PROOF. Since $\nu > 0$ and $p, q \in C[0, \infty)$, there exists $\epsilon > 0$ such that $(p+q)(r) > 0$ for all $0 \leqslant r < \epsilon$. Let $0 < R < \epsilon$ be arbitrary. There exists a positive radial large solution of the problem

$$\Delta \psi_R = (p+q)(x)(f+g)(\psi_R) \quad \text{in } B(0, R).$$

Moreover, for any $0 \leqslant r < R$,

$$\psi_R(r) = \psi_R(0) + \int_0^r t^{1-N} \int_0^t s^{N-1}(p+q)(s)(f+g)\big(\psi_R(s)\big)\,\mathrm{d}s\,\mathrm{d}t.$$

It is clear that $\psi_R'(r) \geqslant 0$. Thus, we find

$$\psi_R'(r) = r^{1-N} \int_0^r s^{N-1}(p+q)(s)(f+g)\big(\psi_R(s)\big)\,\mathrm{d}s \leqslant C(f+g)\big(\psi_R(r)\big)$$

where $C > 0$ is a positive constant such that $\int_0^\epsilon (p+q)(s)\,\mathrm{d}s \leqslant C$.

Since $f+g$ satisfies (A_1) and (A_2), we may invoke Remark 1 in Section 2 to conclude that

$$\int_1^\infty \frac{dt}{(f+g)(t)} < \infty.$$

Therefore, we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}r} \int_{\psi_R(r)}^\infty \frac{\mathrm{d}s}{(f+g)(s)} = \frac{\psi_R'(r)}{(f+g)(\psi_R(r))} \leqslant C \quad \text{for any } 0 < r < R.$$

Integrating from 0 to R and recalling that $\psi_R(r) \to \infty$ as $r \nearrow R$, we obtain

$$\int_{\psi_R(0)}^\infty \frac{\mathrm{d}s}{(f+g)(s)} \leqslant CR.$$

Letting $R \searrow 0$ we conclude that

$$\lim_{R \searrow 0} \int_{\psi_R(0)}^\infty \frac{\mathrm{d}s}{(f+g)(s)} = 0.$$

This implies that $\psi_R(0) \to \infty$ as $R \searrow 0$. So, there exists $0 < \widetilde{R} < \epsilon$ such that $0 < c, d \leqslant \psi_{\widetilde{R}}(0)$. Set

$$u_k(r) = c + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(v_{k-1}(s)\big)\,\mathrm{d}s\,\mathrm{d}t,$$
$$\forall r \in [0, \infty),\ \forall k \geqslant 1, \tag{5.90}$$

$$v_k(r) = d + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(u_k(s)\big)\,\mathrm{d}s\,\mathrm{d}t,$$
$$\forall r \in [0,\infty),\ \forall k \geqslant 1, \tag{5.91}$$

where $v_0(r) = d$ for all $r \in [0,\infty)$. As in Lemma 5.5, we find that (u_k) resp., (v_k) are nondecreasing and

$$u_k(r) \leqslant \psi_{\widetilde{R}}(r) \quad \text{and} \quad v_k(r) \leqslant \psi_{\widetilde{R}}(r), \quad \forall r \in [0, \widetilde{R}),\ \forall k \geqslant 1.$$

Thus, for any $r \in [0, \widetilde{R})$, there exists $(u(r), v(r)) = \lim_{k\to\infty}(u_k(r), v_k(r))$ which is, moreover, a radial solution of (5.68) in $B(0, \widetilde{R})$ such that $(u(0), v(0)) = (c, d)$. This shows that $R_{c,d} \geqslant \widetilde{R} > 0$. By the definition of $R_{c,d}$ we also derive

$$\lim_{r\nearrow R_{c,d}} u(r) = \infty \quad \text{and} \quad \lim_{r\nearrow R_{c,d}} v(r) = \infty. \tag{5.92}$$

On the other hand, since $(c, d) \notin \mathcal{G}$, we conclude that $R_{c,d}$ is finite. □

PROOF OF THEOREM 5.4 COMPLETED. Let $(a, b) \in F(\mathcal{G})$ be arbitrary. By Lemma 5.8, $(a, b) \in \mathcal{G}$ so that we can define (U, V) an entire radial solution of (5.68) with

$$\big(U(0), V(0)\big) = (a, b).$$

Obviously, for any $n \geqslant 1$, $(a + 1/n, b + 1/n) \in (\mathbb{R}^+ \times \mathbb{R}^+) \setminus \mathcal{G}$. By Lemma 5.9, $R_{a+1/n,b+1/n}$ (in short, R_n) defined by (5.89) is a positive number. Let (U_n, V_n) be the radial solution of (5.68) in $B(0, R_n)$ with the central values $(a + 1/n, b + 1/n)$. Thus,

$$U_n(r) = a + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} p(s) g\big(V_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0, R_n), \tag{5.93}$$

$$V_n(r) = b + \frac{1}{n} + \int_0^r t^{1-N} \int_0^t s^{N-1} q(s) f\big(U_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall r \in [0, R_n). \tag{5.94}$$

In view of (5.92) we have

$$\lim_{r\nearrow R_n} U_n(r) = \infty \quad \text{and} \quad \lim_{r\nearrow R_n} V_n(r) = \infty, \quad \forall n \geqslant 1.$$

We claim that $(R_n)_{n\geqslant 1}$ is a nondecreasing sequence. Indeed, if (u_k), (v_k) denote the sequences of functions defined by (5.90) and (5.91) with $c = a + 1/(n + 1)$ and $d = b + 1/(n + 1)$, then

$$u_k(r) \leqslant u_{k+1}(r) \leqslant U_n(r), \quad v_k(r) \leqslant v_{k+1}(r) \leqslant V_n(r),$$
$$\forall r \in [0, R_n),\ \forall k \geqslant 1. \tag{5.95}$$

This implies that $(u_k(r))_{k\geqslant 1}$ and $(v_k(r))_{k\geqslant 1}$ converge for any $r \in [0, R_n)$. Moreover, $(U_{n+1}, V_{n+1}) = \lim_{k\to\infty}(u_k, v_k)$ is a radial solution of (5.68) in $B(0, R_n)$ with central values $(a + 1/(n+1), b + 1/(n+1))$. By the definition of R_{n+1}, it follows that $R_{n+1} \geqslant R_n$ for any $n \geqslant 1$.

Set $R := \lim_{n\to\infty} R_n$ and let $0 \leqslant r < R$ be arbitrary. Then, there exists $n_1 = n_1(r)$ such that $r < R_n$ for all $n \geqslant n_1$. From (5.95) we see that $U_{n+1} \leqslant U_n$ (resp., $V_{n+1} \leqslant V_n$) on $[0, R_n)$ for all $n \geqslant 1$. So, there exists $\lim_{n\to\infty}(U_n(r), V_n(r))$ which, by (5.93) and (5.94), is a radial solution of (5.68) in $B(0, R)$ with central values (a, b). Consequently,

$$\lim_{n\to\infty} U_n(r) = U(r) \quad \text{and} \quad \lim_{n\to\infty} V_n(r) = V(r) \quad \text{for any } r \in [0, R). \tag{5.96}$$

Since $U_n'(r) \geqslant 0$, from (5.94) we find

$$V_n(r) \leqslant b + \frac{1}{n} + f\big(U_n(r)\big) \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\,\mathrm{d}s\,\mathrm{d}t.$$

This yields

$$V_n(r) \leqslant C_1 U_n(r) + C_2 f\big(U_n(r)\big) \tag{5.97}$$

where C_1 is an upper bound of $(V(0) + 1/n)/(U(0) + 1/n)$ and

$$C_2 = \int_0^\infty t^{1-N} \int_0^t s^{N-1} q(s)\,\mathrm{d}s\,\mathrm{d}t \leqslant \frac{1}{N-2} \int_0^\infty s q(s)\,\mathrm{d}s < \infty.$$

Define $h(t) = g(C_1 t + C_2 f(t))$ for $t \geqslant 0$. It is easy to check that h satisfies (A_1) and (A_2). Define

$$\Gamma(s) = \int_s^\infty \frac{\mathrm{d}t}{h(t)}, \quad \text{for all } s > 0.$$

But U_n verifies

$$\Delta U_n = p(x) g(V_n)$$

which combined with (5.97) implies

$$\Delta U_n \leqslant p(x) h(U_n).$$

A simple calculation shows that

$$\begin{aligned}\Delta\Gamma(U_n) &= \Gamma'(U_n)\Delta U_n + \Gamma''(U_n)|\nabla U_n|^2 = \frac{-1}{h(U_n)}\Delta U_n + \frac{h'(U_n)}{[h(U_n)]^2}|\nabla U_n|^2 \\ &\geqslant \frac{-1}{h(U_n)} p(r) h(U_n) = -p(r)\end{aligned}$$

which we rewrite as

$$\left(r^{N-1}\frac{\mathrm{d}}{\mathrm{d}r}\Gamma(U_n)\right)' \geqslant -r^{N-1}p(r) \quad \text{for any } 0 < r < R_n.$$

Fix $0 < r < R$. Then $r < R_n$ for all $n \geqslant n_1$ provided n_1 is large enough. Integrating the above inequality over $[0, r]$, we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\Gamma(U_n) \geqslant -r^{1-N}\int_0^r s^{N-1}p(s)\,\mathrm{d}s.$$

Integrating this new inequality over $[r, R_n]$ we obtain

$$-\Gamma\big(U_n(r)\big) \geqslant -\int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall n \geqslant n_1,$$

since $U_n(r) \to \infty$ as $r \nearrow R_n$ implies $\Gamma(U_n(r)) \to 0$ as $r \nearrow R_n$. Therefore,

$$\Gamma\big(U_n(r)\big) \leqslant \int_r^{R_n} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t, \quad \forall n \geqslant n_1.$$

Letting $n \to \infty$ and using (5.96) we find

$$\Gamma\big(U(r)\big) \leqslant \int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t,$$

or, equivalently

$$U(r) \geqslant \Gamma^{-1}\left(\int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t\right).$$

Passing to the limit as $r \nearrow R$ and using the fact that $\lim_{s\searrow 0}\Gamma^{-1}(s) = \infty$ we deduce

$$\lim_{r\nearrow R} U(r) \geqslant \lim_{r\nearrow R}\Gamma^{-1}\left(\int_r^{R} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t\right) = \infty.$$

But (U, V) is an entire solution so that we conclude $R = \infty$ and $\lim_{r\to\infty} U(r) = \infty$. Since (5.70) holds and $V'(r) \geqslant 0$ we find

$$\begin{aligned} U(r) &\leqslant a + g\big(V(r)\big)\int_0^{\infty} t^{1-N}\int_0^t s^{N-1}p(s)\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant a + g\big(V(r)\big)\frac{1}{N-2}\int_0^{\infty} tp(t)\,\mathrm{d}t, \quad \forall r \geqslant 0. \end{aligned}$$

We deduce $\lim_{r\to\infty} V(r) = \infty$, otherwise we obtain that $\lim_{r\to\infty} U(r)$ is finite, a contradiction. Consequently, (U, V) is an entire large solution of (5.68). This concludes our proof. □

6. Bifurcation problems for singular Lane–Emden–Fowler equations

In this section we study the bifurcation problem

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\lambda \in \mathbb{R}$ is a parameter and $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is a bounded domain with smooth boundary $\partial\Omega$. The main feature of this boundary value problem is the presence of the "smooth" nonlinearity f combined with the "singular" nonlinearity g. More exactly, we assume that $0 < f \in C^{0,\beta}[0,\infty)$ and $0 \leqslant g \in C^{0,\beta}(0,\infty)$ $(0 < \beta < 1)$ fulfill the hypotheses

(f1) f is nondecreasing on $(0,\infty)$ while $f(s)/s$ is nonincreasing for $s > 0$;

(g1) g is nonincreasing on $(0,\infty)$ with $\lim_{s \searrow 0} g(s) = +\infty$;

(g2) there exists $C_0, \eta_0 > 0$ and $\alpha \in (0,1)$ so that $g(s) \leqslant C_0 s^{-\alpha}$, $\forall s \in (0,\eta_0)$.

The assumption (g2) implies the following Keller–Osserman-type growth condition around the origin

$$\int_0^1 \left(\int_0^t g(s)\,\mathrm{d}s \right)^{-1/2} dt < +\infty. \tag{6.98}$$

As proved by Bénilan, Brezis and Crandall in [11], condition (6.98) is equivalent to the *property of compact support*, that is, for any $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and

$$-\Delta u + g(u) = h \quad \text{a.e. in } \mathbb{R}^N.$$

In many papers (see, e.g., Dalmasso [35], Kusano and Swanson [63]) the potential $a(x)$ is assumed to depend "almost" radially on x, in the sense that $C_1 p(|x|) \leqslant a(x) \leqslant C_2 p(|x|)$, where C_1, C_2 are positive constants and $p(|x|)$ is a positive function satisfying some integrability condition. We do not impose any growth assumption on a, but we suppose throughout this section that the variable potential $a(x)$ satisfies $a \in C^{0,\beta}(\overline{\Omega})$ and $a > 0$ in Ω.

If $\lambda = 0$ this equation is called the Lane–Emden–Fowler equation and arises in the boundary-layer theory of viscous fluids (see Wong [91]). Problems of this type, as well as the associated evolution equations, describe naturally certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes [36] as a model for long range Van der Waals interactions in thin films spreading on solid surfaces.

Our purpose is to study the effect of the asymptotically linear perturbation $f(u)$ in (P_λ), as well as to describe the set of values of the positive parameter λ such that problem (P_λ) admits a solution. In this case, we also prove a uniqueness result. Due to the singular character of (P_λ), we cannot expect to find solutions in $C^2(\overline{\Omega})$. However, under the above assumptions we will show that (P_λ) has solutions in the class

$$\mathcal{E} := \left\{ u \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega});\ \Delta u \in L^1(\Omega) \right\}.$$

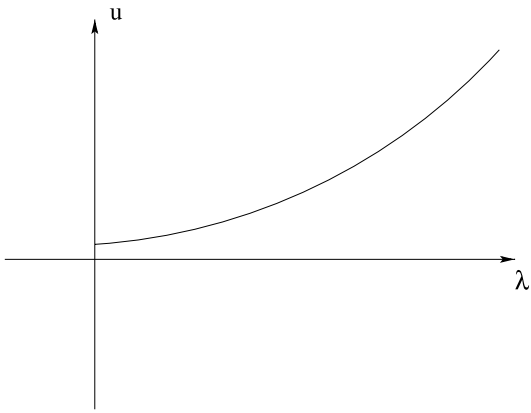

Fig. 1. The "sublinear" case $m = 0$.

We first observe that, in view of the assumption $(f1)$, there exists

$$m := \lim_{s\to\infty} \frac{f(s)}{s} \in [0, \infty).$$

This number plays a crucial role in our analysis. More precisely, the existence of the solutions to (P_λ) will be separately discussed for $m > 0$ and $m = 0$. Let $a_* = \min_{x\in\overline{\Omega}} a(x)$.

Theorems 6.1–6.4 have been established by Cîrstea, Ghergu, and Rădulescu [23].

THEOREM 6.1. *Assume* (f1), (g1), (g2) *and* $m = 0$. *If* $a_* > 0$ (*resp.*, $a_* = 0$), *then* (P_λ) *has a unique solution* $u_\lambda \in \mathcal{E}$ *for all* $\lambda \in \mathbb{R}$ (*resp.*, $\lambda \geqslant 0$) *with the properties*:

(i) u_λ *is strictly increasing with respect to* λ;

(ii) *there exist two positive constant* $c_1, c_2 > 0$ *depending on* λ *such that* $c_1 d(x) \leqslant u_\lambda \leqslant c_2 d(x)$ *in* Ω.

The bifurcation diagram in the "sublinear" case $m = 0$ is depicted in Figure 1.

PROOF. We first recall some auxiliary results that we need in the proof.

LEMMA 6.2 (Shi and Yao [86]). *Let* $F : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ *be a Hölder continuous function with exponent* $\beta \in (0, 1)$, *on each compact subset of* $\overline{\Omega} \times (0, \infty)$, *which satisfies*

(F1) $\limsup_{s\to+\infty}(s^{-1} \max_{x\in\overline{\Omega}} F(x, s)) < \lambda_1$;

(F2) *for each* $t > 0$, *there exists a constant* $D(t) > 0$, *such that*

$$F(x, r) - F(x, s) \geqslant -D(t)(r - s), \quad \text{for } x \in \overline{\Omega} \text{ and } r \geqslant s \geqslant t;$$

(F3) *there exists* $\eta_0 > 0$, *and an open subset* $\Omega_0 \subset \Omega$, *such that*

$$\min_{x\in\overline{\Omega}} F(x, s) \geqslant 0 \quad \text{for } s \in (0, \eta_0),$$

and

$$\lim_{s\searrow 0}\frac{F(x,s)}{s}=+\infty \quad \textit{uniformly for } x\in\Omega_0.$$

Then for any nonnegative function $\phi_0\in C^{2,\beta}(\partial\Omega)$, the problem

$$\begin{cases} -\Delta u = F(x,u) & \textit{in } \Omega,\\ u>0 & \textit{in } \Omega,\\ u=\phi_0 & \textit{on } \partial\Omega, \end{cases}$$

has at least one positive solution $u\in C^{2,\beta}(G)\cap C(\overline{\Omega})$, for any compact set $G\subset\Omega\cup\{x\in\partial\Omega;\ \phi_0(x)>0\}$.

LEMMA 6.3 (Shi and Yao [86]). *Let $F:\overline{\Omega}\times(0,\infty)\to\mathbb{R}$ be a continuous function such that the mapping $(0,\infty)\ni s\mapsto F(x,s)/s$, is strictly decreasing at each $x\in\Omega$. Assume that there exists $v, w\in C^2(\Omega)\cap C(\overline{\Omega})$ such that*

(a) $\Delta w+F(x,w)\leqslant 0\leqslant \Delta v+F(x,v)$ *in* Ω;
(b) $v, w>0$ *in* Ω *and* $v\leqslant w$ *on* $\partial\Omega$;
(c) $\Delta v\in L^1(\Omega)$.
Then $v\leqslant w$ *in* Ω.

Now, we are ready to give the proof of Theorem 6.1. This will be divided into four steps.

Step 1. *Existence of solutions to problem* (P_λ).

For any $\lambda\in\mathbb{R}$, define the function

$$\Phi_\lambda(x,s)=\lambda f(s)+a(x)g(s),\quad (x,s)\in\overline{\Omega}\times(0,\infty). \tag{6.99}$$

Taking into account the assumptions of Theorem 6.1, it follows that Φ_λ verifies the hypotheses of Lemma 6.2 for $\lambda\in\mathbb{R}$ if $a_*>0$ and $\lambda\geqslant 0$ if $a_*=0$. Hence, for λ in the above range, (P_λ) has at least one solution $u_\lambda\in C^{2,\beta}(\Omega)\cap C(\overline{\Omega})$.

Step 2. *Uniqueness of solution*.

Fix $\lambda\in\mathbb{R}$ (resp., $\lambda\geqslant 0$) if $a_*>0$ (resp., $a_*=0$). Let u_λ be a solution of (P_λ). Denote $\lambda^-=\min\{0,\lambda\}$ and $\lambda^+=\max\{0,\lambda\}$. We claim that $\Delta u_\lambda\in L^1(\Omega)$. Since $a\in C^{0,\beta}(\overline{\Omega})$, by [54, Theorem 6.14], there exists a unique nonnegative solution $\zeta\in C^{2,\beta}(\overline{\Omega})$ of

$$\begin{cases} -\Delta\zeta=a(x) & \text{in } \Omega,\\ \zeta=0 & \text{on } \partial\Omega. \end{cases}$$

By the weak maximum principle (see e.g., [54, Theorem 2.2]), $\zeta>0$ in Ω. Moreover, we are going to prove that

(a) $z(x):=c\zeta(x)$ is a sub-solution of (P_λ), for $c>0$ small enough;
(b) $z(x)\geqslant c_1 d(x)$ in $\overline{\Omega}$, for some positive constant $c_1>0$;
(c) $u_\lambda\geqslant z$ in $\overline{\Omega}$.

Therefore, by (b) and (c), $u_\lambda \geqslant c_1 d(x)$ in $\overline{\Omega}$. Using (g2), we obtain $g(u_\lambda) \leqslant C d^{-\alpha}(x)$ in Ω, where $C > 0$ is a constant. So, $g(u_\lambda) \in L^1(\Omega)$. This implies

$$\Delta u_\lambda \in L^1(\Omega).$$

Proof of (a). Using (f1) and (g1), we have

$$\begin{aligned}\Delta z(x) + \Phi_\lambda(x,z) &= -ca(x) + \lambda f(c\zeta) + a(x) g(c\zeta) \\ &\geqslant -ca(x) + \lambda^- f\big(c\|\zeta\|_\infty\big) + a(x) g\big(c\|\zeta\|_\infty\big) \\ &\geqslant ca(x)\left[\frac{g(c\|\zeta\|_\infty)}{2c} - 1\right] \\ &\quad + f\big(c\|\zeta\|_\infty\big)\left[a_* \frac{g(c\|\zeta\|_\infty)}{2 f(c\|\zeta\|_\infty)} + \lambda^-\right]\end{aligned}$$

for each $x \in \Omega$. Since $\lambda < 0$ corresponds to $a_* > 0$, using $\lim_{t \searrow 0} g(t) = +\infty$ and $\lim_{t \to 0} f(t) \in (0, \infty)$, we can find $c > 0$ small such that

$$\Delta z + \Phi_\lambda(x,z) \geqslant 0, \quad \forall x \in \Omega.$$

This concludes (a).

Proof of (b). Since $\zeta \in C^{2,\beta}(\overline{\Omega})$, $\zeta > 0$ in Ω and $\zeta = 0$ on $\partial\Omega$, by Lemma 3.4 in Gilbarg and Trudinger [54], we have

$$\frac{\partial \zeta}{\partial \nu}(y) < 0, \quad \forall y \in \partial\Omega.$$

Therefore, there exists a positive constant c_0 such that

$$\frac{\partial \zeta}{\partial \nu}(y) := \lim_{x \in \Omega, x \to y} \frac{\zeta(y) - \zeta(x)}{|x - y|} \leqslant -c_0, \quad \forall y \in \partial\Omega.$$

So, for each $y \in \Omega$, there exists $r_y > 0$ such that

$$\frac{\zeta(x)}{|x - y|} \geqslant \frac{c_0}{2}, \quad \forall x \in B_{r_y}(y) \cap \Omega. \tag{6.100}$$

Using the compactness of $\partial\Omega$, we can find a finite number k of balls $B_{r_{y_i}}(y_i)$ such that $\partial\Omega \subset \bigcup_{i=1}^k B_{r_{y_i}}(y_i)$. Moreover, we can assume that for small $d_0 > 0$,

$$\big\{x \in \Omega\colon\ d(x) < d_0\big\} \subset \bigcup_{i=1}^k B_{r_{y_i}}(y_i).$$

Therefore, by (6.100) we obtain

$$\zeta(x) \geqslant \frac{c_0}{2}\, d(x), \quad \forall x \in \Omega \text{ with } d(x) < d_0.$$

This fact, combined with $\zeta > 0$ in Ω, shows that for some constant $\tilde{c} > 0$

$$\zeta(x) \geqslant \tilde{c} d(x), \quad \forall x \in \Omega.$$

Thus, (b) follows by the definition of z.

Proof of (c). We distinguish two cases:

CASE 1. $\lambda \geqslant 0$. We see that Φ_λ verifies the hypotheses in Lemma 6.3. Since

$$\begin{aligned}
&\Delta u_\lambda + \Phi_\lambda(x, u_\lambda) \leqslant 0 \leqslant \Delta z + \Phi_\lambda(x, z) \quad \text{in } \Omega,\\
&u_\lambda, z > 0 \quad \text{in } \Omega,\\
&u_\lambda = z \quad \text{on } \partial\Omega,\\
&\Delta z \in L^1(\Omega),
\end{aligned}$$

by Lemma 6.3 it follows that $u_\lambda \geqslant z$ in $\overline{\Omega}$.

Now, if u_1 and u_2 are two solutions of (P_λ), we can use Lemma 6.3 in order to deduce that $u_1 = u_2$.

CASE 2. $\lambda < 0$ (corresponding to $a_* > 0$). Let $\varepsilon > 0$ be fixed. We prove that

$$z \leqslant u_\lambda + \varepsilon\big(1 + |x|^2\big)^\tau \quad \text{in } \Omega, \tag{6.101}$$

where $\tau < 0$ is chosen such that $\tau|x|^2 + 1 > 0$, $\forall x \in \Omega$. This is always possible since $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is bounded.

We argue by contradiction. Suppose that there exists $x_0 \in \Omega$ such that $u_\lambda(x_0) + \varepsilon(1 + |x_0|)^\tau < z(x_0)$. Then

$$\min_{x \in \overline{\Omega}} \big\{u_\lambda(x) + \varepsilon\big(1 + |x|^2\big)^\tau - z(x)\big\} < 0$$

is achieved at some point $x_1 \in \Omega$. Since $\Phi_\lambda(x, z)$ is nonincreasing in z, we have

$$\begin{aligned}
0 &\geqslant -\Delta\big[u_\lambda(x) - z(x) + \varepsilon\big(1 + |x|^2\big)^\tau\big]|_{x=x_1}\\
&= \Phi_\lambda\big(x_1, u_\lambda(x_1)\big) - \Phi_\lambda\big(x_1, z(x_1)\big) - \varepsilon\Delta\big[\big(1 + |x|^2\big)^\tau\big]|_{x=x_1}\\
&\geqslant -\varepsilon\Delta\big[(1 + |x|^2)^\tau\big]|_{x=x_1} = -2\varepsilon\tau(1 + |x_1|^2)^{\tau-2}\big[(N + 2\tau - 2)|x_1|^2 + N\big]\\
&\geqslant -4\varepsilon\tau\big(1 + |x_1|^2\big)^{\tau-2}\big(\tau|x_1|^2 + 1\big) > 0.
\end{aligned}$$

This contradiction proves (6.101). Passing to the limit $\varepsilon \to 0$, we obtain (c).

In a similar way we can prove that (P_λ) has a unique solution.

Step 3. *Dependence on* λ.

We fix $\lambda_1 < \lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ if $a_* > 0$ resp., $\lambda_1, \lambda_2 \in [0, \infty)$ if $a_* = 0$. Let $u_{\lambda_1}, u_{\lambda_2}$ be the corresponding solutions of (P_{λ_1}) and (P_{λ_2}) respectively.

If $\lambda_1 \geqslant 0$, then Φ_{λ_1} verifies the hypotheses in Lemma 6.3. Furthermore, we have

$$\begin{aligned}
&\Delta u_{\lambda_2} + \Phi_{\lambda_1}(x, u_{\lambda_2}) \leqslant 0 \leqslant \Delta u_{\lambda_1} + \Phi_{\lambda_1}(x, u_{\lambda_1}) \quad \text{in } \Omega,\\
&u_{\lambda_1}, u_{\lambda_2} > 0 \quad \text{in } \Omega,\\
&u_{\lambda_1} = u_{\lambda_2} \quad \text{on } \partial\Omega,\\
&\Delta u_{\lambda_1} \in L^1(\Omega).
\end{aligned}$$

Again by Lemma 6.3, we conclude that $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$. Moreover, by the maximum principle, $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

Let $\lambda_2 \leqslant 0$; we show that $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$. Indeed, supposing the contrary, there exists $x_0 \in \Omega$ such that $u_{\lambda_1}(x_0) > u_{\lambda_2}(x_0)$. We conclude now that $\max_{x\in\overline{\Omega}}\{u_{\lambda_1}(x) - u_{\lambda_2}(x)\} > 0$ is achieved at some point in Ω. At that point, say $\bar{x}$, we have

$$0 \leqslant -\Delta(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = \Phi_{\lambda_1}\big(\bar{x}, u_{\lambda_1}(\bar{x})\big) - \Phi_{\lambda_2}\big(\bar{x}, u_{\lambda_2}(\bar{x})\big) < 0,$$

which is a contradiction. It follows that $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$, and by maximum principle we have $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

If $\lambda_1 < 0 < \lambda_2$, then $u_{\lambda_1} < u_0 < u_{\lambda_2}$ in Ω. This finishes the proof of Step 3.

Step 4. *Regularity of the solution.*

Fix $\lambda \in \mathbb{R}$ and let $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$ be the unique solution of (P_λ). An important result in our approach is the following estimate

$$c_1 d(x) \leqslant u_\lambda(x) \leqslant c_2 d(x), \quad \text{for all } x \in \Omega, \tag{6.102}$$

where c_1, c_2 are positive constants. The first inequality in (6.102) was established in Step 2. For the second one, we apply an idea found in Gui and Lin [56].

Using the smoothness of $\partial\Omega$, we can find $\delta \in (0,1)$ such that for all $x_0 \in \Omega_\delta := \{x \in \Omega; d(x) \leqslant \delta\}$, there exists $y \in \mathbb{R}^N \setminus \overline{\Omega}$ with $d(y, \partial\Omega) = \delta$ and $d(x_0) = |x_0 - y| - \delta$.

Let $K > 1$ be such that $\operatorname{diam}(\Omega) < (K-1)\delta$ and let w be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta w = \lambda^+ f(w) + g(w) & \text{in } B_K(0) \setminus \overline{B_1}(0),\\ w > 0 & \text{in } B_K(0) \setminus \overline{B_1}(0),\\ w = 0 & \text{on } \partial(B_K(0) \setminus \overline{B_1}(0)), \end{cases} \tag{6.103}$$

where $B_r(0)$ is the open ball in $\mathbb{R}^N$ of radius r and centered at the origin. By uniqueness, w is radially symmetric. Hence $w(x) = \widetilde{w}(|x|)$ and

$$\begin{cases} \widetilde{w}'' + \frac{N-1}{r}\widetilde{w}' + \lambda^+ f(\widetilde{w}) + g(\widetilde{w}) = 0 & \text{for } r \in (1, K),\\ \widetilde{w} > 0 & \text{in } (1, K),\\ \widetilde{w}(1) = \widetilde{w}(K) = 0. \end{cases} \tag{6.104}$$

Integrating in (6.104) we have

$$\begin{aligned}\widetilde{w}'(t) &= \widetilde{w}'(a)a^{N-1}t^{1-N} - t^{1-N}\int_a^t r^{N-1}\big[\lambda^+ f\big(\widetilde{w}(r)\big) + g\big(\widetilde{w}(r)\big)\big]\,\mathrm{d}r,\\ &= \widetilde{w}'(b)b^{N-1}t^{1-N} + t^{1-N}\int_t^b r^{N-1}\big[\lambda^+ f\big(\widetilde{w}(r)\big) + g\big(\widetilde{w}(r)\big)\big]\,\mathrm{d}r,\end{aligned}$$

where $1 < a < t < b < K$. Since $g(\widetilde{w}) \in L^1(1, K)$, we deduce that both $\widetilde{w}'(1)$ and $\widetilde{w}'(K)$ are finite, so $\widetilde{w} \in C^2(1, K) \cap C^1[1, K]$. Furthermore,

$$w(x) \leqslant C\min\big\{K - |x|, |x| - 1\big\}, \quad \text{for any } x \in B_K(0) \setminus B_1(0). \tag{6.105}$$

Let us fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ with $d(y_0, \partial\Omega) = \delta$ and $d(x_0) = |x_0 - y| - \delta$. Thus, $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define $v(x) = cw((x - y_0)/\delta)$, $x \in \overline{\Omega}$. We show that v is a super-solution of (P_λ), provided that c is large enough. Indeed, if $c > \max\{1, \delta^2\|a\|_\infty\}$, then for all $x \in \Omega$ we have

$$\begin{aligned}\Delta v + \lambda f(v) + a(x)g(v) &\leqslant \frac{c}{\delta^2}\left(\widetilde{w}''(r) + \frac{N-1}{r}\widetilde{w}'(r)\right)\\ &\quad + \lambda^+ f\big(c\widetilde{w}(r)\big) + a(x)g\big(c\widetilde{w}(r)\big),\end{aligned}$$

where $r = |x - y_0|/\delta \in (1, K)$. Using the assumption (f1) we get $f(c\widetilde{w}) \leqslant cf(\widetilde{w})$ in $(1, K)$. The above relations lead us to

$$\begin{aligned}\Delta v + \lambda f(v) + a(x)g(v) &\leqslant \frac{c}{\delta^2}\left(\widetilde{w}'' + \frac{N-1}{r}\widetilde{w}'\right) + \lambda^+ cf(\widetilde{w}) + \|a\|_\infty g(\widetilde{w})\\ &\leqslant \frac{c}{\delta^2}\left(\widetilde{w}'' + \frac{N-1}{r}\widetilde{w}' + \lambda^+ f(\widetilde{w}) + g(\widetilde{w})\right)\\ &= 0.\end{aligned}$$

Since $\Delta u_\lambda \in L^1(\Omega)$, with a similar proof as in Step 2 we get $u_\lambda \leqslant v$ in Ω. This combined with (6.105) yields

$$u_\lambda(x_0) \leqslant v(x_0) \leqslant C\min\left\{K - \frac{|x_0 - y_0|}{\delta}, \frac{|x_0 - y_0|}{\delta} - 1\right\} \leqslant \frac{C}{\delta}d(x_0).$$

Hence $u_\lambda \leqslant (C/\delta)d(x)$ in Ω_δ and the last inequality in (6.102) follows.

Let G be the Green's function associated with the Laplace operator in Ω. Then, for all $x \in \Omega$ we have

$$u_\lambda(x) = -\int_\Omega G(x, y)\big[\lambda f\big(u_\lambda(y)\big) + a(y)g\big(u_\lambda(y)\big)\big]\,\mathrm{d}y,$$

and

$$\nabla u_\lambda(x) = -\int_\Omega G_x(x,y)\big[\lambda f\big(u_\lambda(y)\big) + a(y)g\big(u_\lambda(y)\big)\big]\,\mathrm{d}y.$$

If $x_1, x_2 \in \Omega$, using (g2) we obtain

$$\begin{aligned}\big|\nabla u_\lambda(x_1) - \nabla u_\lambda(x_2)\big| \leqslant |\lambda| \int_\Omega \big|G_x(x_1,y) - G_x(x_2,y)\big| \cdot f\big(u_\lambda(y)\big)\,\mathrm{d}y \\ + \tilde{c}\int_\Omega \big|G_x(x_1,y) - G_x(x_2,y)\big| \cdot u_\lambda^{-\alpha}(y)\,\mathrm{d}y.\end{aligned}$$

Now, taking into account that $u_\lambda \in C(\overline{\Omega})$, by the standard regularity theory (see Gilbarg and Trudinger [54]) we get

$$\int_\Omega \big|G_x(x_1,y) - G_x(x_2,y)\big| \cdot f\big(u_\lambda(y)\big) \leqslant \tilde{c}_1|x_1 - x_2|.$$

On the other hand, with the same proof as in [56, Theorem 1], we deduce

$$\int_\Omega \big|G_x(x_1,y) - G_x(x_2,y)\big| \cdot u_\lambda^{-\alpha}(y) \leqslant \tilde{c}_2|x_1 - x_2|^{1-\alpha}.$$

The above inequalities imply $u_\lambda \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega})$. The proof of Theorem 6.1 is now complete. □

Next, consider the case $m > 0$. The results in this case are different from those presented in Theorem 6.1. A careful examination of (P_λ) reveals the fact that the singular term $g(u)$ is not significant. Actually, the conclusions are close to those established in Mironescu and Rădulescu [77, Theorem A], where an elliptic problem associated to an asymptotically linear function is studied.

Let λ_1 be the first Dirichlet eigenvalue of $(-\Delta)$ in Ω and $\lambda^* = \lambda_1/m$. Our result in this case is the following.

THEOREM 6.4. *Assume* (f1), (g1), (g2) *and* $m > 0$. *Then the following hold.*

(i) *If* $\lambda \geqslant \lambda^*$, *then* (P_λ) *has no solutions in* $\mathcal{E}$.

(ii) *If* $a_* > 0$ (*resp.* $a_* = 0$) *then* (P_λ) *has a unique solution* $u_\lambda \in \mathcal{E}$ *for all* $-\infty < \lambda < \lambda^*$ (*resp.* $0 < \lambda < \lambda^*$) *with the properties*:

- (ii1) u_λ *is strictly increasing with respect to* λ;
- (ii2) *there exists two positive constants* $c_1, c_2 > 0$ *depending on* λ *such that* $c_1 d(x) \leqslant u_\lambda \leqslant c_2 d(x)$ *in* Ω;
- (ii3) $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, *uniformly on compact subsets of* Ω.

The bifurcation diagram in the "linear" case $m > 0$ is depicted in Figure 2.

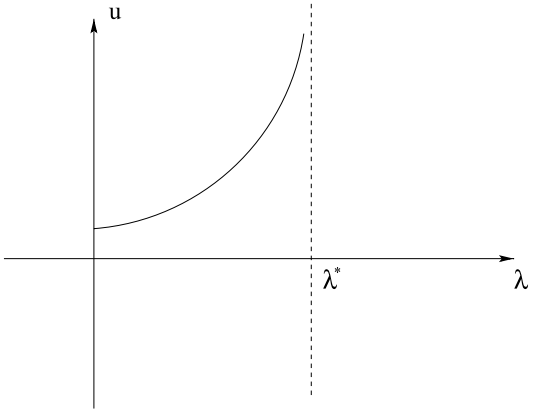

Fig. 2. The "linear" case $m > 0$.

PROOF. (i) Let ϕ_1 be the first eigenfunction of the Laplace operator in Ω with Dirichlet boundary condition. Arguing by contradiction, let us suppose that there exists $\lambda \geqslant \lambda^*$ such that (P_λ) has a solution $u_\lambda \in \mathcal{E}$.

Multiplying by ϕ_1 in (P_λ) and then integrating over Ω we get

$$-\int_\Omega \phi_1 \Delta u_\lambda = \lambda \int_\Omega f(u_\lambda)\phi_1 + \int_\Omega a(x)g(u_\lambda)\phi_1. \tag{6.106}$$

Since $\lambda \geqslant \lambda_1/m$, in view of the assumption (f1) we get $\lambda f(u_\lambda) \geqslant \lambda_1 u_\lambda$ in Ω. Using this fact in (6.106) we obtain

$$-\int_\Omega \phi_1 \Delta u_\lambda > \lambda_1 \int_\Omega u_\lambda \phi_1.$$

The regularity of u_λ yields $-\int_\Omega u_\lambda \Delta\phi_1 > \lambda_1 \int_\Omega u_\lambda\phi_1$. This is clearly a contradiction since $-\Delta\phi_1 = \lambda_1\phi_1$ in Ω. Hence (P_λ) has no solutions in $\mathcal{E}$ for any $\lambda \geqslant \lambda^*$.

(ii) From now on, the proof of the existence, uniqueness and regularity of solution is the same as in Theorem 6.1.

(ii3) In what follows we shall apply some ideas developed in Mironescu and Rădulescu [77]. Due to the special character of our problem, we will be able to prove that, in certain cases, L^2-boundedness implies H_0^1-boundedness!

Let $u_\lambda \in \mathcal{E}$ be the unique solution of (P_λ) for $0 < \lambda < \lambda^*$. We prove that $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. Suppose the contrary. Since $(u_\lambda)_{0<\lambda<\lambda^*}$ is a sequence of nonnegative super-harmonic functions in Ω, by Theorem 4.1.9 in Hörmander [60], there exists a subsequence of $(u_\lambda)_{\lambda<\lambda^*}$ (still denoted by $(u_\lambda)_{\lambda<\lambda^*}$) which is convergent in $L^1_{\mathrm{loc}}(\Omega)$.

We first prove that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We argue by contradiction. Suppose that $(u_\lambda)_{\lambda<\lambda^*}$ is not bounded in $L^2(\Omega)$. Thus, passing eventually at a subsequence we have

$u_\lambda = M(\lambda) w_\lambda$, where

$$\begin{aligned} &M(\lambda) = \|u_\lambda\|_{L^2(\Omega)} \to \infty \quad \text{as } \lambda \nearrow \lambda^* \quad \text{and} \\ &w_\lambda \in L^2(\Omega), \quad \|w_\lambda\|_{L^2(\Omega)} = 1. \end{aligned} \tag{6.107}$$

Using (f1), (g2) and the monotonicity assumption on g, we deduce the existence of $A, B, C, D > 0$ $(A > m)$ such that

$$f(t) \leqslant At + B, \quad g(t) \leqslant Ct^{-\alpha} + D, \quad \text{for all } t > 0. \tag{6.108}$$

This implies

$$\frac{1}{M(\lambda)}\big(\lambda f(u_\lambda) + a(x) g(u_\lambda)\big) \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*$$

that is,

$$-\Delta w_\lambda \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*. \tag{6.109}$$

By Green's first identity, we have

$$\begin{aligned} \int_\Omega \nabla w_\lambda \cdot \nabla \phi \, \mathrm{d}x &= -\int_\Omega \phi \Delta w_\lambda \, \mathrm{d}x \\ &= -\int_{\operatorname{Supp}\phi} \phi \Delta w_\lambda \, \mathrm{d}x \quad \forall \phi \in C_0^\infty(\Omega). \end{aligned} \tag{6.110}$$

Using (6.109) we derive that

$$\begin{aligned} \left| \int_{\operatorname{Supp}\phi} \phi \Delta w_\lambda \, \mathrm{d}x \right| &\leqslant \int_{\operatorname{Supp}\phi} |\phi| |\Delta w_\lambda| \, \mathrm{d}x \\ &\leqslant \|\phi\|_{L^\infty} \int_{\operatorname{Supp}\phi} |\Delta w_\lambda| \, \mathrm{d}x \to 0 \quad \text{as } \lambda \nearrow \lambda^*. \end{aligned} \tag{6.111}$$

Combining (6.110) and (6.111), we arrive at

$$\int_\Omega \nabla w_\lambda \cdot \nabla \phi \, \mathrm{d}x \to 0 \quad \text{as } \lambda \nearrow \lambda^*, \ \forall \phi \in C_0^\infty(\Omega). \tag{6.112}$$

By definition, the sequence $(w_\lambda)_{0<\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$.

We claim that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, using (6.108) and Hölder's inequality, we have

$$\int_\Omega |\nabla w_\lambda|^2 = -\int_\Omega w_\lambda \Delta w_\lambda = \frac{-1}{M(\lambda)} \int_\Omega w_\lambda \Delta u_\lambda$$

$$
\begin{aligned}
&= \frac{1}{M(\lambda)} \int_{\Omega} \left[\lambda w_\lambda f(u_\lambda) + a(x) g(u_\lambda) w_\lambda\right] \\
&\leqslant \frac{\lambda}{M(\lambda)} \int_{\Omega} w_\lambda (A u_\lambda + B) + \frac{\|a\|_\infty}{M(\lambda)} \int_{\Omega} w_\lambda (C u_\lambda^{-\alpha} + D) \\
&= \lambda A \int_{\Omega} w_\lambda^2 + \frac{\|a\|_\infty C}{M(\lambda)^{1+\alpha}} \int_{\Omega} w_\lambda^{1-\alpha} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)} \int_{\Omega} w_\lambda \\
&\leqslant \lambda^* A + \frac{\|a\|_\infty C}{M(\lambda)^{1+\alpha}} |\Omega|^{(1+\alpha)/2} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)} |\Omega|^{1/2}.
\end{aligned}
$$

From the above estimates, it is easy to see that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$, so the claim is proved. Then, there exists $w \in H_0^1(\Omega)$ such that (up to a subsequence)

$$
w_\lambda \rightharpoonup w \quad \text{weakly in } H_0^1(\Omega) \text{ as } \lambda \nearrow \lambda^* \tag{6.113}
$$

and, because $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$,

$$
w_\lambda \to w \quad \text{strongly in } L^2(\Omega) \text{ as } \lambda \nearrow \lambda^*. \tag{6.114}
$$

On the one hand, by (6.107) and (6.114), we derive that $\|w\|_{L^2(\Omega)} = 1$. Furthermore, using (6.112) and (6.113), we infer that

$$
\int_{\Omega} \nabla w \cdot \nabla \phi \, \mathrm{d}x = 0, \quad \forall \phi \in C_0^\infty(\Omega).
$$

Since $w \in H_0^1(\Omega)$, using the above relation and the definition of $H_0^1(\Omega)$, we get $w = 0$. This contradiction shows that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. As above for w_λ, we can derive that u_λ is bounded in $H_0^1(\Omega)$. So, there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence,

$$
\begin{cases}
u_\lambda \rightharpoonup u^* & \text{weakly in } H_0^1(\Omega) \text{ as } \lambda \nearrow \lambda^*, \\
u_\lambda \to u^* & \text{strongly in } L^2(\Omega) \text{ as } \lambda \nearrow \lambda^*, \\
u_\lambda \to u^* & \text{a.e. in } \Omega \text{ as } \lambda \nearrow \lambda^*.
\end{cases} \tag{6.115}
$$

Now we can proceed to get a contradiction. Multiplying by ϕ_1 in (P_λ) and integrating over Ω we have

$$
\begin{aligned}
&-\int_{\Omega} \varphi_1 \Delta u_\lambda = \lambda \int_{\Omega} f(u_\lambda)\varphi_1 + \int_{\Omega} a(x) g(u_\lambda)\varphi_1, \\
&\quad \text{for all } 0 < \lambda < \lambda^*.
\end{aligned} \tag{6.116}
$$

On the other hand, by (f1) it follows that $f(u_\lambda) \geqslant m u_\lambda$ in Ω, for all $0 < \lambda < \lambda^*$. Combining this with (6.116) we obtain

$$\lambda_1 \int_\Omega u_\lambda \varphi_1 \geqslant \lambda m \int_\Omega u_\lambda \varphi_1 + \int_\Omega a(x) g(u_\lambda) \varphi_1, \quad \text{for all } 0 < \lambda < \lambda^*. \tag{6.117}$$

Notice that by (g1), (6.115) and the monotonicity of u_λ with respect to λ we can apply the Lebesgue convergence theorem to find

$$\int_\Omega a(x) g(u_\lambda) \varphi_1 \, \mathrm{d}x \to \int_\Omega a(x) g(u^*) \varphi_1 \, \mathrm{d}x \quad \text{as } \lambda \nearrow \lambda_1.$$

Passing to the limit in (6.117) as $\lambda \nearrow \lambda^*$, and using (6.115), we get

$$\lambda_1 \int_\Omega u^* \varphi_1 \geqslant \lambda_1 \int_\Omega u^* \varphi_1 + \int_\Omega a(x) g(u^*) \varphi_1. \tag{6.118}$$

Hence $\int_\Omega a(x) g(u^*) \varphi_1 = 0$, which is a contradiction. This fact shows that $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. This ends the proof. □

7. Sublinear singular elliptic problems with two bifurcation parameters

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$). In this section we study the existence or the nonexistence of solutions to the following boundary value problem

$$\begin{cases} -\Delta u + K(x) g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathrm{P}_{\lambda,\mu}$}$$

Here $K, h \in C^{0,\gamma}(\overline{\Omega})$, with $h > 0$, on Ω, and λ, μ, are positive real numbers. We suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$, is a Hölder continuous function which is positive on $\overline{\Omega} \times (0,\infty)$. We also assume that f, is nondecreasing with respect to the second variable and is sublinear, that is,

(f1) the mapping $(0,\infty) \ni s \mapsto f(x,s)/s$ is nonincreasing for all $x \in \overline{\Omega}$;

(f2) $\lim_{s \downarrow 0} f(x,s)/s = +\infty$ and $\lim_{s \to \infty} f(x,s)/s = 0$, uniformly for $x \in \overline{\Omega}$.

We assume that $g \in C^{0,\gamma}(0,\infty)$, is a nonnegative and nonincreasing function satisfying

(g1) $\lim_{s \downarrow 0} g(s) = +\infty$;

(g2) there exists $C, \delta_0 > 0$ and $\alpha \in (0,1)$, such that $g(s) \leqslant C s^{-\alpha}$, for all $s \in (0, \delta_0)$.

Our framework includes the Emden–Fowler equation that corresponds to $g(s) = s^{-\gamma}$, $\gamma > 0$ (see Wong [91]).

Denote $\mathcal{E} = \{u \in C^2(\Omega) \cap C(\overline{\Omega});\ g(u) \in L^1(\Omega)\}$.

We show in this section that ($\mathrm{P}_{\lambda,\mu}$) has at least one solution in $\mathcal{E}$, for λ, μ, belonging to a certain range. We also prove that in some cases ($\mathrm{P}_{\lambda,\mu}$), has no solutions in $\mathcal{E}$, provided that λ and μ are sufficiently small.

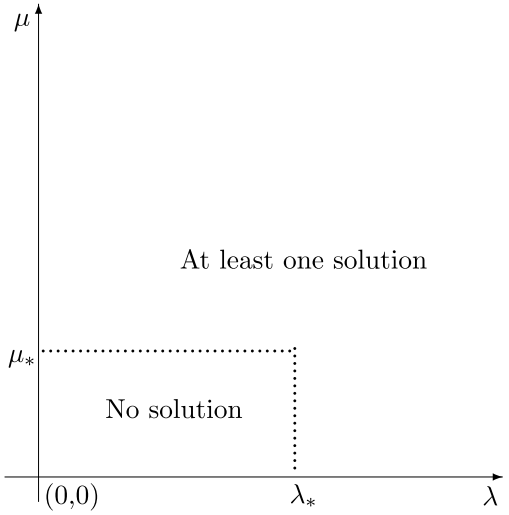

Fig. 3. The dependence on λ, and μ in Theorem 7.2.

REMARK 5.

(i) If $u \in \mathcal{E}$, $v \in C^2(\Omega) \cap C(\overline{\Omega})$, and $0 < u < v$, in Ω, then $v \in \mathcal{E}$.

(ii) Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$, be a solution of $(P_{\lambda,\mu})$. Then $u \in \mathcal{E}$, if and only if $\Delta u \in L^1(\Omega)$.

A fundamental role will be played in our analysis by the numbers

$$K^* = \max_{x \in \overline{\Omega}} K(x), \qquad K_* = \min_{x \in \overline{\Omega}} K(x).$$

Our main results (see Ghergu and Rădulescu [46]) are the following.

THEOREM 7.1. *Assume that* $K_* > 0$, *and* f, *satisfies* (f1), (f2).
If $\int_0^1 g(s)\,ds = +\infty$, *then* $(P_{\lambda,\mu})$, *has no solution in* $\mathcal{E}$ *for any* $\lambda, \mu > 0$.

THEOREM 7.2. *Assume that* $K_* > 0$, f, *satisfies* (f1), (f2), *and* g, *satisfies* (g1), (g2). *Then there exists* $\lambda_*, \mu_* > 0$, *such that*

$(P_{\lambda,\mu})$ *has at least one solution in* $\mathcal{E}$, *if* $\lambda > \lambda_*$, *or* $\mu > \mu_*$.

$(P_{\lambda,\mu})$ *has no solution in* $\mathcal{E}$, *if* $\lambda < \lambda_*$, *and* $\mu < \mu_*$.

Moreover, if $\lambda > \lambda_*$, *or* $\mu > \mu_*$, *then* $(P_{\lambda,\mu})$, *has a maximal solution in* $\mathcal{E}$, *which is increasing with respect to* λ, *and* μ.

THEOREM 7.3. *Assume that* $K^* \leqslant 0$, f, *satisfies* (f1), (f2), *and* g, *satisfies* (g1), (g2). *Then* $(P_{\lambda,\mu})$, *has a unique solution* $u_{\lambda,\mu} \in \mathcal{E}$, *for any* $\lambda, \mu > 0$. *Moreover,* $u_{\lambda,\mu}$, *is increasing with respect to* λ, *and* μ.

Theorems 7.2 and 7.3 also show the role played by the sublinear term f and the sign of $K(x)$. Indeed, if f becomes linear then the situation changes radically. First, by the results established by Crandall, Rabinowitz and Tartar [34] (see also Gomes [55]), the problem

$$\begin{cases} -\Delta u - u^{-\alpha} = -u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$. Next, as showed in Chen [19], the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has no solution, provided $0 < \alpha < 1$ and $\lambda_1 \geqslant 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

THEOREM 7.4. *Assume that $K^* > 0 > K_*$, f satisfies* (f1), (f2) *and g verifies* (g1), (g2). *Then there exists $\lambda_*, \mu_* > 0$ such that* $(P_{\lambda,\mu})$, *has at least one solution $u_{\lambda,\mu} \in \mathcal{E}$, if $\lambda > \lambda_*$, or $\mu > \mu_*$. Moreover, for $\lambda > \lambda_*$ or $\mu > \mu_*$, $u_{\lambda,\mu}$ is increasing with respect to λ and μ.*

Before giving the proofs, we state some auxiliary results.

Let ϕ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.119}$$

LEMMA 7.5 (Lazer and McKenna [67]). $\int_\Omega \phi_1^{-s}\,dx < +\infty$, *if and only if* $s < 1$.

Next, we observe that the hypotheses of Lemmas 6.2 and 6.3 are fulfilled for

$$\Phi_{\lambda,\mu}(x,s) = \lambda f(x,s) + \mu h(x), \tag{7.120}$$

$$\Psi_{\lambda,\mu}(x,s) = \lambda f(x,s) - K(x)g(s) + \mu h(x), \quad \text{provided } K^* \leqslant 0. \tag{7.121}$$

LEMMA 7.6. *Let f, satisfying* (f1), (f2), *and g, satisfying* (g1), (g2). *Then there exists $\bar{\lambda} > 0$, such that the problem*

$$\begin{cases} -\Delta v + g(v) = \lambda f(x,v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.122}$$

has at least one solution $v_{\lambda,\mu} \in \mathcal{E}$, for all $\lambda > \bar{\lambda}$, and for any $\mu > 0$.

PROOF. Let $\lambda, \mu > 0$. According to Lemmas 6.2 and 6.3, the boundary value problem

$$\begin{cases} -\Delta U = \lambda f(x, U) + \mu h(x) & \text{in } \Omega, \\ U > 0 & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{7.123}$$

has a unique solution $U_{\lambda,\mu} \in C^{2,\gamma}(\Omega) \cap C(\overline{\Omega})$. Then $\bar{v}_{\lambda,\mu} = U_{\lambda,\mu}$, is a super-solution of (7.122). The main point is to find a sub-solution of (7.122). For this purpose, let $H : [0, \infty) \to [0, \infty)$, be such that

$$\begin{cases} H''(t) = g(H(t)), \quad \text{for all } t > 0, \\ H'(0) = H(0) = 0. \end{cases} \tag{7.124}$$

Obviously, $H \in C^2(0, \infty) \cap C^1[0, \infty)$ exists by our assumption (g2). From (7.124) it follows that H'', is nonincreasing, while H and H' are nondecreasing on $(0, \infty)$. Using this fact and applying the mean value theorem, we deduce that for all $t > 0$, there exists $\xi_t^1,\ \xi_t^2 \in (0, t)$, such that

$$\frac{H(t)}{t} = \frac{H(t) - H(0)}{t - 0} = H'\big(\xi_t^1\big) \leqslant H'(t);$$
$$\frac{H'(t)}{t} = \frac{H'(t) - H'(0)}{t - 0} = H''\big(\xi_t^2\big) \geqslant H''(t).$$

The above inequalities imply

$$H(t) \leqslant tH'(t) \leqslant 2H(t), \quad \text{for all } t > 0.$$

Hence

$$1 \leqslant \frac{tH'(t)}{H(t)} \leqslant 2, \quad \text{for all } t > 0. \tag{7.125}$$

On the other hand, by (g2), and (7.124), there exists $\eta > 0$, such that

$$\begin{cases} H(t) \leqslant \delta_0, \quad \text{for all } t \in (0, \eta), \\ H''(t) \leqslant CH^{-\alpha}(t), \quad \text{for all } t \in (0, \eta), \end{cases} \tag{7.126}$$

which yields

$$H(t) \leqslant ct^{2/(\alpha+1)}, \quad \text{for all } t \in (0, \eta), \tag{7.127}$$

where $c > 0$, is a constant.

Now we look for a sub-solution of the form $\underline{v}_{\lambda,\mu} = MH(\phi_1)$, for some constant $M > 0$. We have

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) = \lambda_1 MH'(\phi_1)\phi_1 + g\big(MH(\phi_1)\big) - Mg\big(H(\phi_1)\big)|\nabla\phi_1|^2 \quad \text{in } \Omega. \tag{7.128}$$

Take $M \geqslant 1$. The monotonicity of g, leads to

$$g\big(MH(\phi_1)\big) \leqslant g\big(H(\phi_1)\big) \quad \text{in } \Omega,$$

and, by (7.128),

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant \lambda_1 M H'(\phi_1)\phi_1 + g\big(H(\phi_1)\big)\big(1 - M|\nabla \phi_1|^2\big) \quad \text{in } \Omega. \tag{7.129}$$

We claim that

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant 2\lambda_1 M H'(\phi_1)\phi_1 \quad \text{in } \Omega. \tag{7.130}$$

Indeed, by Hopf's maximum principle, there exists $\delta > 0$, and $\omega \Subset \Omega$, such that

$$|\nabla \phi_1| \geqslant \delta \quad \text{in } \Omega \setminus \omega,$$
$$\phi_1 \geqslant \delta \quad \text{in } \omega.$$

On $\Omega \setminus \omega$, we choose $M \geqslant M_1 = \max\{1, \delta^{-2}\}$. Then, by (7.129) we obtain

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant \lambda_1 M H'(\phi_1)\phi_1 \quad \text{in } \Omega \setminus \omega. \tag{7.131}$$

Fix $M \geqslant \max\{M_1, g(H(\delta))/(\lambda_1 H'(\delta)\delta)\}$. Then

$$g\big(H(\phi_1)\big) \leqslant g\big(H(\delta)\big) \leqslant \lambda_1 M H'(\delta)\delta \leqslant \lambda_1 M H'(\phi_1)\phi_1 \quad \text{in } \omega.$$

From (7.129) we deduce

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant 2\lambda_1 M H'(\phi_1)\phi_1 \quad \text{in } \omega. \tag{7.132}$$

Hence our claim (7.130) follows from (7.131) and (7.132).

Since $\phi_1 > 0$, in Ω, from (7.125) we have

$$1 \leqslant \frac{H'(\phi_1)\phi_1}{H(\phi_1)} \leqslant 2 \quad \text{in } \Omega. \tag{7.133}$$

Thus, (7.130) and (7.133) yield

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant 4\lambda_1 M H(\phi_1) = 4\lambda_1 \underline{v}_{\lambda,\mu} \quad \text{in } \Omega. \tag{7.134}$$

Take $\bar{\lambda} = 4\lambda_1 c^{-1} |\underline{v}_{\lambda,\mu}|_\infty$, where $c = \inf_{x \in \overline{\Omega}} f(x, |\underline{v}_{\lambda,\mu}|_\infty) > 0$. If $\lambda > \bar{\lambda}$, the assumption (f1), produces

$$\lambda \frac{f(x, \underline{v}_{\lambda,\mu})}{\underline{v}_{\lambda,\mu}} \geqslant \bar{\lambda} \frac{f(x, |\underline{v}_{\lambda,\mu}|_\infty)}{|\underline{v}_{\lambda,\mu}|_\infty} \geqslant 4\lambda_1, \quad \text{for all } x \in \Omega.$$

This combined with (7.134) gives

$$-\Delta \underline{v}_{\lambda,\mu} + g(\underline{v}_{\lambda,\mu}) \leqslant \lambda f(x, \underline{v}_{\lambda,\mu}) \quad \text{in } \Omega.$$

Hence $\underline{v}_{\lambda,\mu}$, is a sub-solution of (7.122), for all $\lambda > \bar{\lambda}$, and $\mu > 0$.

We now prove that $\underline{v}_{\lambda,\mu} \in \mathcal{E}$, that is $g(\underline{v}_{\lambda,\mu}) \in L^1(\Omega)$. Denote

$$\Omega_0 = \big\{x \in \Omega;\ \phi_1(x) < \eta\big\}.$$

By (7.126) and (7.127) it follows that

$$g(\underline{v}_{\lambda,\mu}) = g\big(MH(\phi_1)\big) \leqslant g\big(H(\phi_1)\big) \leqslant CH^{-\alpha}(\phi_1) \leqslant C_0 \phi_1^{-2\alpha/(1+\alpha)} \quad \text{in } \Omega_0,$$
$$g(\underline{v}_{\lambda,\mu}) \leqslant g\big(MH(\eta)\big) \quad \text{in } \Omega \setminus \Omega_0.$$

These estimates combined with Lemma 7.5 yield $g(\underline{v}_{\lambda,\mu}) \in L^1(\Omega)$, and so $\Delta \underline{v}_{\lambda,\mu} \in L^1(\Omega)$. Hence

$$\Delta \bar{v}_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, \bar{v}_{\lambda,\mu}) \leqslant 0 \leqslant \Delta \underline{v}_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, \underline{v}_{\lambda,\mu}) \quad \text{in } \Omega,$$
$$\underline{v}_{\lambda,\mu}, \bar{v}_{\lambda,\mu} > 0 \quad \text{in } \Omega,$$
$$\underline{v}_{\lambda,\mu} = \bar{v}_{\lambda,\mu} \quad \text{on } \partial\Omega,$$
$$\Delta \underline{v}_{\lambda,\mu} \in L^1(\Omega).$$

By Lemma 6.3, it follows that $\underline{v}_{\lambda,\mu} \leqslant \bar{v}_{\lambda,\mu}$ on $\overline{\Omega}$. Now, standard elliptic arguments guarantee the existence of a solution $v_{\lambda,\mu} \in C^2(\Omega) \cap C(\overline{\Omega})$, for (7.122) such that $\underline{v}_{\lambda,\mu} \leqslant v_{\lambda,\mu} \leqslant \bar{v}_{\lambda,\mu}$, in $\overline{\Omega}$. Since $\underline{v}_{\lambda,\mu} \in \mathcal{E}$, by Remark 5 we deduce that $v_{\lambda,\mu} \in \mathcal{E}$. Hence, for all $\lambda > \bar{\lambda}$, and $\mu > 0$, problem (7.122) has at least a solution in $\mathcal{E}$. The proof of Lemma 7.6 is now complete. □

We shall often refer in what follows to the following approaching problem of $(P_{\lambda,\mu})$:

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \frac{1}{k} & \text{on } \partial\Omega, \end{cases} \qquad (P^k_{\lambda,\mu})$$

where k, is a positive integer. We observe that any solution of $(P_{\lambda,\mu})$, is a sub-solution of $(P^k_{\lambda,\mu})$.

PROOF OF THEOREM 7.1. Suppose to the contrary that there exists λ and μ such that $(P_{\lambda,\mu})$, has a solution $u_{\lambda,\mu} \in \mathcal{E}$ and let $U_{\lambda,\mu}$, be the solution of (7.123). Since

$$\Delta U_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, U_{\lambda,\mu}) \leqslant 0 \leqslant \Delta u_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, u_{\lambda,\mu}) \quad \text{in } \Omega,$$

by Lemma 6.3 we get $u_{\lambda,\mu} \leqslant U_{\lambda,\mu}$ in $\overline{\Omega}$.

Consider the perturbed problem

$$\begin{cases} -\Delta u + K_* g(u+\varepsilon) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7.135}$$

Since $K_* > 0$, it follows that $u_{\lambda,\mu}$, and $U_{\lambda,\mu}$, are sub and super-solution for (7.135), respectively. So, by elliptic regularity, there exists $u_\varepsilon \in C^{2,\gamma}(\overline{\Omega})$, a solution of (7.135) such that

$$u_{\lambda,\mu} \leqslant u_\varepsilon \leqslant U_{\lambda,\mu} \quad \text{in } \Omega. \tag{7.136}$$

Integrating in (7.135) we deduce

$$-\int_\Omega \Delta u_\varepsilon \,\mathrm{d}x + K_* \int_\Omega g(u_\varepsilon + \varepsilon)\,\mathrm{d}x = \int_\Omega \big[\lambda f(x,u_\varepsilon) + \mu h(x)\big]\,\mathrm{d}x.$$

Hence

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}\,\mathrm{d}s + K_* \int_\Omega g(u_\varepsilon+\varepsilon)\,\mathrm{d}x \leqslant M, \tag{7.137}$$

where $M > 0$, is a constant. Since $\partial u_\varepsilon/\partial n \leqslant 0$ on $\partial\Omega$, relation (7.137) yields

$$K_* \int_\Omega g(u_\varepsilon+\varepsilon)\,\mathrm{d}x \leqslant M,$$

and so $K_* \int_\Omega g(U_{\lambda,\mu}+\varepsilon)\,\mathrm{d}x \leqslant M$. Thus, for any compact subset $\omega \Subset \Omega$, we have

$$K_* \int_\omega g(U_{\lambda,\mu}+\varepsilon)\,\mathrm{d}x \leqslant M.$$

Letting $\varepsilon \to 0$, the above relation leads to $K_* \int_\omega g(U_{\lambda,\mu})\,\mathrm{d}x \leqslant M$. Therefore

$$K_* \int_\Omega g(U_{\lambda,\mu})\,\mathrm{d}x \leqslant M. \tag{7.138}$$

Choose $\delta > 0$, sufficiently small and define $\Omega_\delta := \{x \in \Omega;\ \operatorname{dist}(x,\partial\Omega) \leqslant \delta\}$. Taking into account the regularity of domain, there exists $k > 0$, such that

$$U_{\lambda,\mu} \leqslant k \operatorname{dist}(x,\partial\Omega) \quad \text{for all } x \in \Omega_\delta.$$

Then

$$\int_\Omega g(U_{\lambda,\mu})\,\mathrm{d}x \geqslant \int_{\Omega_\delta} g(U_{\lambda,\mu})\,\mathrm{d}x \geqslant \int_{\Omega_\delta} g\big(k \operatorname{dist}(x,\partial\Omega)\big)\,\mathrm{d}x = +\infty,$$

which contradicts (7.138). It follows that the problem $(P_{\lambda,\mu})$, has no solutions in $\mathcal{E}$, and the proof of Theorem 7.1 is now complete. □

Using the same method as in Zhang [92, Theorem 2], we can prove that $(P_{\lambda,\mu})$, has no solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$, as it was pointed out in Choi, Lazer and McKenna [21, Remark 2].

PROOF OF THEOREM 7.2. We split the proof into several steps.

Step I. *Existence of the solutions of* $(P_{\lambda,\mu})$, *for* λ *large*. By Lemma 7.6, there exists $\bar{\lambda}$, such that for all $\lambda > \bar{\lambda}$, and $\mu > 0$, the problem

$$\begin{cases} -\Delta v + K^* g(v) = \lambda f(x, v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one solution $v_{\lambda,\mu} \in \mathcal{E}$. Then $v_k = v_{\lambda,\mu} + 1/k$, is a sub-solution of $(P^k_{\lambda,\mu})$, for all positive integers $k \geqslant 1$.

From Lemma 6.2, let $w \in C^{2,\gamma}(\overline{\Omega})$, be the solution of

$$\begin{cases} -\Delta w = \lambda f(x, w) + \mu h(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 1 & \text{on } \partial\Omega. \end{cases}$$

It follows that w, is a super-solution of $(P^k_{\lambda,\mu})$, for all $k \geqslant 1$, and

$$\Delta w + \Phi_{\lambda,\mu}(x, w) \leqslant 0 \leqslant \Delta v_1 + \Phi_{\lambda,\mu}(x, v_1) \quad \text{in } \Omega,$$

$$w, v_1 > 0 \quad \text{in } \Omega,$$

$$w = v_1 \quad \text{on } \partial\Omega,$$

$$\Delta v_1 \in L^1(\Omega).$$

Therefore, by Lemma 6.3, $1 \leqslant v_1 \leqslant w$ in $\overline{\Omega}$. Standard elliptic arguments imply that there exists a solution $u^1_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(P^1_{\lambda,\mu})$ such that $v_1 \leqslant u^1_{\lambda,\mu} \leqslant w$, in $\overline{\Omega}$. Now, taking $u^1_{\lambda,\mu}$, and v_2, as a pair of super and sub-solutions for $(P^2_{\lambda,\mu})$, we obtain a solution $u^2_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(P^2_{\lambda,\mu})$, such that $v_2 \leqslant u^2_{\lambda,\mu} \leqslant u^1_{\lambda,\mu}$, in $\overline{\Omega}$. In this manner we find a sequence $\{u^n_{\lambda,\mu}\}$, such that

$$v_n \leqslant u^n_{\lambda,\mu} \leqslant u^{n-1}_{\lambda,\mu} \leqslant w \quad \text{in } \overline{\Omega}. \tag{7.139}$$

Define $u_{\lambda,\mu}(x) = \lim_{n\to\infty} u^n_{\lambda,\mu}(x)$, for all $x \in \overline{\Omega}$. Standard bootstrap arguments imply that $u_{\lambda,\mu}$, is a solution of $(P_{\lambda,\mu})$. From (7.139) we have $v_{\lambda,\mu} \leqslant u_{\lambda,\mu} \leqslant w$ in $\overline{\Omega}$. Since $v_{\lambda,\mu} \in \mathcal{E}$, by Remark 5 it follows that $u_{\lambda,\mu} \in \mathcal{E}$. Consequently, problem $(P_{\lambda,\mu})$, has at least a solution in $\mathcal{E}$, for all $\lambda > \bar{\lambda}$, and $\mu > 0$.

Step II. *Existence of the solutions of* $(P_{\lambda,\mu})$*, for* μ*, large*. Let us first notice that g verifies the hypotheses of Theorem 2 in Díaz, Morel, and Oswald [38]. We also remark that the assumption (g2), and Lemma 7.5 are essential to find a sub-solution in the proof of Theorem 2 in Díaz, Morel and Oswald [38].

According to this result, there exists $\overline{\mu} > 0$, such that the problem

$$\begin{cases} -\Delta v + K^* g(v) = \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least a solution $v_\mu \in \mathcal{E}$, provided that $\mu > \overline{\mu}$. Fix $\lambda > 0$, and denote $v_k = v_\mu + 1/k$, $k \geqslant 1$. Hence v_k, is a sub-solution of $(P^k_{\lambda,\mu})$, for all $k \geqslant 1$. Similarly to the previous step we obtain a solution $u_{\lambda,\mu} \in \mathcal{E}$, for all $\lambda > 0$, and $\mu > \overline{\mu}$.

Step III. *Nonexistence for* λ, μ*, small*. Let $\lambda, \mu > 0$. Since $K_* > 0$, the assumption (g1), implies $\lim_{s\downarrow 0} \Psi_{\lambda,\mu}(x, s) = -\infty$, uniformly for $x \in \overline{\Omega}$. So, there exists $c > 0$, such that

$$\Psi_{\lambda,\mu}(x, s) < 0 \quad \text{for all } (x, s) \in \overline{\Omega} \times (0, c). \tag{7.140}$$

Let $s \geqslant c$. From (f1), we deduce

$$\frac{\Psi_{\lambda,\mu}(x, s)}{s} \leqslant \lambda \frac{f(x, s)}{s} + \mu \frac{h(x)}{s} \leqslant \lambda \frac{f(x, c)}{c} + \mu \frac{|h|_\infty}{s},$$

for all $x \in \overline{\Omega}$. Fix $\mu < c\lambda_1/(2|h|_\infty)$ and let $M = \sup_{x\in\overline{\Omega}}(f(x, c)/c) > 0$. From the above inequality we have

$$\frac{\Psi_{\lambda,\mu}(x, s)}{s} \leqslant \lambda M + \frac{\lambda_1}{2}, \quad \text{for all } (x, s) \in \overline{\Omega} \times [c, +\infty). \tag{7.141}$$

Thus, (7.140) and (7.141) yield

$$\Psi_{\lambda,\mu}(x, s) \leqslant a(\lambda)s + \frac{\lambda_1}{2}s, \quad \text{for all } (x, s) \in \overline{\Omega} \times (0, +\infty). \tag{7.142}$$

Moreover, $a(\lambda) \to 0$, as $\lambda \to 0$. If $(P_{\lambda,\mu})$, has a solution $u_{\lambda,\mu}$, then

$$\begin{aligned} \lambda_1 \int_\Omega u^2_{\lambda,\mu}(x)\,dx &\leqslant \int_\Omega |\nabla u_{\lambda,\mu}|^2\,dx = -\int_\Omega u_{\lambda,\mu}(x)\Delta u_{\lambda,\mu}(x)\,dx \\ &\leqslant \int_\Omega u_{\lambda,\mu}(x)\Psi\big(x, u_{\lambda,\mu}(x)\big)\,dx. \end{aligned}$$

Using (7.142), we get

$$\lambda_1 \int_\Omega u^2_{\lambda,\mu}(x)\,dx \leqslant \left[a(\lambda) + \frac{\lambda_1}{2}\right] \int_\Omega u^2_{\lambda,\mu}(x)\,dx.$$

Since $a(\lambda) \to 0$, as $\lambda \to 0$, the above relation leads to a contradiction for $\lambda, \mu > 0$, sufficiently small.

Step IV. *Existence of a maximal solution of* $(P_{\lambda,\mu})$. We show that if $(P_{\lambda,\mu})$, has a solution $u_{\lambda,\mu} \in \mathcal{E}$, then it has a maximal solution. Let $\lambda, \mu > 0$, be such that $(P_{\lambda,\mu})$, has a solution $u_{\lambda,\mu} \in \mathcal{E}$. If $U_{\lambda,\mu}$, is the solution of (7.123), by Lemma 6.3 we have $u_{\lambda,\mu} \leqslant U_{\lambda,\mu}$, in $\overline{\Omega}$. For any $j \geqslant 1$, denote

$$\Omega_j = \left\{ x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > \frac{1}{j} \right\}.$$

Let $U_0 = U_{\lambda,\mu}$, and U_j, be the solution of

$$\begin{cases} -\Delta\zeta + K(x)g(U_{j-1}) = \lambda f(x, U_{j-1}) + \mu h(x) & \text{in } \Omega_j, \\ \zeta = U_{j-1} & \text{in } \Omega \setminus \Omega_j. \end{cases}$$

Using the fact that $\Psi_{\lambda,\mu}$, is nondecreasing with respect to the second variable, we get

$$u_{\lambda,\mu} \leqslant U_j \leqslant U_{j-1} \leqslant U_0 \quad \text{in } \overline{\Omega}.$$

If $\bar{u}_{\lambda,\mu}(x) = \lim_{j\to\infty} U_j(x)$, for all $x \in \overline{\Omega}$, by standard elliptic arguments (see Gilbarg and Trudinger [54]) it follows that $\bar{u}_{\lambda,\mu}$, is a solution of $(P_{\lambda,\mu})$. Since $u_{\lambda,\mu} \leqslant \bar{u}_{\lambda,\mu}$, in $\overline{\Omega}$, by Remark 5 we have $\bar{u}_{\lambda,\mu} \in \mathcal{E}$. Moreover, $\bar{u}_{\lambda,\mu}$, is a maximal solution of $(P_{\lambda,\mu})$.

Step V. *Dependence on* λ, *and* μ. We first show the dependence on λ, of the maximal solution $\bar{u}_{\lambda,\mu} \in \mathcal{E}$, of $(P_{\lambda,\mu})$. For this purpose, fix $\mu > 0$, and define

$$A := \left\{ \lambda > 0;\ (P_{\lambda,\mu}) \text{ has at least a solution } u_{\lambda,\mu} \in \mathcal{E} \right\}.$$

Let $\lambda_* = \inf A$. From the previous steps we have $A \neq \emptyset$, and $\lambda_* > 0$. Let $\lambda_1 \in A$, and $\bar{u}_{\lambda_1,\mu}$, be the maximal solution of $(P_{\lambda_1,\mu})$. We prove that $(\lambda_1, +\infty) \subset A$. If $\lambda_2 > \lambda_1$, then $\bar{u}_{\lambda_1,\mu}$, is a sub-solution of $(P_{\lambda_2,\mu})$. On the other hand,

$$\begin{aligned} &\Delta U_{\lambda_2,\mu} + \Phi_{\lambda_2,\mu}(x, U_{\lambda_2,\mu}) \leqslant 0 \leqslant \Delta\bar{u}_{\lambda_1,\mu} + \Phi_{\lambda_2,\mu}(x, \bar{u}_{\lambda_1,\mu}) \quad \text{in } \Omega, \\ &U_{\lambda_2,\mu}, \bar{u}_{\lambda_1,\mu} > 0 \quad \text{in } \Omega, \\ &U_{\lambda_2,\mu} \geqslant \bar{u}_{\lambda_1,\mu} \quad \text{on } \partial\Omega, \\ &\Delta\bar{u}_{\lambda_1,\mu} \in L^1(\Omega). \end{aligned}$$

By Lemma 6.3, $\bar{u}_{\lambda_1,\mu} \leqslant U_{\lambda_2,\mu}$, in $\overline{\Omega}$. In the same way as in Step IV we find a solution $u_{\lambda_2,\mu} \in \mathcal{E}$, of $(P_{\lambda_2,\mu})$ such that

$$\bar{u}_{\lambda_1,\mu} \leqslant u_{\lambda_2,\mu} \leqslant U_{\lambda_2,\mu} \quad \text{in } \overline{\Omega}.$$

Hence $\lambda_2 \in A$, and so $(\lambda_*, +\infty) \subset A$. If $\bar{u}_{\lambda_2,\mu} \in \mathcal{E}$, is the maximal solution of $(P_{\lambda_2,\mu})$, the above relation implies $\bar{u}_{\lambda_1,\mu} \leqslant \bar{u}_{\lambda_2,\mu}$, in $\overline{\Omega}$. By the maximum principle, it follows that $\bar{u}_{\lambda_1,\mu} < \bar{u}_{\lambda_2,\mu}$, in Ω. So, $\bar{u}_{\lambda,\mu}$, is increasing with respect to λ.

To prove the dependence on μ, we fix $\lambda > 0$, and define

$$B := \{\mu > 0;\ (\mathrm{P}_{\lambda,\mu}) \text{ has at least one solution } u_{\lambda,\mu} \in \mathcal{E}\}.$$

Let $\mu_* = \inf B$. The conclusion follows in the same manner as above. The proof of Theorem 7.2 is now complete. □

PROOF OF THEOREM 7.3. Let $\lambda, \mu > 0$. We recall that the function $\Psi_{\lambda,\mu}$, defined in (7.121) verifies the hypotheses of Lemma 6.2, since $K^* \leqslant 0$. So, there exists $u_{\lambda,\mu} \in C^{2,\gamma}(\Omega) \cap C(\overline{\Omega})$ a solution of $(\mathrm{P}_{\lambda,\mu})$. If $U_{\lambda,\mu}$, is the solution of (7.123), then

$$\Delta u_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, u_{\lambda,\mu}) \leqslant 0 \leqslant \Delta U_{\lambda,\mu} + \Phi_{\lambda,\mu}(x, U_{\lambda,\mu}) \quad \text{in } \Omega,$$
$$u_{\lambda,\mu}, U_{\lambda,\mu} > 0 \quad \text{in } \Omega,$$
$$u_{\lambda,\mu} = U_{\lambda,\mu} = 0 \quad \text{on } \partial\Omega.$$

Since $\Delta U_{\lambda,\mu} \in L^1(\Omega)$, by Lemma 6.3 we get $u_{\lambda,\mu} \geqslant U_{\lambda,\mu}$, in $\overline{\Omega}$.

We claim that there exists $c > 0$, such that

$$U_{\lambda,\mu} \geqslant c\phi_1 \quad \text{in } \Omega. \tag{7.143}$$

Indeed, if not, there exists $\{x_n\} \subset \Omega$, and $\varepsilon_n \to 0$, such that

$$(U_{\lambda,\mu} - \varepsilon_n \phi_1)(x_n) < 0. \tag{7.144}$$

Moreover, we can choose the sequence $\{x_n\}$, with the additional property

$$\nabla(U_{\lambda,\mu} - \varepsilon_n \phi_1)(x_n) = 0. \tag{7.145}$$

Passing eventually at a subsequence, we can assume that $x_n \to x_0 \in \overline{\Omega}$. From (7.144) it follows that $U_{\lambda,\mu}(x_0) \leqslant 0$, which implies $U_{\lambda,\mu}(x_0) = 0$, that is $x_0 \in \partial\Omega$. Furthermore, from (7.145) we have $\nabla U_{\lambda,\mu}(x_0) = 0$. This is a contradiction since $(\partial U_{\lambda,\mu}/\partial n)(x_0) < 0$, by Hopf's strong maximum principle. Our claim follows and so

$$u_{\lambda,\mu} \geqslant U_{\lambda,\mu} \geqslant c\phi_1 \quad \text{in } \Omega. \tag{7.146}$$

Then, $g(u_{\lambda,\mu}) \leqslant g(U_{\lambda,\mu}) \leqslant g(c\phi_1)$ in Ω. From the assumption (g2), and Lemma 2.2 (using the same method as in the proof of Lemma 7.6) it follows that $g(c\phi_1) \in L^1(\Omega)$. Hence $u_{\lambda,\mu} \in \mathcal{E}$.

Let us now assume that $u^1_{\lambda,\mu}, u^2_{\lambda,\mu} \in \mathcal{E}$, are two solutions of $(\mathrm{P}_{\lambda,\mu})$. In order to prove the uniqueness, it is enough to show that $u^1_{\lambda,\mu} \geqslant u^2_{\lambda,\mu}$, in $\overline{\Omega}$. This follows by Lemma 6.3.

Let us show now the dependence on λ, of the solution of $(\mathrm{P}_{\lambda,\mu})$. For this purpose, let $0 < \lambda_1 < \lambda_2$, and $u_{\lambda_1,\mu}, u_{\lambda_2,\mu}$, be the unique solutions of $(\mathrm{P}_{\lambda_1,\mu})$, and $(\mathrm{P}_{\lambda_2,\mu})$, respectively, with $\mu > 0$, fixed. Since $u_{\lambda_1,\mu}, u_{\lambda_2,\mu} \in \mathcal{E}$, and

$$\Delta u_{\lambda_2,\mu} + \Phi_{\lambda_2,\mu}(x, u_{\lambda_2,\mu}) \leqslant 0 \leqslant \Delta u_{\lambda_1,\mu} + \Phi_{\lambda_2,\mu}(x, u_{\lambda_1,\mu}) \quad \text{in } \Omega,$$

in virtue of Lemma 6.3 we find $u_{\lambda_1,\mu} \leqslant u_{\lambda_2,\mu}$, in $\overline{\Omega}$. So, by the maximum principle, $u_{\lambda_1,\mu} < u_{\lambda_2,\mu}$, in Ω.

The dependence on μ, follows similarly. The proof of Theorem 7.3 is now complete. □

PROOF OF THEOREM 7.4. *Step* I. *Existence.* Using the fact that $K^* > 0$, from Theorem 7.2 it follows that there exists $\lambda_*, \mu_* > 0$, such that the problem

$$\begin{cases} -\Delta v + K^* g(v) = \lambda f(x, v) + \mu h(x) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

has a maximal solution $v_{\lambda,\mu} \in \mathcal{E}$, provided $\lambda > \lambda_*$, or $\mu > \mu_*$. Moreover, $v_{\lambda,\mu}$, is increasing with respect to λ, and μ. Then $v_k = v_{\lambda,\mu} + 1/k$, is a sub-solution of $(\mathrm{P}^k_{\lambda,\mu})$, for all $k \geqslant 1$. On the other hand, by Lemma 6.2, the boundary value problem

$$\begin{cases} -\Delta w + K_* g(w) = \lambda f(x, w) + \mu h(x) & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = \frac{1}{k} & \text{on } \partial\Omega. \end{cases}$$

has a solution $w_k \in C^{2,\gamma}(\overline{\Omega})$. Obviously, w_k, is a super-solution of $(\mathrm{P}^k_{\lambda,\mu})$.

Since $K^* > 0 > K_*$, we have

$$\Delta w_k + \Phi_{\lambda,\mu}(x, w_k) \leqslant 0 \leqslant \Delta v_k + \Phi_{\lambda,\mu}(x, v_k) \quad \text{in } \Omega,$$

and

$$\begin{aligned} &w_k, v_k > 0 \quad \text{in } \Omega, \\ &w_k = v_k \quad \text{on } \partial\Omega, \\ &\Delta v_k \in L^1(\Omega). \end{aligned}$$

From Lemma 6.3 it follows that $v_k \leqslant w_k$, in $\overline{\Omega}$. By standard super and sub-solution argument, there exists a minimal solution $u^1_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(\mathrm{P}^1_{\lambda,\mu})$ such that $v_1 \leqslant u^1_{\lambda,\mu} \leqslant w_1$, in $\overline{\Omega}$. Now, taking $u^1_{\lambda,\mu}$, and v_2, as a pair of super and sub-solutions for $(\mathrm{P}^2_{\lambda,\mu})$, we deduce that there exists a minimal solution $u^2_{\lambda,\mu} \in C^{2,\gamma}(\overline{\Omega})$ of $(\mathrm{P}^2_{\lambda,\mu})$, such that $v_2 \leqslant u^2_{\lambda,\mu} \leqslant u^1_{\lambda,\mu}$, in $\overline{\Omega}$. Arguing in the same manner, we obtain a sequence $\{u^k_{\lambda,\mu}\}$, such that

$$v_k \leqslant u^k_{\lambda,\mu} \leqslant u^{k-1}_{\lambda,\mu} \leqslant w_1 \quad \text{in } \overline{\Omega}. \tag{7.147}$$

Define $u_{\lambda,\mu}(x) = \lim_{k\to\infty} u^k_{\lambda,\mu}(x)$, for all $x \in \overline{\Omega}$. With a similar argument to that used in the proof of Theorem 7.2, we find that $u_{\lambda,\mu} \in \mathcal{E}$, is a solution of $(\mathrm{P}_{\lambda,\mu})$. Hence, problem $(\mathrm{P}_{\lambda,\mu})$, has at least a solution in $\mathcal{E}$, provided that $\lambda > \lambda_*$, or $\mu > \mu_*$.

Step II. *Dependence on* λ, *and* μ. As above, it is enough to justify only the dependence on λ. Fix $\lambda_* < \lambda_1 < \lambda_2$, $\mu > 0$ and let $u_{\lambda_1,\mu}$, $u_{\lambda_2,\mu} \in \mathcal{E}$ be the solutions of $(\mathrm{P}_{\lambda_1,\mu})$,

and $(P_{\lambda_2,\mu})$, respectively that we have obtained in Step I. It follows that $u^k_{\lambda_2,\mu}$, is a supersolution of $(P^k_{\lambda_1,\mu})$. So, Lemma 6.3 combined with the fact that $v_{\lambda,\mu}$, is increasing with respect to $\lambda > \lambda_*$, yield

$$u^k_{\lambda_2,\mu} \geqslant v_{\lambda_2,\mu} + \frac{1}{k} \geqslant v_{\lambda_1,\mu} + \frac{1}{k} \quad \text{in } \overline{\Omega}.$$

Thus, $u^k_{\lambda_2,\mu} \geqslant u^k_{\lambda_1,\mu}$, in $\overline{\Omega}$, since $u^k_{\lambda_1,\mu}$, is the minimal solution of $(P^k_{\lambda_1,\mu})$, which satisfies $u^k_{\lambda_1,\mu} \geqslant v_{\lambda_1,\mu} + 1/k$ in $\overline{\Omega}$. It follows that $u_{\lambda_2,\mu} \geqslant u_{\lambda_1,\mu}$, in $\overline{\Omega}$. By the maximum principle we deduce that $u_{\lambda_2,\mu} > u_{\lambda_1,\mu}$ in Ω. This concludes the proof. □

8. Bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with a convection term

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded domain with a smooth boundary. In this section we are concerned with singular elliptic problems of the following type

$$\begin{cases} -\Delta u = g(u) + \lambda |\nabla u|^p + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.148}$$

where $0 < p \leqslant 2$ and $\lambda, \mu \geqslant 0$. As remarked by Choquet-Bruhat and Leray [22] and by Kazdan and Warner [61], the requirement that the nonlinearity grows at most quadratically in $|\nabla u|$ is natural in order to apply the maximum principle.

Throughout this section we suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and is positive on $\overline{\Omega} \times (0,\infty)$. We assume that $g : (0,\infty) \to (0,\infty)$ is a Hölder continuous function which is nonincreasing and $\lim_{s \searrow 0} g(s) = +\infty$.

Many papers have been devoted to the case $\lambda = 0$, where the problem (8.148) becomes

$$\begin{cases} -\Delta u = g(u) + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.149}$$

If $\mu = 0$, then (8.149) has a unique solution (see Crandall, Rabinowitz and Tartar [34], Lazer and McKenna [67]). When $\mu > 0$, the study of (8.149) emphasizes the role played by the nonlinear term $f(x,u)$. For instance, if one of the following assumptions are fulfilled

(f1) there exists $c > 0$ such that $f(x,s) \geqslant cs$ for all $(x,s) \in \overline{\Omega} \times [0,\infty)$;

(f2) the mapping $(0,\infty) \ni s \mapsto f(x,s)/s$ is nondecreasing for all $x \in \overline{\Omega}$, then problem (8.149) has solutions only if $\mu > 0$ is small enough (see Coclite and Palmieri [33]). In turn, when f satisfies the following assumptions

(f3) the mapping $(0,\infty) \ni s \mapsto f(x,s)/s$ is nonincreasing for all $x \in \overline{\Omega}$;

(f4) $\lim_{s\to\infty} f(x,s)/s = 0$, uniformly for $x \in \overline{\Omega}$, then problem (8.149) has at least one solutions for all $\mu > 0$ (see Coclite and Palmieri [33], Shi and Yao [86] and the references therein). The same assumptions will be used in the study of (8.148).

By the monotonicity of g, there exists

$$a = \lim_{s\to\infty} g(s) \in [0,\infty).$$

The main results in this section have been obtained by Ghergu and Rădulescu [52,53].

We are first concerned with the case $\lambda = 1$ and $1 < p \leqslant 2$. In the statement of the following result we do not need assumptions (f1)–(f4); we just require that f is a Hölder continuous function which is nondecreasing with respect to the second variable and is positive on $\overline{\Omega} \times (0,\infty)$.

THEOREM 8.1. *Assume* $\lambda = 1$ *and* $1 < p \leqslant 2$.

(i) *If* $p = 2$ *and* $a \geqslant \lambda_1$, *then* (8.148) *has no solutions*;

(ii) *If* $p = 2$ *and* $a < \lambda_1$ *or* $1 < p < 2$, *then there exists* $\mu^* > 0$ *such that* (8.148) *has at least one classical solution for* $\mu < \mu^*$ *and no solutions exist if* $\mu > \mu^*$.

If $\lambda = 1$ and $0 < p \leqslant 1$ the study of existence is close related to the asymptotic behavior of the nonlinear term $f(x,u)$. In this case we prove

THEOREM 8.2. *Assume* $\lambda = 1$ *and* $0 < p \leqslant 1$.

(i) *If* f *satisfies* (f1) *or* (f2), *then there exists* $\mu^* > 0$ *such that* (8.148) *has at least one classical solution for* $\mu < \mu^*$ *and no solutions exist if* $\mu > \mu^*$;

(ii) *If* $0 < p < 1$ *and* f *satisfies* (f3), (f4), *then* (8.148) *has at least one solution for all* $\mu \geqslant 0$.

Next we are concerned with the case $\mu = 1$. Our result is the following

THEOREM 8.3. *Assume* $\mu = 1$ *and* f *satisfies assumptions* (f3) *and* (f4). *Then the following properties hold true*.

(i) *If* $0 < p < 1$, *then* (8.148) *has at least one classical solution for all* $\lambda \geqslant 0$;

(ii) *If* $1 \leqslant p \leqslant 2$, *then there exists* $\lambda^* \in (0,\infty]$ *such that* (8.148) *has at least one classical solution for* $\lambda < \lambda^*$ *and no solution exists if* $\lambda > \lambda^*$. *Moreover, if* $1 < p \leqslant 2$, *then* λ^* *is finite*.

Related to the above result we raise the following *open problem*: if $p = 1$ and $\mu = 1$, is λ^* a finite number?

Theorem 8.3 shows the importance of the convection term $\lambda|\nabla u|^p$ in (8.148). Indeed, according to Theorem 7.3 and for any $\mu > 0$, the boundary value problem

$$\begin{cases} -\Delta u = u^{-\alpha} + \lambda|\nabla u|^p + \mu u^\beta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{8.150}$$

has a unique solution, provided $\lambda = 0$, $\alpha, \beta \in (0,1)$. The above theorem shows that if λ is not necessarily 0, then the following situations may occur: (i) problem (8.150) has solutions if $p \in (0,1)$ and for all $\lambda \geqslant 0$; (ii) if $p \in (1,2)$ then there exists $\lambda^* > 0$ such that problem (8.150) has a solution for any $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$.

To see the dependence between λ and μ in (8.148), we consider the special case $f \equiv 1$ and $p = 2$. In this case we can say more about the problem (8.148). More precisely we have

THEOREM 8.4. *Assume that $p = 2$ and $f \equiv 1$.*

(i) *The problem* (8.148) *has solution if and only if $\lambda(a + \mu) < \lambda_1$;*

(ii) *Assume $\mu > 0$ is fixed, g is decreasing and let $\lambda^* = \lambda_1/(a + \mu)$. Then* (8.148) *has a unique solution u_λ for all $\lambda < \lambda^*$ and the sequence $(u_\lambda)_{\lambda<\lambda^*}$ is increasing with respect to λ. Moreover, if $\limsup_{s \searrow 0} s^\alpha g(s) < +\infty$, for some $\alpha \in (0,1)$, then the sequence of solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ has the following properties*

(ii1) *For all $0 < \lambda < \lambda^*$ there exist two positive constants c_1, c_2 depending on λ such that $c_1 \operatorname{dist}(x, \partial\Omega) \leqslant u_\lambda \leqslant c_2 \operatorname{dist}(x, \partial\Omega)$ in Ω;*

(ii2) *$u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;*

(ii3) *$u_\lambda \to +\infty$ as $\lambda \nearrow \lambda^*$, uniformly on compact subsets of Ω.*

As regards the uniqueness of the solutions to problem (8.148), we may say that this does not seem to be a feature easy to achieve. Only when $f(x,u)$ is constant in u we can use classical methods in order to prove the uniqueness. It is worth pointing out here that the uniqueness of the solution is a delicate issue even for the simpler problem (8.149). We have already observed that if f fulfills (f3), (f4) and g satisfies the same growth condition as in Theorem 8.4, then this solution is unique, provided that problem (8.149) has a solution. On the other hand, if f satisfies (f2), the uniqueness generally does not occur. In that sense we refer the interested reader to Haitao [57]. In the case $f(x,u) = u^q$, $g(u) = u^{-\gamma}$, $0 < \gamma < 1/N$ and $1 < q < (N+2)/(N-2)$, we learn from [57] that problem (8.149) has at least two classical solutions provided μ belongs to a certain range.

Our approach relies on finding of appropriate sub- and super-solutions of (8.148). This allows us to enlarge the study of bifurcation to a class of problems more generally to that studied in Zhang and Yu [94]. However, neither the method used in [94], nor our method gives a precise answer if λ^* is finite or not in the case $p = 1$ and $\mu = 1$.

We start with some auxiliary results.

Let φ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of $(-\Delta)$ in $H_0^1(\Omega)$. As it is well known $\lambda_1 > 0$, $\varphi_1 \in C^2(\overline{\Omega})$ and

$$C_1 \operatorname{dist}(x, \partial\Omega) \leqslant \varphi_1 \leqslant C_2 \operatorname{dist}(x, \partial\Omega) \quad \text{in } \Omega, \tag{8.151}$$

for some positive constants $C_1, C_2 > 0$. From the characterization of λ_1 and φ_1 we state the following elementary result. For the convenience of the reader we shall give a complete proof.

LEMMA 8.5. *Let* $F:\overline{\Omega}\times(0,\infty)\to\mathbb{R}$ *be a continuous function such that* $F(x,s)\geqslant \lambda_1 s+b$ *for some* $b>0$ *and for all* $(x,s)\in\overline{\Omega}\times(0,\infty)$. *Then the problem*

$$\begin{cases} -\Delta u = F(x,u) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{8.152}$$

has no solutions.

PROOF. By contradiction, suppose that (8.152) admits a solution. This will provide a super-solution of the problem

$$\begin{cases} -\Delta u = \lambda_1 u+b & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{8.153}$$

Since 0 is a sub-solution, by the sub and super-solution method and classical regularity theory it follows that (8.152) has a solution $u\in C^2(\overline{\Omega})$. Multiplying by φ_1 in (8.153) and then integrating over Ω, we get

$$-\int_\Omega \varphi_1\Delta u=\lambda_1\int_\Omega \varphi_1 u+b\int_\Omega \varphi_1,$$

that is $\lambda_1\int_\Omega\varphi_1 u=\lambda_1\int_\Omega\varphi_1 u+b\int_\Omega\varphi_1$, which implies $\int_\Omega\varphi_1=0$. This is clearly a contradiction since $\varphi_1>0$ in Ω. Hence (8.152) has no solutions. □

According to Lemma 6.2, there exists $\zeta\in C^2(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta\zeta = g(\zeta) & \text{in } \Omega,\\ \zeta>0 & \text{in } \Omega,\\ \zeta=0 & \text{on } \partial\Omega. \end{cases} \tag{8.154}$$

Clearly ζ is a sub-solution of (8.148) for all $\lambda\geqslant 0$. It is worth pointing out here that the sub-super solution method still works for the problem (8.148). With the same proof as in Zhang and Yu [94, Lemma 2.8] that goes back to the pioneering work of Amann [3] we state the following result.

LEMMA 8.6. *Let* $\lambda,\mu\geqslant 0$. *If* (8.148) *has a super-solution* $\bar{u}\in C^2(\Omega)\cap C(\overline{\Omega})$ *such that* $\zeta\leqslant\bar{u}$ *in* Ω, *then* (8.148) *has at least a solution*.

LEMMA 8.7 (Alaa and Pierre [1]). *If* $p>1$, *then there exists a real number* $\bar{\sigma}>0$ *such that the problem*

$$\begin{cases} -\Delta u = |\nabla u|^p+\sigma & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{8.155}$$

has no solutions for $\sigma>\bar{\sigma}$.

LEMMA 8.8. *Let $F:\overline{\Omega}\times(0,\infty)\to[0,\infty)$ and $G:(0,\infty)\to(0,\infty)$ be two Hölder continuous functions that verify*

(A1) *$F(x,s)>0$, for all $(x,s)\in\overline{\Omega}\times(0,\infty)$;*

(A2) *The mapping $[0,\infty)\ni s\mapsto F(x,s)$ is nondecreasing for all $x\in\overline{\Omega}$;*

(A3) *G is nonincreasing and $\lim_{s\searrow 0}G(s)=+\infty$.*

Assume that $\tau>0$ is a positive real number. Then the following holds.

(i) *If $\tau\lim_{s\to\infty}G(s)\geqslant\lambda_1$, then the problem*

$$\begin{cases} -\Delta u = G(u)+\tau|\nabla u|^2+\mu F(x,u) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{8.156}$$

has no solutions.

(ii) *If $\tau\lim_{s\to\infty}G(s)<\lambda_1$, then there exists $\bar{\mu}>0$ such that the problem* (8.156) *has at least one solution for all $0\leqslant\mu<\bar{\mu}$.*

PROOF. (i) With the change of variable $v=\mathrm{e}^{\tau u}-1$, the problem (8.156) takes the form

$$\begin{cases} -\Delta v=\Psi_\mu(x,u) & \text{in } \Omega,\\ v>0 & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega, \end{cases} \tag{8.157}$$

where

$$\Psi_\mu(x,s)=\tau(s+1)G\left(\frac{1}{\tau}\ln(s+1)\right)+\mu\tau(s+1)F\left(x,\frac{1}{\tau}\ln(s+1)\right),$$

for all $(x,s)\in\overline{\Omega}\times(0,\infty)$.

Taking into account the fact that G is nonincreasing and $\tau\lim_{s\to\infty}G(s)\geqslant\lambda_1$, we get

$$\Psi_\mu(x,s)\geqslant\lambda_1(s+1)\quad\text{in }\overline{\Omega}\times(0,\infty),\ \text{for all }\mu\geqslant 0.$$

By Lemma 8.5 we conclude that (8.157) has no solutions. Hence (8.156) has no solutions.

(ii) Since

$$\lim_{s\to+\infty}\frac{\tau(s+1)G((1/\tau)\ln(s+1))+1}{s}<\lambda_1$$

and

$$\lim_{s\searrow 0}\frac{\tau(s+1)G((1/\tau)\ln(s+1))+1}{s}=+\infty,$$

we deduce that the mapping $(0,\infty) \ni s \mapsto \tau(s+1)G((1/\tau)\ln(s+1)) + 1$ fulfills the hypotheses in Lemma 6.2. According to this one, there exists $\bar{v} \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta v = \tau(v+1)G((1/\tau)\ln(v+1)) + 1 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

Define

$$\bar{\mu} := \frac{1}{\tau(\|\bar{v}\|_\infty + 1)} \cdot \frac{1}{\max_{x\in\overline{\Omega}} F(x, (1/\tau)\ln(\|\bar{v}\|_\infty + 1))}.$$

It follows that $\bar{v}$ is a super-solution of (8.157) for all $0 \leqslant \mu < \bar{\mu}$.

Next we provide a sub-solution $\underline{v}$ of (8.157) such that $\underline{v} \leqslant \bar{v}$ in Ω. To this aim, we apply Lemma 6.2 to get that there exists $\underline{v} \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution of the problem

$$\begin{cases} -\Delta v = \tau G((1/\tau)\ln(v+1)) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly, $\underline{v}$ is a sub-solution of (8.157) for all $0 \leqslant \mu < \bar{\mu}$. Let us prove now that $\underline{v} \leqslant \bar{v}$ in Ω. Assuming the contrary, it follows that $\max_{x\in\overline{\Omega}}\{\underline{v} - \bar{v}\} > 0$ is achieved in Ω. At that point, say x_0, we have

$$\begin{aligned} 0 &\leqslant -\Delta(\underline{v} - \bar{v})(x_0) \\ &\leqslant \tau\left[G\left(\frac{1}{\tau}\ln\big(\underline{v}(x_0)+1\big)\right) - G\left(\frac{1}{\tau}\ln\big(\bar{v}(x_0)+1\big)\right)\right] - 1 < 0, \end{aligned}$$

which is a contradiction. Thus, $\underline{v} \leqslant \bar{v}$ in Ω. We have proved that $(\underline{v}, \bar{v})$ is an ordered pair of sub-super solutions of (8.157) provided $0 \leqslant \mu < \bar{\mu}$. It follows that (8.156) has at least one classical solution for all $0 \leqslant \mu < \bar{\mu}$ and the proof of Lemma 8.8 is now complete. □

PROOF OF THEOREM 8.1. According to Lemma 8.8(i) we deduce that (8.148) has no solutions if $p = 2$ and $a \geqslant \lambda_1$. Furthermore, if $p = 2$ and $a < \lambda_1$, in view of Lemma 8.8(ii), we deduce that (8.148) has at least one classical solution if μ is small enough. Assume now $1 < p < 2$ and let us fix $C > 0$ such that

$$aC^{p/2} + C^{p-1} < \lambda_1. \tag{8.158}$$

Define

$$\psi : [0,\infty) \to [0,\infty), \quad \psi(s) = \frac{s^p}{s^2 + C}.$$

A careful examination reveals the fact that ψ attains its maximum at $\bar{s} = (Cp/(2-p))^{2-p}$. Hence

$$\psi(s) \leqslant \psi(\bar{s}) = \frac{p^{p/2}(2-p)^{(2-p)/2}}{2C^{1-p/2}}, \quad \text{for all } s \geqslant 0.$$

By the classical Young's inequality we deduce

$$p^{p/2}(2-p)^{(2-p)/2} \leqslant 2,$$

which yields $\psi(s) \leqslant C^{p/2-1}$, for all $s \geqslant 0$. Thus, we have proved

$$s^p \leqslant C^{p/2}s^2 + C^{p/2-1}, \quad \text{for all } s \geqslant 0. \tag{8.159}$$

Consider the problem

$$\begin{cases} -\Delta u = g(u) + C^{p/2-1} + C^{p/2}|\nabla u|^2 + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.160}$$

By virtue of (8.159), any solution of (8.160) is a super-solution of (8.148).

Using now (8.158) we get

$$\lim_{s\to\infty} C^{p/2}\big(g(u) + C^{p/2-1}\big) < \lambda_1.$$

The above relation enables us to apply Lemma 8.8(ii) with $G(s) = g(s) + C^{p/2-1}$ and $\tau = C^{p/2}$. It follows that there exists $\bar{\mu} > 0$ such that (8.160) has at least a solution u. With a similar argument to that used in the proof of Lemma 8.8, we obtain $\zeta \leqslant u$ in Ω, where ζ is defined in (8.154). By Lemma 8.6 we get that (8.148) has at least one solution if $0 \leqslant \mu < \bar{\mu}$.

We have proved that (8.148) has at least one classical solution for both cases $p = 2$ and $a < \lambda_1$ or $1 < p < 2$, provided μ is nonnegative small enough. Define next

$$A = \{\mu \geqslant 0;\ \text{problem (8.148) has at least one solution}\}.$$

The above arguments implies that A is nonempty. Let $\mu^* = \sup A$. We first show that $[0, \mu^*) \subseteq A$. For this purpose, let $\mu_1 \in A$ and $0 \leqslant \mu_2 < \mu_1$. If u_{μ_1} is a solution of (8.148) with $\mu = \mu_1$, then u_{μ_1} is a super-solution of (8.148) with $\mu = \mu_2$. It is easy to prove that $\zeta \leqslant u_{\mu_1}$ in Ω and by virtue of Lemma 8.6 we conclude that the problem (8.148) with $\mu = \mu_2$ has at least one solution.

Thus we have proved $[0, \mu^*) \subseteq A$. Next we show $\mu^* < +\infty$.

Since $\lim_{s\searrow 0} g(s) = +\infty$, we can choose $s_0 > 0$ such that $g(s) > \bar{\sigma}$ for all $s \leqslant s_0$. Let

$$\mu_0 = \frac{\bar{\sigma}}{\min_{x\in\overline{\Omega}} f(x, s_0)}.$$

Using the monotonicity of f with respect to the second argument, the above relation yields

$$g(s) + \mu f(x, s) \geqslant \bar{\sigma}, \quad \text{for all } (x, s) \in \overline{\Omega} \times (0, \infty) \text{ and } \mu > \mu_0.$$

If (8.148) has a solution for $\mu > \mu_0$, this would be a super-solution of the problem

$$\begin{cases} -\Delta u = |\nabla u|^p + \bar{\sigma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.161}$$

Since 0 is a sub-solution, we deduce that (8.161) has at least one solution. According to Lemma 8.7, this is a contradiction. Hence $\mu^* \leqslant \mu_0 < +\infty$. This concludes the proof of Theorem 8.1. □

PROOF OF THEOREM 8.2. (i) We fix $p \in (0, 1]$ and define

$$q = q(p) = \begin{cases} p + 1 & \text{if } 0 < p < 1, \\ 3/2 & \text{if } p = 1. \end{cases}$$

Consider the problem

$$\begin{cases} -\Delta u = g(u) + 1 + |\nabla u|^q + \mu f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.162}$$

Since $s^p \leqslant s^q + 1$, for all $s \geqslant 0$, we deduce that any solution of (8.162) is a super-solution of (8.148). Furthermore, taking into account the fact that $1 < q < 2$, we can apply Theorem 8.1(ii) in order to get that (8.162) has at least one solution if μ is small enough. Thus, by Lemma 8.6 we deduce that (8.148) has at least one classical solution. Following the method used in the proof of Theorem 8.1, we set

$$A = \{\mu \geqslant 0;\ \text{problem (8.148) has at least one solution}\}$$

and let $\mu^* = \sup A$. With the same arguments we prove that $[0, \mu^*) \subseteq A$. It remains only to show that $\mu^* < +\infty$.

Let us assume first that f satisfies (f1). Since $\lim_{s \searrow 0} g(s) = +\infty$, we can choose $\mu_0 > 2\lambda_1/c$, such that $\frac{1}{2}\mu_0 cs + g(s) \geqslant 1$ for all $s > 0$. Then

$$g(s) + \mu f(x, s) \geqslant \lambda_1 s + 1, \quad \text{for all } (x, s) \in \overline{\Omega} \times (0, \infty) \text{ and } \mu \geqslant \mu_0.$$

By virtue of Lemma 8.5 we obtain that (8.148) has no classical solutions if $\mu \geqslant \mu_0$, so μ^* is finite.

Assume now that f satisfies (f2). Since $\lim_{s \searrow 0} g(s) = +\infty$, there exists $s_0 > 0$ such that

$$g(s) \geqslant \lambda_1(s + 1) \quad \text{for all } 0 < s < s_0. \tag{8.163}$$

On the other hand, the assumption (f2) and the fact that Ω is bounded implies that the mapping

$$(0,\infty) \ni s \mapsto \frac{\min_{x\in\overline{\Omega}} f(x,s)}{s+1}$$

is nondecreasing, so we can choose $\tilde{\mu} > 0$ with the property

$$\tilde{\mu} \cdot \frac{\min_{x\in\overline{\Omega}} f(x,s)}{s+1} \geqslant \lambda_1 \quad \text{for all } s \geqslant s_0. \tag{8.164}$$

Now (8.163) combined with (8.164) yields

$$g(s) + \mu f(x,s) \geqslant \lambda_1(s+1), \quad \text{for all } (x,s) \in \overline{\Omega} \times (0,\infty) \text{ and } \mu \geqslant \tilde{\mu}.$$

Using Lemma 8.5, we deduce that (8.148) has no solutions if $\mu > \tilde{\mu}$, that is, μ^* is finite.

The first part in Theorem 8.2 is therefore established.

(ii) The strategy is to find a super-solution $\bar{u}_\mu \in C^2(\Omega) \cap C(\overline{\Omega})$ of (8.148) such that $\zeta \leqslant \bar{u}_\mu$ in Ω. To this aim, let $h \in C^2(0,\eta] \cap C[0,\eta]$ be such that

$$\begin{cases} h''(t) = -g(h(t)), & \text{for all } 0 < t < \eta, \\ h(0) = 0, \\ h > 0 \quad \text{in } (0,\eta]. \end{cases} \tag{8.165}$$

The existence of h follows by classical arguments of ODE. Since h is concave, there exists $h'(0+) \in (0,+\infty]$. By taking $\eta > 0$ small enough, we can assume that $h' > 0$ in $(0,\eta]$, so h is increasing on $[0,\eta]$.

LEMMA 8.9.

(i) $h \in C^1[0,\eta]$ *if and only if* $\int_0^1 g(s)\,\mathrm{d}s < +\infty$;

(ii) *If* $0 < p \leqslant 2$, *then there exist* $c_1, c_2 > 0$ *such that*

$$(h')^p(t) \leqslant c_1 g\big(h(t)\big) + c_2, \quad \textit{for all } 0 < t < \eta.$$

PROOF. (i) Multiplying by h' in (8.165) and then integrating on $[t,\eta]$, $0 < t < \eta$, we get

$$(h')^2(t) - (h')^2(\eta) = 2\int_t^\eta g\big(h(s)\big)h'(s)\,\mathrm{d}s = 2\int_{h(t)}^{h(\eta)} g(\tau)\,\mathrm{d}\tau. \tag{8.166}$$

This gives

$$(h')^2(t) = 2G\big(h(t)\big) + (h')^2(\eta) \quad \text{for all } 0 < t < \eta, \tag{8.167}$$

where $G(t) = \int_t^{h(\eta)} g(s)\,\mathrm{d}s$. From (8.167) we deduce that $h'(0+)$ is finite if and only if $G(0+)$ is finite, so (i) follows.

(ii) Let $p \in (0, 2]$. Taking into account the fact that g is nonincreasing, the inequality (8.167) leads to

$$(h')^2(t) \leqslant 2h(\eta)g\big(h(t)\big) + (h')^2(\eta), \quad \text{for all } 0 < t < \eta. \tag{8.168}$$

Since $s^p \leqslant s^2 + 1$, for all $s \geqslant 0$, from (8.168) we have

$$(h')^p(t) \leqslant c_1 g\big(h(t)\big) + c_2, \quad \text{for all } 0 < t < \eta \tag{8.169}$$

where $c_1 = 2h(\eta)$ and $c_2 = (h')^2(\eta) + 1$. This completes the proof of our lemma. □

PROOF OF THEOREM 8.2 COMPLETED. Let $p \in (0, 1)$ and $\mu \geqslant 0$ be fixed. We also fix $c > 0$ such that $c\|\varphi_1\|_\infty < \eta$. By Hopf's maximum principle, there exist $\delta > 0$ small enough and $\theta_1 > 0$ such that

$$|\nabla\varphi_1| > \theta_1 \quad \text{in } \Omega_\delta, \tag{8.170}$$

where $\Omega_\delta := \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \leqslant \delta\}$.

Moreover, since $\lim_{s \searrow 0} g(h(s)) = +\infty$, we can pick δ with the property

$$(c\theta_1)^2 g\big(h(c\varphi_1)\big) - 3\mu f\big(x, h(c\varphi_1)\big) > 0 \quad \text{in } \Omega_\delta. \tag{8.171}$$

Let $\theta_2 := \inf_{\Omega\setminus\Omega_\delta} \varphi_1 > 0$. We choose $M > 1$ with

$$M(c\theta_1)^2 > 3, \tag{8.172}$$

$$Mc\lambda_1\theta_2 h'\big(c\|\varphi_1\|_\infty\big) > 3g\big(h(c\theta_2)\big). \tag{8.173}$$

Since $p < 1$, we also may assume

$$(Mc)^{1-p}\lambda_1(h')^{1-p}\big(c\|\varphi_1\|_\infty\big) \geqslant 3\|\nabla\varphi_1\|_\infty^p. \tag{8.174}$$

On the other hand, by Lemma 8.9(ii) we can choose $M > 1$ such that

$$3\big(h'(c\varphi_1)\big)^p \leqslant M^{1-p}(c\theta_1)^{2-p} g\big(h(c\varphi_1)\big) \quad \text{in } \Omega_\delta. \tag{8.175}$$

The assumption (f4) yields

$$\lim_{s\to\infty} \frac{3\mu f(x, sh(c\|\varphi_1\|_\infty))}{sh(c\|\varphi_1\|_\infty)} = 0.$$

So we can choose $M > 1$ large enough such that

$$\frac{3\mu f(x, Mh(c\|\varphi_1\|_\infty))}{Mh(c\|\varphi_1\|_\infty)} < \frac{c\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty)}{h(c\|\varphi_1\|_\infty)},$$

uniformly in Ω. This leads us to

$$3\mu f\big(x, Mh(c\|\varphi_1\|_\infty)\big) < Mc\lambda_1\theta_2 h'\big(c\|\varphi_1\|_\infty\big), \quad \text{for all } x \in \Omega. \tag{8.176}$$

For M satisfying (8.172)–(8.176), we prove that $\bar{u}_\mu = Mh(c\varphi_1)$ is a super-solution of (8.148). We have

$$-\Delta\bar{u}_\lambda = Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) \quad \text{in } \Omega. \tag{8.177}$$

First we prove that

$$Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant g(\bar{u}_\mu) + |\nabla\bar{u}_\mu|^p + \mu f(x, \bar{u}_\mu) \quad \text{in } \Omega_\delta. \tag{8.178}$$

From (8.170) and (8.172) we get

$$\begin{aligned}\frac{1}{3}Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geqslant g\big(h(c\varphi_1)\big) \geqslant g\big(Mh(c\varphi_1)\big)\\ &= g(\bar{u}_\mu) \quad \text{in } \Omega_\delta.\end{aligned} \tag{8.179}$$

By (8.170) and (8.175) we also have

$$\frac{1}{3}Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant (Mc)^p (h')^p\big(c\varphi_1\big))|\nabla\varphi_1|^p = |\nabla\bar{u}_\mu|^p \quad \text{in } \Omega_\delta. \tag{8.180}$$

The assumption (f3) and (8.171) produce

$$\begin{aligned}\frac{1}{3}Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 &\geqslant \mu M f\big(x, h(c\varphi_1)\big)\\ &\geqslant \mu f\big(x, Mh(c\varphi_1)\big) \quad \text{in } \Omega_\delta.\end{aligned} \tag{8.181}$$

Now, by (8.179), (8.180) and (8.181) we conclude that (8.178) is fulfilled.

Next we prove

$$Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant g(\bar{u}_\mu) + |\nabla\bar{u}_\mu|^p + \mu f(x, \bar{u}_\mu) \quad \text{in } \Omega \setminus \Omega_\delta. \tag{8.182}$$

From (8.173) we obtain

$$\frac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant g\big(h(c\varphi_1)\big) \geqslant g\big(Mh(c\varphi_1)\big) = g(\bar{u}_\mu) \quad \text{in } \Omega \setminus \Omega_\delta. \tag{8.183}$$

From (8.174) we get

$$\frac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant (Mc)^p (h')^p(c\varphi_1)|\nabla\varphi_1|^p = |\nabla\bar{u}_\mu|^p \quad \text{in } \Omega \setminus \Omega_\delta. \tag{8.184}$$

By (8.176) we deduce

$$\frac{1}{3}Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant \mu f\big(x, Mh(c\varphi_1)\big) = \mu f(x, \bar{u}_\mu) \quad \text{in } \Omega \setminus \Omega_\delta. \tag{8.185}$$

Obviously, (8.182) follows now by (8.183), (8.184) and (8.185).

Combining (8.177) with (8.178) and (8.182) we find that $\bar{u}_\mu$ is a super-solution of (8.148). Moreover, $\zeta \leqslant \bar{u}_\mu$ in Ω. Applying Lemma 8.6, we deduce that (8.148) has at least one solution for all $\mu \geqslant 0$. This finishes the proof of Theorem 8.2. □

PROOF OF THEOREM 8.3. The proof case relies on the same arguments used in the proof of Theorem 8.2. In fact, the main point is to find a super-solution $\bar{u}_\lambda \in C^2(\Omega) \cap (\overline{\Omega})$ of (8.148), while ζ defined in (8.154) is a sub-solution. Since g is nonincreasing, the inequality $\zeta \leqslant \bar{u}_\lambda$ in Ω can be proved easily and the existence of solutions to (8.148) follows by Lemma 8.6.

Define c, δ and θ_1, θ_2 as in the proof of Theorem 8.2. Let M satisfying (8.172) and (8.173). Since $g(h(s)) \to +\infty$ as $s \searrow 0$, we can choose $\delta > 0$ such that

$$(c\theta_1)^2 g\big(h(c\varphi_1)\big) - 3f\big(x, h(c\varphi_1)\big) > 0 \quad \text{in } \Omega_\delta. \tag{8.186}$$

The assumption (f4) produces

$$\lim_{s\to\infty} \frac{f(x, sh(c\|\varphi_1\|_\infty))}{sh(c\|\varphi_1\|_\infty)} = 0, \quad \text{uniformly for } x \in \overline{\Omega}.$$

Thus, we can take $M > 3$ large enough, such that

$$\frac{f(x, Mh(c\|\varphi_1\|_\infty))}{Mh(c\|\varphi_1\|_\infty)} < \frac{c\lambda_1\theta_2 h'(c\|\varphi_1\|_\infty)}{3h(c\|\varphi_1\|_\infty)}.$$

The above relation yields

$$3f\big(x, Mh\big(c\|\varphi_1\|_\infty\big)\big) < Mc\lambda_1\theta_2 h'\big(c\|\varphi_1\|_\infty\big), \quad \text{for all } x \in \overline{\Omega}. \tag{8.187}$$

Using Lemma 8.9(ii) we can take $\lambda > 0$ small enough such that the following inequalities hold

$$3\lambda M^{p-1}(h')^p(c\varphi_1) \leqslant g\big(h(c\varphi_1)\big)(c\theta_1)^{2-p} \quad \text{in } \Omega_\delta, \tag{8.188}$$

$$\lambda_1\theta_2 h'\big(c\|\varphi_1\|_\infty\big) > 3\lambda(Mc)^{p-1}(h')^p(c\theta_2)\|\nabla\varphi_1\|_\infty^p. \tag{8.189}$$

For M and λ satisfying (8.172)–(8.173) and (8.186)–(8.189), we claim that $\bar{u}_\lambda = Mh(c\varphi_1)$ is a super-solution of (8.148). First we have

$$-\Delta\bar{u}_\lambda = Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 + Mc\lambda_1\varphi_1 h'(c\varphi_1) \quad \text{in } \Omega. \tag{8.190}$$

Arguing as in the proof of Theorem 8.2, from (8.170), (8.172), (8.186), (8.188) and the assumption (f3) we obtain

$$Mc^2 g\big(h(c\varphi_1)\big)|\nabla\varphi_1|^2 \geqslant g(\bar{u}_\lambda) + \lambda|\nabla\bar{u}_\lambda|^p + f(x,\bar{u}_\lambda) \quad \text{in } \Omega_\delta. \tag{8.191}$$

On the other hand, (8.173), (8.187) and (8.189) gives

$$Mc\lambda_1\varphi_1 h'(c\varphi_1) \geqslant g(\bar{u}_\lambda) + \lambda|\nabla\bar{u}_\lambda|^p + f(x,\bar{u}_\lambda) \quad \text{in } \Omega \setminus \Omega_\delta. \tag{8.192}$$

Using now (8.190) and (8.191)–(8.192) we find that $\bar{u}_\lambda$ is a super-solution of (8.148) so our claim follows.

As we have already argued at the beginning of this case, we easily get that $\zeta \leqslant \bar{u}_\lambda$ in Ω and by Lemma 8.6 we deduce that problem (8.148) has at least one solution if $\lambda > 0$ is sufficiently small.

Set

$$A = \{\lambda \geqslant 0;\ \text{problem (8.148) has at least one classical solution}\}.$$

From the above arguments, A is nonempty. Let $\lambda^* = \sup A$. First we claim that if $\lambda \in A$, then $[0,\lambda) \subseteq A$. For this purpose, let $\lambda_1 \in A$ and $0 \leqslant \lambda_2 < \lambda_1$. If u_{λ_1} is a solution of (8.148) with $\lambda = \lambda_1$, then u_{λ_1} is a super-solution for (8.148) with $\lambda = \lambda_2$ while ζ defined in (8.154) is a sub-solution. Using Lemma 8.6 once more, we have that (8.148) with $\lambda = \lambda_2$ has at least one classical solution. This proves the claim. Since $\lambda \in A$ was arbitrary chosen, we conclude that $[0,\lambda^*) \subset A$.

Let us assume now $p \in (1,2]$. We prove that $\lambda^* < +\infty$. Set

$$m := \inf_{(x,s)\in\overline{\Omega}\times(0,\infty)} \big(g(s) + f(x,s)\big).$$

Since $\lim_{s\searrow 0} g(s) = +\infty$ and the mapping $(0,\infty) \ni s \mapsto \min_{x\in\overline{\Omega}} f(x,s)$ is positive and nondecreasing, we deduce that m is a positive real number. Let $\lambda > 0$ be such that (8.148) has a solution u_λ. If $v = \lambda^{1/(p-1)}u_\lambda$, then v verifies

$$\begin{cases} -\Delta v \geqslant |\nabla v|^p + \lambda^{1/(p-1)}m & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.193}$$

It follows that v is a super-solution of (8.155) for $\sigma = \lambda^{1/(p-1)}m$. Since 0 is a sub-solution, we obtain that (8.155) has at least one classical solution for σ defined above. According to Lemma 8.7, we have $\sigma \leqslant \bar{\sigma}$, and so $\lambda \leqslant (\bar{\sigma}/m)^{p-1}$. This means that λ^* is finite.

Assume now $p \in (0,1)$ and let us prove that $\lambda^* = +\infty$. Recall that ζ defined in (8.154) is a sub-solution. To get a super-solution, we proceed in the same manner. Fix $\lambda > 0$. Since $p < 1$ we can find $M > 1$ large enough such that (8.172), (8.173) and (8.187)–(8.189) hold. From now on, we follow the same steps as above. The proof of Theorem 8.3 is now complete. □

We remark that if $\int_0^1 g(s)\,ds < \infty$, then the above method can be applied in order to extend the study of (8.148) to the case $\mu = 1$ and $p > 2$. Indeed, by Lemma 8.9(i) it follows $h \in C^1[0, \eta]$. Using this fact, we can choose $c_1, c_2 > 0$ large enough such that the conclusion of Lemma 8.9(ii) holds. Repeating the above arguments we prove that if $p > 2$ then there exists a real number $\lambda^* > 0$ such that (8.148) has at least one solution if $\lambda < \lambda^*$ and no solutions exist if $\lambda > \lambda^*$.

PROOF OF THEOREM 8.4. (i) If $\lambda = 0$, the existence of the solution follows by using Lemma 6.2. Next we assume that $\lambda > 0$ and let us fix $\mu \geqslant 0$. With the change of variable $v = e^{\lambda u} - 1$, the problem (8.148) becomes

$$\begin{cases} -\Delta v = \Phi_\lambda(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.194}$$

where

$$\Phi_\lambda(s) = \lambda(s+1)g\left(\frac{1}{\lambda}\ln(s+1)\right) + \lambda\mu(s+1),$$

for all $s \in (0, \infty)$. Obviously Φ_λ is not monotone but we still have that the mapping $(0, \infty) \ni s \mapsto \Phi_\lambda(s)/s$, is decreasing for all $\lambda > 0$ and

$$\lim_{s\to+\infty} \frac{\Phi_\lambda(s)}{s} = \lambda(a+\mu) \quad \text{and} \quad \lim_{s\searrow 0} \frac{\Phi_\lambda(s)}{s} = +\infty,$$

uniformly for $\lambda > 0$.

We first remark that Φ_λ satisfies the hypotheses in Lemma 6.2 provided $\lambda(a+\mu) < \lambda_1$. Hence (8.194) has at least one solution.

On the other hand, since $g \geqslant a$ on $(0, \infty)$, we get

$$\Phi_\lambda(s) \geqslant \lambda(a+\mu)(s+1), \quad \text{for all } \lambda, s \in (0, \infty). \tag{8.195}$$

Using now Lemma 8.5 we deduce that (8.194) has no solutions if $\lambda(a+\mu) \geqslant \lambda_1$. The proof of the first part in Theorem 8.4 is therefore complete.

(ii) We split the proof into several steps.

Step 1. *Existence of solutions*. This follows directly from (i).

Step 2. *Uniqueness of the solution*. Fix $\lambda \geqslant 0$. Let u_1 and u_2 be two classical solutions of (8.148) with $\lambda < \lambda^*$. We show that $u_1 \leqslant u_2$ in Ω. Supposing the contrary, we deduce that $\max_{\overline{\Omega}}\{u_1 - u_2\} > 0$ is achieved in a point $x_0 \in \Omega$. This yields $\nabla(u_1 - u_2)(x_0) = 0$ and

$$0 \leqslant -\Delta(u_1 - u_2)(x_0) = g\big(u_1(x_0)\big) - g\big(u_2(x_0)\big) < 0,$$

a contradiction. We conclude that $u_1 \leqslant u_2$ in Ω; similarly $u_2 \leqslant u_1$. Therefore $u_1 = u_2$ in Ω and the uniqueness is proved.

Step 3. *Dependence on* λ. Fix $0 \leqslant \lambda_1 < \lambda_2 < \lambda^*$ and let $u_{\lambda_1}, u_{\lambda_2}$ be the unique solutions of (8.148) with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively. If $\{x \in \Omega;\ u_{\lambda_1} > u_{\lambda_2}\}$ is nonempty, then $\max_{\overline{\Omega}}\{u_{\lambda_1} - u_{\lambda_2}\} > 0$ is achieved in Ω. At that point, say $\bar{x}$, we have $\nabla(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = 0$ and

$$0 \leqslant -\Delta(u_{\lambda_1} - u_{\lambda_2})(\bar{x}) = g\big(u_{\lambda_1}(\bar{x})\big) - g\big(u_{\lambda_2}(\bar{x})\big) + (\lambda_1 - \lambda_2)|\nabla u_{\lambda_1}|^p(\bar{x}) < 0,$$

which is a contradiction.

Hence $u_{\lambda_1} \leqslant u_{\lambda_2}$ in $\overline{\Omega}$. The maximum principle also gives $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

Step 4. *Regularity*. We fix $0 < \lambda < \lambda^*$, $\mu > 0$ and assume that $\limsup_{s \searrow 0} s^{\alpha} g(s) < +\infty$. This means that $g(s) \leqslant cs^{-\alpha}$ in a small positive neighborhood of the origin. To prove the regularity, we will use again the change of variable $v = \mathrm{e}^{\lambda u} - 1$. Thus, if u_λ is the unique solution of (8.148), then $v_\lambda = \mathrm{e}^{\lambda u_\lambda} - 1$ is the unique solution of (8.194). Since $\lim_{s \searrow 0}(\mathrm{e}^{\lambda s} - 1)/s = \lambda$, we conclude that (ii1) and (ii2) in Theorem 8.4 are established if we prove

(a) $\tilde{c}_1 \operatorname{dist}(x, \partial\Omega) \leqslant v_\lambda(x) \leqslant \tilde{c}_2 \operatorname{dist}(x, \partial\Omega)$ in Ω, for some positive constants $\tilde{c}_1, \tilde{c}_2 > 0$.

(b) $v_\lambda \in C^{1,1-\alpha}(\overline{\Omega})$.

Proof of (a). By the monotonicity of g and the fact that $g(s) \leqslant cs^{-\alpha}$ near the origin, we deduce the existence of $A, B, C > 0$ such that

$$\Phi_\lambda(s) \leqslant As + Bs^{-\alpha} + C, \quad \text{for all } 0 < \lambda < \lambda^* \text{ and } s > 0. \tag{8.196}$$

Let us fix $m > 0$ such that $m\lambda_1 \|\varphi_1\|_\infty < \lambda\mu$. Combining this with (8.195) we deduce

$$-\Delta(v_\lambda - m\varphi_1) = \Phi_\lambda(v_\lambda) - m\lambda_1\varphi_1 \geqslant \lambda\mu - m\lambda_1\varphi_1 \geqslant 0 \tag{8.197}$$

in Ω. Since $v_\lambda - m\varphi_1 = 0$ on $\partial\Omega$, we conclude

$$v_\lambda \geqslant m\varphi_1 \quad \text{in } \Omega. \tag{8.198}$$

Now, (8.198) and (8.151) imply $v_\lambda \geqslant \tilde{c}_1 \operatorname{dist}(x, \partial\Omega)$ in Ω, for some positive constant $\tilde{c}_1 > 0$. The first inequality in the statement of (a) is therefore established. For the second one, we apply an idea found in Gui and Lin [56]. Using (8.198) and the estimate (8.196), by virtue of Lemma 7.5 we deduce $\Phi_\lambda(v_\lambda) \in L^1(\Omega)$, that is, $\Delta v_\lambda \in L^1(\Omega)$.

Using the smoothness of $\partial\Omega$, we can find $\delta \in (0, 1)$ such that for all $x_0 \in \Omega_\delta := \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) \leqslant \delta\}$, there exists $y \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\operatorname{dist}(y, \partial\Omega) = \delta$ and $\operatorname{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$.

Let $K > 1$ be such that $\operatorname{diam}(\Omega) < (K - 1)\delta$ and let ξ be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta\xi = \Phi_\lambda(\xi) & \text{in } B_K(0) \setminus B_1(0), \\ \xi > 0 & \text{in } B_K(0) \setminus B_1(0), \\ \xi = 0 & \text{on } \partial(B_K(0) \setminus B_1(0)), \end{cases}$$

where $B_r(0)$ denotes the open ball in $\mathbb{R}^N$ of radius r and centered at the origin. By uniqueness, ξ is radially symmetric. Hence $\xi(x) = \tilde{\xi}(|x|)$ and

$$\begin{cases} \tilde{\xi}'' + ((N-1)/r)\tilde{\xi}' + \Phi_\lambda(\tilde{\xi}) = 0 & \text{in } (1, K), \\ \tilde{\xi} > 0 & \text{in } (1, K), \\ \tilde{\xi}(1) = \tilde{\xi}(K) = 0. \end{cases} \tag{8.199}$$

Integrating in (8.199) we have

$$\begin{aligned} \tilde{\xi}'(t) &= \tilde{\xi}'(a)a^{N-1}t^{1-N} - t^{1-N}\int_a^t r^{N-1}\Phi_\lambda\big(\tilde{\xi}(r)\big)\,\mathrm{d}r \\ &= \tilde{\xi}'(b)b^{N-1}t^{1-N} + t^{1-N}\int_t^b r^{N-1}\Phi_\lambda\big(\tilde{\xi}(r)\big)\,\mathrm{d}r, \end{aligned}$$

where $1 < a < t < b < K$. With the same arguments as above we have $\Phi_\lambda(\tilde{\xi}) \in L^1(1, K)$ which implies that both $\tilde{\xi}(1)$ and $\tilde{\xi}(K)$ are finite. Hence $\tilde{\xi} \in C^2(1, K) \cap C^1[1, K]$. Furthermore,

$$\xi(x) \leqslant \widetilde{C}\min\big\{K - |x|, |x| - 1\big\}, \quad \text{for any } x \in B_K(0) \setminus B_1(0). \tag{8.200}$$

Let us fix $x_0 \in \Omega_\delta$. Then we can find $y_0 \in \mathbb{R}^N \setminus \overline{\Omega}$ with $\operatorname{dist}(y_0, \partial\Omega) = \delta$ and $\operatorname{dist}(x_0, \partial\Omega) = |x_0 - y| - \delta$. Thus, $\Omega \subset B_{K\delta}(y_0) \setminus B_\delta(y_0)$. Define $\bar{v}(x) = \xi((x - y_0)/\delta)$, for all $x \in \overline{\Omega}$. We show that $\bar{v}$ is a super-solution of (8.194). Indeed, for all $x \in \Omega$ we have

$$\begin{aligned} \Delta\bar{v} + \Phi_\lambda(\bar{v}) &= \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}'\right) + \Phi_\lambda(\tilde{\xi}) \\ &\leqslant \frac{1}{\delta^2}\left(\tilde{\xi}'' + \frac{N-1}{r}\tilde{\xi}' + \Phi_\lambda(\tilde{\xi})\right) \\ &= 0, \end{aligned}$$

where $r = |x - y_0|/\delta$. We have obtained that

$$\begin{aligned} &\Delta\bar{v} + \Phi_\lambda(\bar{v}) \leqslant 0 \leqslant \Delta v_\lambda + \Phi_\lambda(v_\lambda) \quad \text{in } \Omega, \\ &\bar{v}, v_\lambda > 0 \quad \text{in } \Omega, \qquad \bar{v} = v_\lambda \quad \text{on } \partial\Omega \\ &\Delta v_\lambda \in L^1(\Omega). \end{aligned}$$

By Lemma 6.3 we get $v_\lambda \leqslant \bar{v}$ in Ω. Combining this with (8.200) we obtain

$$v_\lambda(x_0) \leqslant \bar{v}(x_0) \leqslant \widetilde{C}\min\left\{K - \frac{|x_0 - y_0|}{\delta}, \frac{|x_0 - y_0|}{\delta} - 1\right\} \leqslant \frac{\widetilde{C}}{\delta}\operatorname{dist}(x_0, \partial\Omega).$$

Hence $v_\lambda \leqslant (\widetilde{C}/\delta)\operatorname{dist}(x, \partial\Omega)$ in Ω_δ and the second inequality in the statement of (a) follows.

Proof of (b). Let G be the Green's function associated with the Laplace operator in Ω. Then, for all $x \in \Omega$ we have

$$v_\lambda(x) = -\int_\Omega G(x, y)\Phi_\lambda\big(v_\lambda(y)\big)\,\mathrm{d}y$$

and

$$\nabla v_\lambda(x) = -\int_\Omega G_x(x, y)\Phi_\lambda\big(v_\lambda(y)\big)\,\mathrm{d}y.$$

If $x_1, x_2 \in \Omega$, using (8.196) we obtain

$$\begin{aligned}\big|\nabla v_\lambda(x_1) - \nabla v_\lambda(x_2)\big| \leqslant &\int_\Omega \big|G_x(x_1, y) - G_x(x_2, y)\big| \cdot (Av_\lambda + C)\,\mathrm{d}y \\ &+ B\int_\Omega \big|G_x(x_1, y) - G_x(x_2, y)\big| \cdot v_\lambda^{-\alpha}(y)\,\mathrm{d}y.\end{aligned}$$

Now, taking into account that $v_\lambda \in C(\overline{\Omega})$, by the standard regularity theory (see Gilbarg and Trudinger [54]) we get

$$\int_\Omega \big|G_x(x_1, y) - G_x(x_2, y)\big| \cdot (Av_\lambda + C)\,\mathrm{d}y \leqslant \tilde{c}_1|x_1 - x_2|.$$

On the other hand, with the same proof as in [56, Theorem 1], we deduce

$$\int_\Omega \big|G_x(x_1, y) - G_x(x_2, y)\big| \cdot v_\lambda^{-\alpha}(y) \leqslant \tilde{c}_2|x_1 - x_2|^{1-\alpha}.$$

The above inequalities imply $u_\lambda \in C^2(\Omega) \cap C^{1,1-\alpha}(\overline{\Omega})$.

Step 5. *Asymptotic behavior of the solution*. This follows with the same lines as in the proof of Theorem 6.4. □

We are concerned in what follows with the closely related Dirichlet problem

$$\begin{cases} -\Delta u + K(x)g(u) + |\nabla u|^a = \lambda f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1_λ}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geqslant 2$), $\lambda > 0$, $0 < a \leqslant 2$ and $K \in C^{0,\gamma}(\overline{\Omega})$, $0 < \gamma < 1$. We assume from now on that $f : \overline{\Omega} \times [0, \infty) \to [0, \infty)$ is a Hölder continuous function which is positive on $\overline{\Omega} \times (0, \infty)$ such that f is nondecreasing with respect to the second variable and is sublinear, in the sense that the mapping

$$(0, \infty) \ni s \mapsto \frac{f(x, s)}{s} \quad \text{is nonincreasing for all } x \in \overline{\Omega}$$

and

$$\lim_{s\to 0^+} \frac{f(x,s)}{s} = +\infty \quad \text{and} \quad \lim_{s\to\infty} \frac{f(x,s)}{s} = 0, \quad \text{uniformly for } x \in \overline{\Omega}.$$

We also assume that $g \in C^{0,\gamma}(0,\infty)$ is a nonnegative and nonincreasing function satisfying

$$\lim_{s\to 0^+} g(s) = +\infty.$$

Problem (1_λ) has been considered in Section 7 in the absence of the gradient term $|\nabla u|^a$ and assuming that the singular term $g(t)$ behaves like $t^{-\alpha}$ around the origin, with $t \in (0,1)$. In this case it has been shown that the sign of the extremal values of K plays a crucial role. In this sense, we have proved in Section 7 that if $K < 0$ in $\overline{\Omega}$, then problem (1_λ) (with $a = 0$) has a unique solution in the class $\mathcal{E} = \{u \in C^2(\Omega) \cap C(\overline{\Omega});\ g(u) \in L^1(\Omega)\}$, for all $\lambda > 0$. On the other hand, if $K > 0$ in $\overline{\Omega}$, then there exists λ^* such that problem (1_λ) has solutions in $\mathcal{E}$ if $\lambda > \lambda^*$ and no solution exists if $\lambda < \lambda^*$. The case where f is asymptotically linear, $K \leqslant 0$, and $a = 0$ has been discussed in Section 6. In this framework, a major role is played by $\lim_{s\to\infty} f(s)/s = m > 0$. More precisely, there exists a solution (which is unique) $u_\lambda \in C^2(\Omega) \cap C^1(\overline{\Omega})$ if and only if $\lambda < \lambda^* := \lambda_1/m$. An additional result asserts that the mapping $(0,\lambda^*) \mapsto u_\lambda$ is increasing and $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$ uniformly on compact subsets of Ω.

Due to the singular character of our problem $(1)_\lambda$, we cannot expect to have solutions in $C^2(\overline{\Omega})$. We are seeking in this paper classical solutions of (1_λ), that is, solutions $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that verify (1_λ). Closely related to our problem is the following one, which has been considered in the first part of this section:

$$\begin{cases} -\Delta u = g(u) + |\nabla u|^a + \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.201}$$

where f and g verifies the above assumptions. We recall that we have proved that if $0 < a < 1$ then problem (8.201) has at least one classical solution for all $\lambda \geqslant 0$. In turn, if $1 < a \leqslant 2$, then problem (8.201) has no solutions for large values of $\lambda > 0$.

The existence results for our problem (1_λ) are quite different to those of (8.201) presented in the first part of this section. More exactly, we prove in what follows that problem (1_λ) has at least one solution only when $\lambda > 0$ is large enough and g satisfies a naturally growth condition around the origin. Thus, we extend the results in Barles, G. Díaz, and J.I. Díaz [10, Theorem 1], corresponding to $K \equiv 0$, $f \equiv f(x)$ and $a \in [0,1)$.

The main difficulty in the treatment of (1_λ) is the lack of the usual maximal principle between super and sub-solutions, due to the singular character of the equation. To overcome it, we state an improved comparison principle that fit to our problem (1_λ) (see Lemma 8.13 below).

In our first result we assume that $K < 0$ in Ω. Note that K may vanish on $\partial\Omega$ which leads us to a competition on the boundary between the potential $K(x)$ and the singular term $g(u)$. We prove the following result.

THEOREM 8.10. *Assume that $K < 0$ in Ω. Then, for all $\lambda > 0$, problem (1_λ) has at least one classical solution.*

Next, we assume that $K > 0$ in $\overline{\Omega}$. In this case, the existence of a solution to (1_λ) is closely related to the decay rate around its singularity. In this sense, we prove that problem (1_λ) has no solution, provided that g has a "strong" singularity at the origin. More precisely, we have

THEOREM 8.11. *Assume that $K > 0$ in $\overline{\Omega}$ and $\int_0^1 g(s)\,\mathrm{d}s = +\infty$. Then problem (1_λ) has no classical solutions.*

In the following result, assuming that $\int_0^1 g(s)\,\mathrm{d}s < +\infty$, we show that problem (1_λ) has at least one solution, provided that $\lambda > 0$ is large enough.

THEOREM 8.12. *Assume that $K > 0$ in $\overline{\Omega}$ and $\int_0^1 g(s)\,\mathrm{d}s < +\infty$. Then there exists $\lambda^* > 0$ such that problem (1_λ) has at least one classical solution if $\lambda > \lambda^*$ and no solution exists if $\lambda < \lambda^*$.*

A very useful auxiliary result in the proofs of the above theorems is the following comparison principle that improves Lemma 6.3. Our proof uses some ideas from Shi and Yao [86], that go back to the pioneering work by Brezis and Kamin [14].

LEMMA 8.13. *Let $\Psi : \overline{\Omega} \times (0, \infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0, \infty) \ni s \mapsto \Psi(x, s)/s$ is strictly decreasing at each $x \in \Omega$. Assume that there exists v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\Delta w + \Psi(x, w) \leqslant 0 \leqslant \Delta v + \Psi(x, v)$ *in* Ω;

(b) $v, w > 0$ *in* Ω *and* $v \leqslant w$ *on* $\partial\Omega$;

(c) $\Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \leqslant w$ in Ω.

PROOF. We argue by contradiction and assume that $v \geqslant w$ is not true in Ω. Then, we can find $\varepsilon_0, \delta_0 > 0$ and a ball $B \Subset \Omega$ such that $v - w \geqslant \varepsilon_0$ in B and

$$\int_B vw\left(\frac{\Psi(x, w)}{w} - \frac{\Psi(x, v)}{v}\right)\mathrm{d}x \geqslant \delta_0. \tag{8.202}$$

The case $\Delta v \in L^1(\Omega)$ was stated in Lemma 6.3. Let us assume now that $\Delta w \in L^1(\Omega)$ and set $M = \max\{1, \|\Delta w\|_{L^1(\Omega)}\}$, $\varepsilon = \min\{1, \varepsilon_0, 2^{-2}\delta_0/M\}$. Consider a nondecreasing function $\theta \in C^1(\mathbb{R})$ such that $\theta(t) = 0$, if $t \leqslant 1/2$, $\theta(t) = 1$, if $t \geqslant 1$, and $\theta(t) \in (0, 1)$ if $t \in (1/2, 1)$. Define

$$\theta_\varepsilon(t) = \theta\left(\frac{t}{\varepsilon}\right), \quad t \in \mathbb{R}.$$

Since $w \geqslant v$ on $\partial\Omega$, we can find a smooth subdomain $\Omega^* \Subset \Omega$ such that

$$B \subset \Omega^* \quad \text{and} \quad v - w < \frac{\varepsilon}{2} \quad \text{in } \Omega \setminus \Omega^*.$$

Using the hypotheses (a) and (b) we deduce

$$\begin{aligned}
&\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,\mathrm{d}x \\
&\quad \geqslant \int_{\Omega^*} vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)\,\mathrm{d}x. \qquad (8.203)
\end{aligned}$$

By (8.202) we have

$$\begin{aligned}
&\int_{\Omega^*} vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)\,\mathrm{d}x \\
&\quad \geqslant \int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\theta_\varepsilon(v-w)\,\mathrm{d}x \\
&\quad = \int_B vw\left(\frac{\Psi(x,w)}{w} - \frac{\Psi(x,v)}{v}\right)\mathrm{d}x \geqslant \delta_0.
\end{aligned}$$

To raise a contradiction, we need only to prove that the left-hand side in (8.203) is smaller than δ_0. For this purpose, we define

$$\Theta_\varepsilon(t) = \int_0^t s\theta_\varepsilon'(s)\,\mathrm{d}s, \quad t \in \mathbb{R}.$$

It is easy to see that

$$\Theta_\varepsilon(t) = 0, \quad \text{if } t < \frac{\varepsilon}{2} \quad \text{and} \quad 0 \leqslant \Theta_\varepsilon(t) \leqslant 2\varepsilon, \quad \text{for all } t \in \mathbb{R}. \qquad (8.204)$$

Now, using the Green theorem, we evaluate the left-hand side of (8.203):

$$\begin{aligned}
&\int_{\Omega^*} (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,\mathrm{d}x \\
&\quad = \int_{\partial\Omega^*} w\theta_\varepsilon(v-w)\frac{\partial v}{\partial n}\,\mathrm{d}\sigma - \int_{\Omega^*} (\nabla w \cdot \nabla v)\theta_\varepsilon(v-w)\,\mathrm{d}x \\
&\qquad - \int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla v \cdot \nabla(v-w)\,\mathrm{d}x - \int_{\partial\Omega^*} v\theta_\varepsilon(v-w)\frac{\partial w}{\partial n}\,\mathrm{d}\sigma \\
&\qquad + \int_{\Omega^*} (\nabla w \cdot \nabla v)\theta_\varepsilon(v-w)\,\mathrm{d}x + \int_{\Omega^*} v\theta_\varepsilon'(v-w)\nabla w \cdot \nabla(v-w)\,\mathrm{d}x \\
&\quad = \int_{\Omega^*} \theta_\varepsilon'(v-w)(v\nabla w - w\nabla v) \cdot \nabla(v-w)\,\mathrm{d}x.
\end{aligned}$$

The above relation can also be rewritten as

$$\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,\mathrm{d}x = \int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla(w-v)\cdot\nabla(v-w)\,\mathrm{d}x + \int_{\Omega^*}(v-w)\theta_\varepsilon'(v-w)\nabla w\cdot\nabla(v-w)\,\mathrm{d}x.$$

Since $\int_{\Omega^*} w\theta_\varepsilon'(v-w)\nabla(w-v)\cdot\nabla(v-w)\,\mathrm{d}x \leqslant 0$, the last equality yields

$$\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,\mathrm{d}x \leqslant \int_{\Omega^*}(v-w)\theta_\varepsilon'(v-w)\nabla w\cdot\nabla(v-w)\,\mathrm{d}x,$$

that is,

$$\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,\mathrm{d}x \leqslant \int_{\Omega^*}\nabla w\cdot\nabla\big(\Theta_\varepsilon(v-w)\big)\,\mathrm{d}x.$$

Again by Green's first formula and by (8.204) we have

$$\begin{aligned}
&\int_{\Omega^*}(w\Delta v - v\Delta w)\theta_\varepsilon(v-w)\,\mathrm{d}x\\
&\quad\leqslant \int_{\partial\Omega^*}\Theta_\varepsilon(v-w)\frac{\partial v}{\partial n}\,\mathrm{d}\sigma - \int_{\Omega^*}\Theta_\varepsilon(v-w)\Delta w\,\mathrm{d}x\\
&\quad\leqslant -\int_{\Omega^*}\Theta_\varepsilon(v-w)\Delta w\,\mathrm{d}x \leqslant 2\varepsilon\int_{\Omega^*}|\Delta w|\,\mathrm{d}x\\
&\quad\leqslant 2\varepsilon M < \frac{\delta_0}{2}.
\end{aligned}$$

Thus, we have obtained a contradiction. Hence $v \leqslant w$ in Ω and the proof of Lemma 8.13 is now complete. □

We are now ready to prove our main results.

PROOF OF THEOREM 8.10. Fix $\lambda > 0$. Obviously, $\Psi(x,s) = \lambda f(x,s) - K(x)g(s)$ satisfies the hypotheses in Lemma 6.2 since $K < 0$ in Ω. Hence, there exists a solution $\bar{u}_\lambda$ of the problem

$$\begin{cases} -\Delta u = \lambda f(x,u) - K(x)g(u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We observe that $\bar{u}_\lambda$ is a super-solution of problem (1_λ). To find a sub-solution, let us denote

$$p(x) = \min\big\{\lambda f(x,1); -K(x)g(1)\big\}, \quad x \in \overline{\Omega}.$$

Using the monotonicity of f and g, we observe that $p(x) \leqslant \lambda f(x,s) - K(x)g(s)$ for all $(x,s) \in \Omega \times (0,\infty)$. We now consider the problem

$$\begin{cases} -\Delta v + |\nabla v|^a = p(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.205}$$

First, we observe that $v = 0$ is a sub-solution of (8.205) while w defined by

$$\begin{cases} -\Delta w = p(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

is a super-solution. Since $p > 0$ in Ω we deduce that $w \geqslant 0$ in Ω. Thus, the problem (8.205) has at least one classical solution v. We claim that v is positive in Ω. Indeed, if v has a minimum in Ω, say at x_0, then $\nabla v(x_0) = 0$ and $\Delta v(x_0) \geqslant 0$. Therefore

$$0 \geqslant -\Delta v(x_0) + |\nabla v|^a(x_0) = p(x_0) > 0,$$

which is a contradiction. Hence $\min_{x\in\overline{\Omega}} v = \min_{x\in\partial\Omega} v = 0$, that is, $v > 0$ in Ω. Now $\underline{u}_\lambda = v$ is a sub-solution of (1_λ) and we have

$$-\Delta \underline{u}_\lambda = p(x) \leqslant \lambda f(x, \bar{u}_\lambda) - K(x)g(\bar{u}_\lambda) = -\Delta \bar{u}_\lambda \quad \text{in } \Omega.$$

Since $\underline{u}_\lambda = \bar{u}_\lambda = 0$ on $\partial\Omega$, from the above relation we may conclude that $\underline{u}_\lambda \leqslant \bar{u}_\lambda$ in Ω and so, there exists at least one classical solution for (1_λ). The proof of Theorem 8.10 is now complete. □

PROOF OF THEOREM 8.11. We give a direct proof, without using any change of variable, as in Zhang [93]. Let us assume that there exists $\lambda > 0$ such that the problem (1_λ) has a classical solution u_λ. By our hypotheses on f, we deduce by Lemma 6.2 that for all $\lambda > 0$ there exists $U_\lambda \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta U_\lambda = \lambda f(x, U_\lambda) & \text{in } \Omega, \\ U_\lambda > 0 & \text{in } \Omega, \\ U_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.206}$$

Moreover, there exist $c_1, c_2 > 0$ such that

$$c_1 \operatorname{dist}(x, \partial\Omega) \leqslant U_\lambda(x) \leqslant c_2 \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \Omega. \tag{8.207}$$

Consider the perturbed problem

$$\begin{cases} -\Delta u + K_* g(u+\varepsilon) = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8.208}$$

where $K_* = \min_{x\in\overline{\Omega}} K(x) > 0$. It is clear that u_λ and U_λ are respectively sub and supersolution of (8.208). Furthermore, we have

$$\Delta U_\lambda + f(x, U_\lambda) \leqslant 0 \leqslant \Delta u_\lambda + f(x, u_\lambda) \quad \text{in } \Omega,$$
$$U_\lambda, u_\lambda > 0 \quad \text{in } \Omega,$$
$$U_\lambda = u_\lambda = 0 \quad \text{on } \partial\Omega,$$
$$\Delta U_\lambda \in L^1(\Omega) \quad (\text{since } U_\lambda \in C^2(\overline{\Omega})).$$

In view of Lemma 8.13 we get $u_\lambda \leqslant U_\lambda$ in Ω. Thus, a standard bootstrap argument (see Gilbarg and Trudinger [54]) implies that there exists a solution $u_\varepsilon \in C^2(\overline{\Omega})$ of (8.208) such that

$$u_\lambda \leqslant u_\varepsilon \leqslant U_\lambda \quad \text{in } \Omega.$$

Integrating in (8.208) we obtain

$$-\int_\Omega \Delta u_\varepsilon \,\mathrm{d}x + K_* \int_\Omega g(u_\varepsilon + \varepsilon)\,\mathrm{d}x = \lambda \int_\Omega f(x, u_\varepsilon)\,\mathrm{d}x.$$

Hence

$$-\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}\,\mathrm{d}s + K_* \int_\Omega g(u_\varepsilon + \varepsilon)\,\mathrm{d}x \leqslant M, \tag{8.209}$$

where $M > 0$ is a positive constant. Taking into account the fact that $\partial u_\varepsilon/\partial n \leqslant 0$ on $\partial\Omega$, relation (8.209) yields $K_* \int_\Omega g(u_\varepsilon + \varepsilon)\,\mathrm{d}x \leqslant M$. Since $u_\varepsilon \leqslant U_\lambda$ in $\overline{\Omega}$, from the last inequality we can conclude that $\int_\Omega g(U_\lambda + \varepsilon)\,\mathrm{d}x \leqslant C$, for some $C > 0$. Thus, for any compact subset $\omega \Subset \Omega$ we have

$$\int_\omega g(U_\lambda + \varepsilon)\,\mathrm{d}x \leqslant C.$$

Letting $\varepsilon \to 0^+$, the above relation produces $\int_\omega g(U_\lambda)\,\mathrm{d}x \leqslant C$. Therefore

$$\int_\Omega g(U_\lambda)\,\mathrm{d}x \leqslant C. \tag{8.210}$$

On the other hand, using (8.207) and the hypothesis $\int_0^1 g(s)\,\mathrm{d}s = +\infty$, it follows

$$\int_\Omega g(U_\lambda)\,\mathrm{d}x \geqslant \int_\Omega g\big(c_2 \operatorname{dist}(x, \partial\Omega)\big)\,\mathrm{d}x = +\infty,$$

which contradicts (8.210). Hence, (1_λ) has no classical solutions and the proof of Theorem 8.11 is now complete. □

PROOF OF THEOREM 8.12. Fix $\lambda > 0$. We first note that U_λ defined in (8.206) is a super-solution of (1_λ). We now focus on finding a sub-solution $\underline{u}_\lambda$ such that $\underline{u}_\lambda \leqslant U_\lambda$ in Ω.

Let $h : [0, \infty) \to [0, \infty)$ be such that

$$\begin{cases} h''(t) = g(h(t)), & \text{for all } t > 0, \\ h > 0 \quad \text{in } (0, \infty), \\ h(0) = 0. \end{cases} \tag{8.211}$$

Multiplying by h' in (8.211) and then integrating over $[s, t]$ we have

$$(h')^2(t) - (h')^2(s) = 2 \int_{h(s)}^{h(t)} g(\tau)\, \mathrm{d}\tau, \quad \text{for all } t > s > 0.$$

Since $\int_0^1 g(\tau)\, \mathrm{d}\tau < \infty$, from the above equality we deduce that we can extend h' in origin by taking $h'(0) = 0$ and so $h \in C^2(0, \infty) \cap C^1[0, \infty)$. Taking into account the fact that h' is increasing and h'' is decreasing on $(0, \infty)$, the mean value theorem implies that

$$\frac{h'(t)}{t} = \frac{h'(t) - h'(0)}{t - 0} \geqslant h''(t), \quad \text{for all } t > 0.$$

Hence $h'(t) \geqslant t h''(t)$, for all $t > 0$. Integrating in the last inequality we get

$$t h'(t) \leqslant 2h(t), \quad \text{for all } t > 0. \tag{8.212}$$

Let ϕ_1 be the normalized positive eigenfunction corresponding to the first eigenvalue λ_1 of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well known that $\phi_1 \in C^2(\overline{\Omega})$. Furthermore, by Hopf's maximum principle there exist $\delta > 0$ and $\Omega_0 \Subset \Omega$ such that $|\nabla \phi_1| \geqslant \delta$ in $\Omega \setminus \Omega_0$. Let

$$M = \max\left\{1, 2K^* \delta^{-2}\right\},$$

where $K^* = \max_{x \in \overline{\Omega}} K(x)$. Since

$$\lim_{\operatorname{dist}(x, \partial\Omega) \to 0^+} \left\{ -K^* g\big(h(\phi_1)\big) + M^a (h')^a(\phi_1) |\nabla \phi_1|^a \right\} = -\infty,$$

by letting Ω_0 close enough to the boundary of Ω we can assume that

$$-K^* g\big(h(\phi_1)\big) + M^a (h')^a(\phi_1) |\nabla \phi_1|^a < 0 \quad \text{in } \Omega \setminus \Omega_0. \tag{8.213}$$

We now are able to show that $\underline{u}_\lambda = M h(\phi_1)$ is a sub-solution of (1_λ) provided $\lambda > 0$ is sufficiently large. Using the monotonicity of g and (8.212) we have

$$
\begin{aligned}
&-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \\
&\quad \leqslant -Mg\big(h(\phi_1)\big)|\nabla\phi_1|^2 + \lambda_1 Mh'(\phi_1)\phi_1 + K^* g\big(Mh(\phi_1)\big) \\
&\qquad + M^a (h')^a(\phi_1)|\nabla\phi_1|^a \\
&\quad \leqslant g\big(h(\phi_1)\big)\big(K^* - M|\nabla\phi_1|^2\big) + \lambda_1 Mh'(\phi_1)\phi_1 + M^a (h')^a(\phi_1)|\nabla\phi_1|^a \\
&\quad \leqslant g\big(h(\phi_1)\big)\big(K^* - M|\nabla\phi_1|^2\big) + 2\lambda_1 Mh(\phi_1) + M^a (h')^a(\phi_1)|\nabla\phi_1|^a. \quad (8.214)
\end{aligned}
$$

The definition of M and (8.213) yield

$$
-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leqslant 2\lambda_1 Mh(\phi_1) = 2\lambda_1 \underline{u}_\lambda \quad \text{in } \Omega \setminus \Omega_0. \quad (8.215)
$$

Let us choose $\lambda > 0$ such that

$$
\lambda \frac{\min_{x\in\overline{\Omega}_0} f(x, Mh(\|\phi_1\|_\infty))}{M\|\phi_1\|_\infty} \geqslant 2\lambda_1. \quad (8.216)
$$

Then, by virtue of the assumptions on f and using (8.216), we have

$$
\lambda \frac{f(x, \underline{u}_\lambda)}{\underline{u}_\lambda} \geqslant \lambda \frac{f(x, Mh(\|\phi_1\|_\infty))}{M\|\phi_1\|_\infty} \geqslant 2\lambda_1 \quad \text{in } \Omega \setminus \Omega_0.
$$

The last inequality combined with (8.215) yield

$$
-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leqslant 2\lambda_1 \underline{u}_\lambda \leqslant \lambda f(x, \underline{u}_\lambda) \quad \text{in } \Omega \setminus \Omega_0. \quad (8.217)
$$

On the other hand, from (8.214) we obtain

$$
\begin{aligned}
&-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \\
&\quad \leqslant K^* g\big(h(\phi_1)\big) + 2\lambda_1 Mh(\phi_1) + M^a (h')^a(\phi_1)|\nabla\phi_1|^a \quad \text{in } \Omega_0. \quad (8.218)
\end{aligned}
$$

Since $\phi_1 > 0$ in $\overline{\Omega}_0$ and f is positive on $\overline{\Omega}_0 \times (0, \infty)$, we may choose $\lambda > 0$ such that

$$
\begin{aligned}
&\lambda \min_{x\in\overline{\Omega}_0} f\big(x, Mh(\phi_1)\big) \\
&\quad \geqslant \max_{x\in\overline{\Omega}_0} \big\{K^* g\big(h(\phi_1)\big) + 2\lambda_1 Mh(\phi_1) + M^a (h')^a(\phi_1)|\nabla\phi_1|^a\big\}. \quad (8.219)
\end{aligned}
$$

From (8.218) and (8.219) we deduce

$$
-\Delta \underline{u}_\lambda + K(x)g(\underline{u}_\lambda) + |\nabla \underline{u}_\lambda|^a \leqslant \lambda f(x, \underline{u}_\lambda) \quad \text{in } \Omega_0. \quad (8.220)
$$

Now, (8.217) together with (8.220) shows that $\underline{u}_\lambda = Mh(\phi_1)$ is a sub-solution of (1_λ) provided $\lambda > 0$ satisfy (8.216) and (8.219). With the same arguments as in the proof of Theorem 8.11 and using Lemma 8.13, one can prove that $\underline{u}_\lambda \leqslant U_\lambda$ in Ω. By a standard

bootstrap argument (see Gilbarg and Trudinger [54]) we obtain a classical solution u_λ such that $\underline{u}_\lambda \leqslant u_\lambda \leqslant U_\lambda$ in Ω.

We have proved that (1_λ) has at least one classical solution when $\lambda > 0$ is large. Set

$$A = \{\lambda > 0;\ \text{problem } (1_\lambda) \text{ has at least one classical solution}\}.$$

From the above arguments we deduce that A is nonempty. Let $\lambda^* = \inf A$. We claim that if $\lambda \in A$, then $(\lambda, +\infty) \subseteq A$. To this aim, let $\lambda_1 \in A$ and $\lambda_2 > \lambda_1$. If u_{λ_1} is a solution of $(1)_{\lambda_1}$, then u_{λ_1} is a sub-solution for $(1)_{\lambda_2}$ while U_{λ_2} defined in (8.206) for $\lambda = \lambda_2$ is a super-solution. Moreover, we have

$$\begin{aligned}
&\Delta U_{\lambda_2} + \lambda_2 f(x, U_{\lambda_2}) \leqslant 0 \leqslant \Delta u_{\lambda_1} + \lambda_2 f(x, u_{\lambda_1}) \quad \text{in } \Omega,\\
&U_{\lambda_2}, u_{\lambda_1} > 0 \quad \text{in } \Omega,\\
&U_{\lambda_2} = u_{\lambda_1} = 0 \quad \text{on } \partial\Omega\\
&\Delta U_{\lambda_2} \in L^1(\Omega).
\end{aligned}$$

Again by Lemma 8.13 we get $u_{\lambda_1} \leqslant U_{\lambda_2}$ in Ω. Therefore, the problem $(1)_{\lambda_2}$ has at least one classical solution. This proves the claim. Since $\lambda \in A$ was arbitrary chosen, we conclude that $(\lambda^*, +\infty) \subset A$.

To end the proof, it suffices to show that $\lambda^* > 0$. In that sense, we will prove that there exists $\lambda > 0$ small enough such that (1_λ) has no classical solutions. We first remark that

$$\lim_{s \to 0^+} \big(f(x,s) - K(x)g(s)\big) = -\infty \quad \text{uniformly for } x \in \Omega.$$

Hence, there exists $c > 0$ such that

$$f(x,s) - K(x)g(s) < 0, \quad \text{for all } (x,s) \in \Omega \times (0,c). \tag{8.221}$$

On the other hand, the assumptions on f yield

$$\begin{aligned}
&\frac{f(x,s) - K(x)g(s)}{s} \leqslant \frac{f(x,s)}{s} \leqslant \frac{f(x,c)}{c},\\
&\quad \text{for all } (x,s) \in \Omega \times [c, +\infty).
\end{aligned} \tag{8.222}$$

Let $m = \max_{x \in \overline{\Omega}} f(x,c)/c$. Combining (8.221) with (8.222) we find

$$f(x,s) - K(x)g(s) < ms, \quad \text{for all } (x,s) \in \Omega \times (0, +\infty). \tag{8.223}$$

Set $\lambda_0 = \min\{1, \lambda_1/2m\}$. We show that problem $(1)_{\lambda_0}$ has no classical solution. Indeed, if u_0 would be a classical solution of $(1)_{\lambda_0}$, then, according to (8.223), u_0 is a sub-solution of

$$\begin{cases} -\Delta u = (\lambda_1/2)u & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{8.224}$$

Obviously, ϕ_1 is a super-solution of (8.224) and by Lemma 8.13 we get $u_0 \leqslant \phi_1$ in Ω. Thus, by standard elliptic arguments, problem (8.224) has a solution $u \in C^2(\overline{\Omega})$. Multiplying by ϕ_1 in (8.224) and then integrating over Ω we have

$$-\int_\Omega \phi_1 \Delta u \, \mathrm{d}x = \frac{\lambda_1}{2} \int_\Omega u\phi_1 \, \mathrm{d}x,$$

that is,

$$-\int_\Omega u \Delta\phi_1 \, \mathrm{d}x = \frac{\lambda_1}{2} \int_\Omega u\phi_1 \, \mathrm{d}x.$$

The above equality yields $\int_\Omega u\phi_1 \, \mathrm{d}x = 0$, which is clearly a contradiction, since u and ϕ_1 are positive in Ω. If follows that problem $(1)_{\lambda_0}$ has no classical solutions which means that $\lambda^* > 0$. This completes the proof of Theorem 8.12. □

We refer to the recent papers [40,45,47–49,51,84] for recent advances in the theory of singular elliptic equations.

References

[1] N.E. Alaa and M. Pierre, *Weak solutions of some quasilinear elliptic equations with data measures*, SIAM J. Math. Anal. **24** (1993), 23–35.

[2] S. Alama and G. Tarantello, *On the solvability of a semilinear elliptic equation via an associated eigenvalue problem*, Math. Z. **221** (1996), 467–493.

[3] H. Amann, *Existence and multiplicity theorems for semilinear elliptic boundary value problems*, Math. Z. **150** (1976), 567–597.

[4] R. Aris, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Clarendon Press, Oxford (1975).

[5] C. Bandle, *Asymptotic behaviour of large solutions of quasilinear elliptic problems*, Z. Angew. Math. Phys. **54** (2003), 731–738.

[6] C. Bandle and M. Essèn, *On the solutions of quasilinear elliptic problems with boundary blow-up*, Partial Differential Equations of Elliptic Type (Cortona, 1992), Sympos. Math., vol. 35, Cambridge Univ. Press, Cambridge (1994), 93–111.

[7] C. Bandle and E. Giarrusso, *Boundary blow-up for semilinear elliptic equations with nonlinear gradient terms*, Adv. Differential Equations **1** (1996), 133–150.

[8] C. Bandle and M. Marcus, *'Large' solutions of semilinear elliptic equations: Existence, uniqueness, and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9–24.

[9] C. Bandle and M. Marcus, *Dependence of blowup rate of large solutions of semilinear elliptic equations on the curvature of the boundary*, Complex Var. Theory Appl. **49** (2004), 555–570.

[10] G. Barles, G. Díaz and J.I. Díaz, *Uniqueness and continuum of foliated solutions for a quasilinear elliptic equation with a non lipschitz nonlinearity*, Comm. Partial Differential Equations **17** (1992), 1037–1050.

[11] P. Bénilan, H. Brezis and M. Crandall, *A semilinear equation in* $L^1(\mathbb{R}^N)$, Ann. Scuola Norm. Sup. Pisa **4** (1975), 523–555.

[12] L. Bieberbach, $\Delta u = \mathrm{e}^u$ *und die automorphen Funktionen*, Math. Ann. **77** (1916), 173–212.

[13] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular Variation*, Cambridge University Press, Cambridge (1987).

[14] H. Brezis and S. Kamin, *Sublinear elliptic equations in* $\mathbb{R}^N$, Manuscripta Math. **74** (1992), 87–106.

[15] H. Brezis and L. Oswald, *Remarks on sublinear elliptic equations*, Nonlinear Anal. TMA **10** (1986), 55–64.

[16] L. Caffarelli, R. Hardt and L. Simon, *Minimal surfaces with isolated singularities*, Manuscripta Math. **48** (1984) 1–18.

[17] A. Callegari and A. Nachman, *Some singular nonlinear equations arising in boundary layer theory*, J. Math. Anal. Appl. **64** (1978), 96–105.

[18] A. Callegari and A. Nachman, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math. **38** (1980), 275–281.

[19] H. Chen, *On a singular nonlinear elliptic equation*, Nonlinear Anal. TMA **29** (1997), 337–345.

[20] M. Chipot, *Elements of Nonlinear Analysis*, Birkhäuser Advanced Texts, Birkhäuser (2000).

[21] Y.S. Choi, A.C. Lazer and P.J. McKenna, *Some remarks on a singular elliptic boundary value problem*, Nonlinear Anal. TMA **3** (1998), 305–314.

[22] Y. Choquet-Bruhat and J. Leray, *Sur le problème de Dirichlet quasilinéaire d'ordre 2*, C. R. Acad. Sci. Paris, Ser. A **274** (1972), 81–85.

[23] F.-C. Cîrstea, M. Ghergu and V. Rădulescu, *Combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane–Emden–Fowler type*, J. Math. Pures Appl. **84** (2005), 493–508.

[24] F.-C. Cîrstea and V. Rădulescu, *Blow-up solutions for semilinear elliptic problems*, Nonlinear Anal. TMA **48** (2002), 541–554.

[25] F.-C. Cîrstea and V. Rădulescu, *Uniqueness of the blow-up boundary solution of logistic equations with absorption*, C. R. Acad. Sci. Paris, Ser. I **335** (2002), 447–452.

[26] F.-C. Cîrstea and V. Rădulescu, *Entire solutions blowing-up at infinity for semilinear elliptic systems*, J. Math. Pures Appl. **81** (2002), 827–846.

[27] F.-C. Cîrstea and V. Rădulescu, *Existence and uniqueness of blow-up solutions for a class of logistic equations*, Comm. Contemp. Math. **4** (2002), 559–586.

[28] F.-C. Cîrstea and V. Rădulescu, *Asymptotics for the blow-up boundary solution of the logistic equation with absorption*, C. R. Acad. Sci. Paris, Ser. I **336** (2003), 231–236.

[29] F.-C. Cîrstea and V. Rădulescu, *Solutions with boundary blow-up for a class of nonlinear elliptic problems*, Houston J. Math. **29** (2003), 821–829.

[30] F.-C. Cîrstea and V. Rădulescu, *Extremal singular solutions for degenerate logistic-type equations in anisotropic media*, C. R. Acad. Sci. Paris, Ser. I **339** (2004), 119–124.

[31] F.-C. Cîrstea and V. Rădulescu, *Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach*, Asymptotic Anal. **46** (2006), 275–298.

[32] F.-C. Cîrstea and V. Rădulescu, *Boundary blow-up in nonlinear elliptic equations of Bieberbach–Rademacher type*, Trans. Amer. Math. Soc., in press.

[33] M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations **14** (1989), 1315–1327.

[34] M.G. Crandall, P.H. Rabinowitz and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), 193–222.

[35] R. Dalmasso, *Solutions d'équations elliptiques semi-linéaires singulières*, Ann. Mat. Pura Appl. **153** (1989), 191–201.

[36] P.G. de Gennes, *Wetting: statics and dynamics*, Rev. Modern Phys. **57** (1985), 827–863.

[37] J.I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries. Vol. I. Elliptic Equations*, Research Notes in Mathematics, vol. 106, Pitman (Advanced Publishing Program), Boston, MA (1985).

[38] J.I. Díaz, J.M. Morel and L. Oswald, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations **12** (1987), 1333–1344.

[39] Y. Du and Q. Huang, *Blow-up solutions for a class of semilinear elliptic and parabolic equations*, SIAM J. Math. Anal. **31** (1999), 1–18.

[40] L. Dupaigne, M. Ghergu and V. Rădulescu, *Lane–Emden–Fowler equations with convection and singular potential*, J. Math. Pures Appl., in press.

[41] W. Fulks and J.S. Maybee, *A singular nonlinear equation*, Osaka J. Math. **12** (1960), 1–19.

[42] V. Galaktionov and J.-L. Vázquez, *The problem of blow-up in nonlinear parabolic equations*, Discrete Contin. Dynam. Systems, Ser. A **8** (2002), 399–433.

[43] J. García-Melián, R. Letelier-Albornoz and J. Sabina de Lis, *Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up*, Proc. Amer. Math. Soc. **129** (2001), 3593–3602.

[44] M. Ghergu, C. Niculescu and V. Rădulescu, *Explosive solutions of elliptic equations with absorption and nonlinear gradient term*, Proc. Indian Acad. Sci. (Math. Sci.) **112** (2002), 441–451.
[45] M. Ghergu and V. Rădulescu, *Bifurcation and asymptotics for the Lane–Emden–Fowler equation*, C. R. Acad. Sci. Paris, Ser. I **337** (2003), 259–264.
[46] M. Ghergu and V. Rădulescu, *Sublinear singular elliptic problems with two parameters*, J. Differential Equations **195** (2003), 520–536.
[47] M. Ghergu and V. Rădulescu, *Explosive solutions of semilinear elliptic systems with gradient term*, RACSAM Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **97** (2003), 437–445.
[48] M. Ghergu and V. Rădulescu, *Existence and non-existence of entire solutions to the logistic differential equation*, Abst. Appl. Anal. **17** (2003), 995–1003.
[49] M. Ghergu and V. Rădulescu, *Bifurcation and asymptotics for the Lane–Emden–Fowler equation*, C. R. Acad. Sci. Paris, Ser. I **337** (2003), 259–264.
[50] M. Ghergu and V. Rădulescu, *Nonradial blow-up solutions of sublinear elliptic equations with gradient term*, Comm. Pure Appl. Anal. **3** (2004), 465–474.
[51] M. Ghergu and V. Rădulescu, *Bifurcation for a class of singular elliptic problems with quadratic convection term*, C. R. Acad. Sci. Paris, Ser. I **338** (2004), 831–836.
[52] M. Ghergu and V. Rădulescu, *Multiparameter bifurcation and asymptotics for the singular Lane–Emden–Fowler equation with a convection term*, Proc. Roy. Soc. Edinburgh Sect. A **135** (2005), 61–84.
[53] M. Ghergu and V. Rădulescu, *On a class of sublinear singular elliptic problems with convection term*, J. Math. Anal. Appl. **311** (2005), 635–646.
[54] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second ed., Springer-Verlag, Berlin (1983).
[55] S.M. Gomes, *On a singular nonlinear elliptic problem*, SIAM J. Math. Anal. **17** (1986) 1359–1369.
[56] C. Gui and F.H. Lin, *Regularity of an elliptic problem with a singular nonlinearity*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 1021–1029.
[57] Y. Haitao, *Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem*, J. Differential Equations **189** (2003), 487–512.
[58] J. Hernández, F.J. Mancebo and J.M. Vega, *On the linearization of some singular nonlinear elliptic problems and applications*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), 777–813.
[59] J. Hernández, F.J. Mancebo and J.M. Vega, *Nonlinear singular elliptic problems: recent results and open problems*, Preprint (2005).
[60] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer-Verlag, Berlin (1983).
[61] J. Kazdan and F.W. Warner, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.
[62] J.B. Keller, *On solutions of* $\Delta u = f(u)$, Comm. Pure Appl. Math. **10** (1957), 503–510.
[63] T. Kusano and C.A. Swanson, *Entire positive solutions of singular elliptic equations*, Japan J. Math. **11** (1985), 145–155.
[64] A.V. Lair and A.W. Shaker, *Existence of entire large positive solutions of semilinear elliptic systems*, J. Differential Equations **164** (2000), 380–394.
[65] A.V. Lair and A.W. Wood, *Large solutions of semilinear elliptic equations with nonlinear gradient terms*, Internat. J. Math. Math. Sci. **22** (1999), 869–883.
[66] J.M. Lasry and P.-L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints; the model problem*, Math. Ann. **283** (1989), 583–630.
[67] A.C. Lazer and P.J. McKenna, *On a singular nonlinear elliptic boundary value problem*, Proc. Amer. Math. Soc. **3** (1991), 720–730.
[68] A.C. Lazer and P.J. McKenna, *On a problem of Bieberbach and Rademacher*, Nonlinear Anal. TMA **21** (1993), 327–335.
[69] A.C. Lazer and P.J. McKenna, *Asymptotic behaviour of solutions of boundary blowup problems*, Differential Integral Equations **7** (1994), 1001–1019.
[70] J.F. Le Gall, *A path-valued Markov process and its connections with partial differential equations*, First European Congress of Mathematics, vol. II (Paris, 1992), Progr. Math., vol. 120, Birkhäuser, Basel (1994), 185–212.
[71] J. Karamata, *Sur un mode de croissance régulière de fonctions. Théorèmes fondamentaux*, Bull. Soc. Math. France **61** (1933), 55–62.

[72] C. Loewner and L. Nirenberg, *Partial differential equations invariant under conformal or projective transformations*, Contribution to Analysis, Academic Press, New York (1974), 245–272.

[73] M. Marcus, *On solutions with blow-up at the boundary for a class of semilinear elliptic equations*, Developments in Partial Differential Equations and Applications to Mathematical Physics, G. Buttazzo et al., eds, Plenum Press, New York (1992), 65–77.

[74] M. Marcus and L. Véron, *Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), 237–274.

[75] M. Marcus and L. Véron, *Existence and uniqueness results for large solutions of general nonlinear elliptic equations*, J. Evol. Equations **3** (2003), 637–652.

[76] A. Meadows, *Stable and singular solutions of the equation* $\Delta u = 1/u$, Indiana Univ. Math. J. **53** (2004), 1681–1703.

[77] P. Mironescu and V. Rădulescu, *The study of a bifurcation problem associated to an asymptotically linear function*, Nonlinear Anal. TMA **26** (1996), 857–875.

[78] R. Osserman, *On the inequality* $\Delta u \geqslant f(u)$, Pacific J. Math. **7** (1957), 1641–1647.

[79] P. Pucci and J. Serrin, *The strong maximum principle revisited*, J. Differential Equations **196** (2004), 1–66; *Erratum*, J. Differential Equations **207** (2004), 226–227.

[80] P. Pucci and J. Serrin, *Maximum principles for elliptic partial differential equations*, Stationary Partial Differential Equations, vol. 4, M. Chipot, ed., Handbook of Differential Equations, Elsevier, Amsterdam (2007), 355-482.

[81] P. Quittner, *Blow-up for semilinear parabolic equations with a gradient term*, Math. Methods Appl. Sci. **14** (1991), 413–417.

[82] H. Rademacher, *Einige besondere Probleme der partiellen Differentialgleichungen*, Die Differential und Integralgleichungen der Mechanik und Physik I, second ed., P. Frank and R. von Mises, eds, Rosenberg, New York (1943), 838–845.

[83] A. Ratto, M. Rigoli and L. Véron, *Scalar curvature and conformal deformation of hyperbolic space*, J. Funct. Anal. **121** (1994), 15–77.

[84] V. Rădulescu, *Bifurcation and asymptotics for elliptic problems with singular nonlinearity*, Studies in Nonlinear Partial Differential Equations: In Honor of Haim Brezis, Fifth European Conference on Elliptic and Parabolic Problems: A special tribute to the work of Haim Brezis, Gaeta, Italy, May 30–June 3, 2004, C. Bandle, H. Berestycki, B. Brighi, A. Brillard, M. Chipot, J.-M. Coron, C. Sbordone, I. Shafrir, V. Valente, G. Vergara Caffarelli, eds, Birkhäuser (2005), 349–362.

[85] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics, vol. 508, Springer-Verlag, Berlin (1976).

[86] J. Shi and M. Yao, *On a singular nonlinear semilinear elliptic problem*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1389–1401.

[87] J. Shi and M. Yao, *Positive solutions for elliptic equations with singular nonlinearity*, Electronic J. Differential Equations **4** (2005), 1–11.

[88] C.A. Stuart, *Existence and approximation of solutions of nonlinear elliptic equations*, Math. Z. **147** (1976), 53–63.

[89] C.A. Stuart and H.-S. Zhou, *A variational problem related to self-trapping of an electromagnetic field*, Math. Methods Appl. Sci. **19** (1996), 1397–1407.

[90] L. Véron, *Singularities of Solutions of Second Order Quasilinear Equations*, Pitman Res. Notes Math. Ser., vol. 353, Longman, Harlow (1996).

[91] J.S.W. Wong, *On the generalized Emden–Fowler equation*, SIAM Rev. **17** (1975), 339–360.

[92] Z. Zhang, *On a Dirichlet problem with a singular nonlinearity*, J. Math. Anal. Appl. **194** (1995), 103–113.

[93] Z. Zhang, *Nonexistence of positive classical solutions of a singular nonlinear Dirichlet problem with a convection term*, Nonlinear Anal. TMA **8** (1996), 957–961.

[94] Z. Zhang and J. Yu, *On a singular nonlinear Dirichlet problem with a convection term*, SIAM J. Math. Anal. **4** (2000), 916–927.

This page intentionally left blank

Author Index

Roman numbers refer to pages on which the author (or his/her work) is mentioned. Italic numbers refer to reference pages. Numbers between brackets are the reference numbers. No distinction is made between the first author and co-author(s).

Adams, R.A. 6, *55* [1]; 213, 256, 291, *297* [1]
Aftalion, A. 457, *480* [1]
Al-Yousef, F. 176, 180, 185, *198* [1]
Alaa, N.E. 567, *590* [1]
Alama, S. 504, *590* [2]
Alberti, G. 98, *113* [1]
Alexandroff, A.D. 446, *480* [2]
Ali, M.A. 157, *201* [97]; 176, 180, 185, *198* [1]
Allegretto, W. 95, *113* [2]
Almgren, F.J. 79, *113* [3]; 381, *480* [3]
Almgren, J.A. 25, 33, *55* [2]
Alvino, A. 6, 28, 33, 36, *56* [3]; *56* [4]; *56* [5]; *56* [6]; *56* [7]; *56* [8]; *56* [9]; *56* [10]; *56* [11]
Aly, E.H. 176, 178, 180, 181, 185, 187, 188, 192, 193, *198* [2]
Amann, H. 198, *198* [3]; 567, *590* [3]
Ambrosio, L. 83, 93, 98, *113* [1]; *113* [4]
Amick, C.J. 301, 302, 304, 315, 317, 352, *352* [1]; *352* [2]
Amin, N. 185, *201* [82]
Andersson, H.I. 157, *198* [4]
Antonini, P. 478, *480* [4]
Aris, R. 488, *590* [4]
Aronsson, G. 48, *56* [12]
Ashbaugh, M. 33, *56* [13]; *56* [14]; *56* [15]; *56* [16]
Atkinson, C. 167, *198* [5]
Attouch, H. 265, 268, *297* [2]
Aziz, A.K. 305, *353* [3]

Babuška, I. 305, *353* [3]
Badiale, M. 4, *56* [17]
Baernstein II, A. 6, 15, 17, 23–25, 27, 29, 30, 32, 33, 36, *56* [18]; *56* [19]
Baiocchi, C. 297, *297* [3]
Bandle, C. 6, 19, 27, 33, *56* [20]; 424, 458, *480* [5]; *480* [6]; 487, 490, 517, 521, *590* [5]; *590* [6]; *590* [7]; *590* [8]; *590* [9]
Banks, W.H.H. 120, 126, 177, 186, 194, *198* [6]; *198* [7]
Barbu, V. 265, *297* [4]
Bardi, M. 368, *480* [7]
Barenblatt, G.I. 123, 146, 178, *198* [8]
Barles, G. 425, *480* [8]; 581, *590* [10]
Barlow, M.T. 83, *113* [5]
Bass, R.F. 83, *113* [5]
Bassom, A.P. 124, 177, 178, *200* [67]
Beckner, W. 25, 30, 31, *56* [21]
Bejan, A. 176, 178–180, 185, 187–189, *198* [9]; *198* [10]; *198* [11]
Belhachmi, Z. 126, 132, 134, 137, 150, 177, 194, *198* [12]; *198* [13]; *198* [14]
Benci, V. 422, 472, *480* [9]
Benguria, R. 33, *56* [14]; *56* [15]; *56* [16]
Bénilan, P. 429, 438, *480* [10]; 541, *590* [11]
Benlahsen, M. 137, 187, 188, 190, *198* [15]; *199* [23]
Benyamini, Y. 15, *56* [22]
Berestycki, H. 4, 36, *56* [23]; *56* [24]; *56* [25]; *56* [26]; 83, 93, *113* [6]; *113* [7]; 420, 444, 451, *480* [11]
Bernstein, S. 75–77, *113* [8]
Bhattacharyya, S. 180, 185, 187, *198* [16]
Bidaut-Véron, M.-F. 444, 470, *480* [12]; *480* [13]
Bieberbach, L. 487, *590* [12]
Bingham, N.H. 515, *590* [13]
Birindelli, I. 368, *480* [14]
Blasius, H. 121, 146, 176, 188, *198* [17]
Bluman, G.W. 178, *199* [18]; *199* [19]
Boerner, J. 178, 180, *201* [95]
Bogovskii, M.E. 305, *353* [4]
Bohme, H. 120, *199* [20]

Bombieri, E. 79, 109, 112, *113* [9]; *113* [10]
Borell, C. 36, *56* [27]
Brascamp, H.J. 25, *56* [28]
Brezis, H. 5, *56* [29]; 72, 73, *113* [11]; 168, 169, *199* [21]; 222, *297* [5]; 429, 438, 444, 451, *480* [10]; *480* [15]; 507, 541, 582, *590* [11]; *590* [14]; *591* [15]
Brighi, B. 126, 132, 134, 137, 150, 176, 177, 180, 187, 188, 190, 194, *198* [12]; *198* [13]; *198* [14]; *199* [22]; *199* [23]; *199* [24]; *199* [25]
Brock, F. 4, 6, 11, 15, 16, 18, 19, 24, 29, 32, 36, 37, 39, 42, 43, 51–54, *56* [30]; *57* [31]; *57* [32]; *57* [33]; *57* [34]; *57* [35]; *57* [36]; *57* [37]
Brothers, J. 3, 27, *57* [38]
Browder, F. 209, 223, 225, *297* [6]; *297* [7]
Brown, S.N. 126, *200* [58]
Burago, Yu.D. 6, 19, *57* [39]
Burchard, A. 25, 33, *57* [40]; *57* [41]
Busca, J. 4, *57* [42]; 457, *480* [1]
Büyük, E. 144, 145, 179, 189, *199* [35]

Cabré, X. 83, 93, 98, *113* [1]; *113* [4]; 359, *481* [16]
Caffarelli, L.A. 93, *113* [6]; 359, *481* [16]; 488, *591* [16]
Calabi, E. 83, *113* [12]
Callegari, A.J. 146, 148, 153–155, *199* [26]; *201* [80]; 488, *591* [17]; *591* [18]
Capelo, A. 297, *297* [3]
Casten, R. 38, *57* [43]
Castro, A. 451, *481* [17]
Cattabriga, L. 322, 325, *353* [5]
Cauchy, A. 63, *113* [13]; *113* [14]
Cellina, A. 438, *481* [18]
Chakrabarti, A. 157, *199* [27]
Chaudhary, M.A. 126, 177, *199* [28]
Chavel, I. 6, 19, 32, *57* [44]
Chen, H. 554, *591* [19]
Chen, T.S. 185, *201* [78]
Cheng, P. 126, 179, 194, *199* [29]; *199* [30]
Cheng, S.Y. 112, *113* [15]
Chern, S.-S. 110, 112, *113* [16]
Chipot, M. 488, *591* [20]
Choi, Y.S. 559, *591* [21]
Chong, K.M. 6, 19, *57* [45]
Choquet-Bruhat, Y. 564, *591* [22]
Cîrstea, F.-C. 490, 492, 506, 508, 515, 528, 533, 542, *591* [23]; *591* [24]; *591* [25]; *591* [26]; *591* [27]; *591* [28]; *591* [29]; *591* [30]; *591* [31]; *591* [32]
Clarkson, P.A. 124, 177, 178, *199* [31]; *200* [67]
Coclite, M. 488, 489, 564, 565, *591* [33]
Coffman, C.V. 154, *199* [32]
Colding, T.H. 103, 104, *114* [17]
Cole, J.D. 178, *199* [18]
Conley, C.H. 378, *481* [19]
Coppel, W.A. 146, 190, *199* [33]
Coron, J.-M. 33, *57* [46]
Cortázar, C. 429, 438, *481* [20]
Coti-Zelati, V. 38, *57* [47]
Courant, R. 362, 363, 368, *481* [21]
Crandall, M. 429, 438, *480* [10]; 541, *590* [11]
Crandall, M.G. 488, 554, 564, *591* [34]
Crowe, J.A. 11, *57* [48]
Cuesta, M. 376, *481* [22]

Da Lio, F. 368, *480* [7]
Dalmasso, R. 541, *591* [35]
Damascelli, L. 4, *57* [49]; *57* [50]; *57* [51]; *57* [52]; *57* [53]; *57* [54]; 451, *481* [23]; *481* [24]
Dancer, E.N. 451, *481* [25]
Danielli, D. 90, 98, *114* [18]
Danilov, V.G. 196, *199* [34]
de Gennes, P.G. 541, *591* [36]
De Giorgi, E. 79, 83, 100, 109, 112, *113* [9]; *113* [10]; *114* [19]; *114* [20]; *114* [21]
De Witt, K.J. 181, 186, 194, *200* [55]
Deimling, K. 223, *297* [8]
Demengel, F. 368, *480* [14]
Díaz, G. 425, *480* [8]; 581, *590* [10]
Díaz, J.I. 3, *57* [55]; 424, 425, 429, 432, 438, 458, 470, *480* [8]; *481* [26]; *481* [27]; *481* [28]; *481* [29]; *481* [30]; 488, 560, 581, *590* [10]; *591* [37]; *591* [38]
Dolbeault, J. 451, *481* [31]
Draghici, C. 25, *57* [56]
Du, Y. 487, 508, 510, *591* [39]
Dubinin, V.N. 6, 15, 18, 27, *57* [57]; *57* [58]; *57* [59]
Dupaigne, L. 590, *591* [40]

Ece, M.C. 144, 145, 179, 189, *199* [35]
Elgueta, M. 429, 438, *481* [20]
Elliott, L. 176, 178, 180, 181, 185, 187, 188, 192, 193, *198* [2]
Eraslan, A.H. 187, *200* [62]
Ericson, L.E. 127, *199* [36]
Essèn, M. 487, *590* [6]
Estabrook, F.B. 178, *200* [52]
Esteban, M. 38, *57* [47]; *58* [60]; *58* [61]
Evans, L.C. 64, *114* [22]; *481* [32]

Faber, G. 32, *58* [62]
Falkner, V.M. 121, 176, *199* [37]
Fan, K. 225, *297* [9]
Fan, L.T. 127, *199* [36]

Farina, A. 83–85, 90, *114* [23]; *114* [24]; *114* [25]; *114* [26]
Felmer, P. 429, 438, 451, *481* [20]; *481* [31]; *481* [33]
Ferone, V. 33, 36, *56* [3]; *56* [4]
Finn, R. 85, 109, 110, *114* [27]; *114* [28]; *114* [29]; *114* [30]; *114* [31]
Fischer-Colbrie, D. 95, *114* [32]
Fisher, A.A. 195, *199* [38]
Fleming, W.H. 419, *481* [34]
Fortunato, D. 422, 472, *480* [9]
Fox, V.G. 127, *199* [36]
Fraenkel, L.E. 359, 362, 364, 390, 444, 451, 456, *481* [35]
Franchi, B. 439, 469, *481* [36]
Friedman, A. 3, 52, *58* [63]
Fujita, H. 303, 304, 309, 310, 312, 314, 318, 343, 345, *353* [6]; *353* [7]; *353* [13]; *353* [14]; *353* [15]; *353* [16]
Fulks, W. 488, *591* [41]

Galaktionov, V. 490, *591* [42]
Galdi, G.P. 305, 310, 311, 329, 332, *353* [8]
García-Melián, J. 487, 517, 521, 523, *591* [43]
Garofalo, N. 90, 98, *114* [18]
Gazzola, F. 97, 98, *114* [33]
Gersten, K. 120, 128, 132, 145, 146, 178, 182, 186, *201* [93]
Ghergu, M. 490, 493, 495, 542, 553, 565, 590, *591* [23]; *591* [40]; *592* [44]; *592* [45]; *592* [46]; *592* [47]; *592* [48]; *592* [49]; *592* [50]; *592* [51]; *592* [52]; *592* [53]
Ghoussoub, N. 83, *114* [34]
Giarrusso, E. 490, *590* [7]
Gidas, B. 4, 40, *58* [64]; *58* [65]; 444, *481* [37]; *481* [38]
Gilbarg, D. 64, 81, 85, 94, 100, 102, 104, 109, *114* [35]; *114* [36]; 358, 359, 361–364, 367, 368, 375, 376, 384, 390, 392–394, 400, 401, 403, 405, 414, 418–421, 429, 462, 474, *481* [39]; *481* [40]; 491, 503, 543, 544, 548, 561, 580, 586, 589, *592* [54]
Gilding, B.H. 166–168, 172, 174, 196, *199* [39]; *199* [40]; *199* [41]
Giusti, E. 32, 34, *58* [66]; 77, 79, 112, *113* [9]; *114* [37]
Glicksberg, I.L. 225, *298* [10]
Goldie, C.M. 515, *590* [13]
Goldstein, S. 178, *199* [42]
Gomes, S.M. 554, *592* [55]
Görtler, H. 181, 186, *199* [43]
Granlund, S. 380, *481* [41]
Grossi, M. 4, *58* [67]
Guedda, M. 121, 126, 132, 136, 137, 166–168, 171, 176–178, 180, 185, 187, 188, 190, 191, 194, *198* [15]; *199* [23]; *199* [44]; *199* [45]; *200* [46]; *200* [47]; *200* [48]; *200* [49]; *200* [50]
Gui, C. 83, *113* [5]; *114* [34]; 546, 548, 578, 580, *592* [56]
Gupta, A.S. 157, *199* [27]; *201* [97]

Hadamard, J. 66, *114* [38]
Hady, F.M. 181, *200* [51]
Haitao, Y. 488, 566, *592* [57]
Hamel, F. 83, *113* [7]
Hammouch, Z. 166–168, 171, 177, 194, *200* [47]; *200* [48]; *200* [49]
Hardt, R. 488, *591* [16]
Hardy, G. 6, 13, 15, 19, 25, *58* [68]
Harrison, B.K. 178, *200* [52]
Hartmann, P. 150, 153, 188, *200* [53]
Hassanien, I.A. 181, 185, *200* [54]; *200* [57]
Hayman, W. 6, 19, 30, *58* [69]
Heinonen, J. 107, *114* [39]
Henrot, A. 33, *58* [70]
Hernández, J. 488, *592* [58]; *592* [59]
Hernández-Bermejo, B. 196, *201* [92]
Herrero, M.A. 429, 432, *481* [28]
Hess, P. 209, 223, *297* [7]
Hilbert, D. 100, *114* [40]; 362, 363, 368, *481* [21]
Hoernel, D. 176, 180, 188, *199* [24]
Holland, C. 38, *57* [43]
Hopf, E. 77, 78, *114* [41]; *114* [42]; 362–366, 388, 390, *481* [42]; *481* [43]
Horgan, C.O. 329, *353* [9]
Hörmander, L. 37, *58* [71]; 549, *592* [60]
Howell, T.G. 181, 186, 194, *200* [55]
Huang, Q. 487, 508, 510, *591* [39]
Hussaini, M.Y. 188, *200* [56]

Ibrahim, F.S. 181, *200* [57]
Ingham, D.B. 126, 176, 178, 180, 181, 185, 187, 188, 192, 193, *198* [2]; *200* [58]

Jeng, D.R. 181, 186, 194, *200* [55]
Jerison, D. 83, *115* [43]
Jörgens, K. 81, *115* [44]

Kakutani, S. 225, *298* [11]
Kamin, S. 507, 582, *590* [14]
Kano, R. 297, *298* [20]
Kaplun, S. 177, *200* [59]
Kapur, J.N. 146, *200* [60]
Karamata, J. 515, *592* [71]
Kato, T. 73, *115* [45]
Kawohl, B. 6, 11, 19, 27, 29, 33, 36, 53, *58* [72]; *58* [73]; *58* [74]; 429, *482* [44]

Kazdan, J. 564, *592* [61]
Keller, H.B. 126, 164–167, 172, 177, 178, 181, 196, *200* [68]; *200* [69]; *200* [70]
Keller, J.B. 69, *115* [46]; 438, *482* [45]; 487, 504, *592* [62]
Kenmochi, N. 214, 223, 297, *298* [12]; *298* [13]; *298* [20]
Kersner, R. 137, 165–168, 171, 172, 174, 177, 188, 196, *198* [15]; *199* [39]; *199* [40]; *199* [41]; *200* [49]; *200* [61]
Kesavan, S. 4, *58* [67]; *58* [75]
Kilpeläinen, T. 107, *114* [39]
Kim, H.H. 187, *200* [62]
Kolmogorov, A. 195, 196, *200* [63]
Kolyada, V.I. 6, *58* [76]
Korevaar, N. 109, *115* [47]
Krahn, E. 32, *58* [77]
Krylov, N.V. 99, 100, *115* [48]; *115* [49]; 359, *482* [46]; *482* [47]
Küçükbursa, A. 178, *201* [85]
Kumari, M. 157, 176, 181, *200* [64]; *200* [65]
Kumei, S. 178, *199* [19]
Kurscal, M.D. 178, *199* [31]
Kusano, T. 541, *592* [63]
Kutev, N. 429, *482* [44]
Kuzin, I. 3, *58* [78]

Lachand-Robert, T. 36, *56* [23]
Ladyzhenskaya, O.A. 87, *115* [50]; 304, 310, 314, *353* [10]; *482* [48]
Lair, A.V. 490, 528, *592* [64]; *592* [65]
Lakin, W.D. 188, *200* [56]
Lanconelli, E. 439, 469, *481* [36]
Lasry, J.M. 490, *592* [66]
Laursen, T.A. 176, 178–180, 185, 187–189, *198* [10]
Lawson, H.B., Jr. 80, *115* [51]
Lazer, A.C. 487, 554, 559, 564, *591* [21]; *592* [67]; *592* [68]; *592* [69]
Le Gall, J.F. 487, *592* [70]
Ledoux, M. 36, *58* [79]
Leray, J. 564, *591* [22]
Letelier-Albornoz, R. 487, 517, 521, 523, *591* [43]
Li, C.-M. 4, *58* [80]; *58* [81]
Li, P. 103, 104, *115* [52]; *115* [53]
Li, Y. 4, 40, *58* [82]; *58* [83]
Libby, P.A. 192, *200* [66]
Lieb, E.H. 6, 25, 33, *55* [2]; *56* [28]; *58* [84]; *58* [85]
Lieberman, G.M. 33, *58* [86]; 477, *482* [49]
Lin, F.H. 546, 548, 578, 580, *592* [56]
Lions, J.L. 254, 283, *298* [14]
Lions, P.-L. 3, 4, 6, 28, 33, 36, *56* [3]; *56* [5]; *56* [6]; *56* [7]; *58* [87]; *59* [88]; *59* [89]; 490, *592* [66]
Liouville, J. 63, *115* [54]; *115* [55]
Littlewood, J. 6, 13, 15, 19, 25, *58* [68]
Littman, W. 390, *482* [50]
Liu, T. 192, *200* [66]
Loewner, C. 487, 517, *593* [72]
Lopes, O. 41, *59* [90]; *59* [91]
Loss, M. 6, *58* [85]
Ludlow, D.K. 124, 177, 178, *200* [67]
Luttinger, J.M. 25, *56* [28]

Magenes, E. 254, 283, *298* [14]
Magyari, E. 126, 164–167, 172, 177, 178, 181, 196, *200* [68]; *200* [69]; *200* [70]
Malek, J. 178, *200* [71]
Mancebo, F.J. 488, *592* [58]; *592* [59]
Mangler, W. 178, *200* [72]
Marcus, M. 487, 488, 493, 506, 517, 521, *590* [8]; *590* [9]; *593* [73]; *593* [74]; *593* [75]
Martio, O. 107, *114* [39]
Maslov, V.P. 196, *199* [34]
Massoudi, M. 178, 179, 181, 184, 187, 188, *201* [73]
Maybee, J.S. 488, *591* [41]
Maz'ya, V.G. 421, *482* [51]
McKenna, P.J. 487, 554, 559, 564, *591* [21]; *592* [67]; *592* [68]; *592* [69]
McNabb, A. 384, *482* [52]
Meadows, A. 488, *593* [76]
Meier, M. 107, *115* [56]
Merkin, J.H. 126, 176, 177, 179–181, 188, *199* [28]; *201* [74]; *201* [75]; *201* [76]
Meyers, N.G. 103, *115* [57]; 370, *482* [53]
Mickle, E.J. 78, *115* [58]
Minicozzi II, W.P. 103, 104, *114* [17]
Minkowycz, W.J. 126, 194, *199* [30]
Miranda, M. 109, *113* [10]
Mironescu, P. 548, 549, *593* [77]
Mitidieri, E. 444, *482* [54]
Moffatt, H.K. 126, *201* [77]
Monneau, R. 83, *113* [7]; *115* [43]; 451, *481* [31]
Monticelli, D.D. 390, *482* [55]
Morel, J.M. 488, 560, *591* [38]
Morimoto, H. 303, 304, 309, 310, 314, 315, 329, 345, 346, *353* [11]; *353* [12]; *353* [13]; *353* [14]; *353* [15]; *353* [16]
Morpurgo, C. 25, *59* [92]
Mosco, U. 268, *298* [15]
Moser, J. 99, 100, 104, 108, *115* [59]; 421, *482* [56]; *482* [57]
Moss, W.F. 95, *115* [60]

Mossino, J. 6, 27, 33, *59* [93]; 478, *482* [58]
Mucoglu, A. 185, *201* [78]
Mugnai, M. 478, *480* [4]
Müntz, Ch.H. 77, *115* [61]
Murase, Y. 297, *298* [20]

Na, T.Y. 157, 178, 181, 187, *201* [79]; *201* [87]; *201* [88]; *201* [94]
Nabana, E. 4, *56* [17]
Nachman, A. 146, 148, 153–155, *199* [26]; *201* [80]; *201* [81]; 488, *591* [17]; *591* [18]
Nadirashvili, N.S. 33, *59* [94]
Nath, G. 157, 176, 181, *200* [64]; *200* [65]
Nazar, R. 185, *201* [82]
Nečas, J. 213, 247, 291, *298* [16]
Nelson, E. 64, *115* [62]
Ni, W.-M. 4, 40, *58* [64]; *58* [65]; *58* [82]; *58* [83]; 75, *115* [63]; 444, *481* [37]
Niculescu, C. 490, 493, *592* [44]
Nield, D.A. 176, 179, *198* [11]
Nirenberg, L. 4, 40, *56* [24]; *56* [25]; *56* [26]; *58* [64]; *58* [65]; 93, *113* [6]; 420, 444, 451, *480* [11]; *481* [37]; 487, 517, *593* [72]
Nitsche, J.C.C. 81, *115* [64]; 443, *482* [59]

Oleinik, O.A. 390, *482* [60]
Olver, P.J. 178, *201* [83]
Osserman, R. 69, 73, 80, 81, *115* [51]; *115* [65]; *115* [66]; *115* [67]; *115* [68]; 438, *482* [61]; 487, 504, *593* [78]
Oswald, L. 488, 507, 560, *591* [15]; *591* [38]
Ovsiannikov, L.V. 178, *201* [84]

Pacella, F. 4, *57* [50]; *57* [51]; *57* [52]; *57* [53]; *58* [67]; *58* [75]; 451, *481* [23]
Pakdemirli, M. 178, *201* [85]
Pal, A. 180, 185, 187, *198* [16]
Palmieri, G. 488, 489, 564, 565, *591* [33]
Pavlov, K.B. 157, *201* [86]
Payne, L.E. 33, *59* [95]
Peletier, L.A. 168, 169, *199* [21]; 443, *482* [62]
Peponas, S. 187, 188, 190, *199* [23]
Perdikis, C. 157, 187, *201* [89]; *201* [98]
Petrovskii, I. 195, 196, *200* [63]
Piepenbrink, J. 95, *115* [60]
Pierre, M. 567, *590* [1]
Pigola, S. 478, *482* [63]
Pisani, L. 422, 472, *480* [9]
Piskunov, I. 195, 196, *200* [63]
Pogorelov, A.V. 83, *115* [69]
Pohozaev, S.I. 3, *58* [78]; 444, *480* [13]; *482* [54]
Polacik, P. 444, *482* [64]
Polya, G. 6, 13, 15, 19, 25, 27, *58* [68]; *59* [96]
Pop, I. 126, 157, 164–167, 172, 176–181, 185, 187, 196, *198* [16]; *199* [28]; *200* [69]; *200* [70]; *201* [75]; *201* [76]; *201* [82]; *201* [87]; *201* [88]; *202* [101]
Porretta, A. 451, *482* [65]
Protter, M.H. 65, *115* [70]; 359, 362, 366, 368, 371, 474, *482* [66]
Pucci, P. 42, 48, *59* [97]; *59* [98]; 378, 403, 405, 414, 419, 420, 424–426, 438, 439, 443, 451, 458, 470, 473, 477–480, *480* [4]; *481* [19]; *482* [67]; *482* [68]; *482* [69]; *482* [70]; *483* [71]; *483* [72]; *483* [73]; *483* [74]; 489, *593* [79]; *593* [80]

Quaas, A. 438, *481* [33]
Quack, H. 178, 180, *201* [95]
Quittner, P. 444, *482* [64]; 490, *593* [81]

Rabier, P. 302, *353* [17]; *353* [18]
Rabinowitz, P.H. 488, 554, 564, *591* [34]
Rademacher, H. 487, *593* [82]
Rado, T. 77, *115* [71]
Rădulescu, V. 490, 492, 493, 495, 506, 508, 515, 528, 533, 542, 548, 549, 553, 565, 590, *591* [23]; *591* [24]; *591* [25]; *591* [26]; *591* [27]; *591* [28]; *591* [29]; *591* [30]; *591* [31]; *591* [32]; *591* [40]; *592* [44]; *592* [45]; *592* [46]; *592* [47]; *592* [48]; *592* [49]; *592* [50]; *592* [51]; *592* [52]; *592* [53]; *593* [77]; *593* [84]
Rajagopal, K.R. 178, *200* [71]
Ramaswamy, M. 4, *57* [52]; *57* [53]; *58* [67]
Raptis, A. 157, 187, *201* [89]; *201* [98]
Ratto, A. 488, *593* [83]
Redheffer, R.M. 68, *115* [72]; 429, 431, 432, *483* [75]; *483* [76]
Reichel, W. 457, *483* [77]
Rellich, F. 368, *483* [78]
Reuter, G.E.H. 167, *198* [5]
Rice, N.M. 6, 19, *57* [45]
Ridler-Rowe, C.J. 167, *198* [5]
Riesz, F. 25, *59* [99]
Rigoli, M. 478, 480, *482* [63]; *482* [67]; 488, *593* [83]
Rishel, R. 419, *481* [34]
Roberts, A.W. 248, 273, *298* [17]
Rosenbloom, P.C. 11, *57* [48]
Rosenhead, L. 164, 178, *201* [90]
Růžička, M. 178, *200* [71]

Saa, J.E. 429, *481* [29]
Sabina de Lis, J. 487, 517, 521, 523, *591* [43]
Safonov, M.V. 99, 100, *115* [49]; *483* [79]
Sakiadis, B.C. 127, *201* [91]

Saks, S. 12, *59* [100]
Saloff-Coste, L. 104, *115* [73]
Sánchez-Valdés, A. 196, *201* [92]
Sari, T. 126, *199* [25]
Sarvas, J. 6, 29, 36, *59* [101]
Sattinger, D.H. 74, *116* [74]
Savin, V.O. 83, *116* [75]; 98, *116* [85]
Schlichting, H. 120, 128, 132, 145, 146, 178, 182, 186, *201* [93]
Schmuckenschläger, M. 25, *57* [41]
Schoen, R. 95, *114* [32]
Sciunzi, B. 4, *57* [54]; 98, *116* [85]; 451, *481* [24]; *482* [68]
Seneta, E. 515, *593* [85]
Serrin, J. 4, 42, 48, *59* [97]; *59* [98]; *59* [102]; *59* [103]; 68, 85, 100, 103–105, *114* [35]; *116* [76]; *116* [77]; *116* [78]; *116* [79]; 362, 378, 380, 381, 390, 391, 403, 405, 414, 419–421, 424–426, 438–440, 443, 444, 446, 451, 454–458, 469, 470, 473, 477–480, *481* [19]; *481* [36]; *482* [62]; *482* [67]; *482* [68]; *482* [69]; *482* [70]; *483* [71]; *483* [72]; *483* [73]; *483* [74]; *483* [80]; *483* [81]; *483* [82]; *483* [83]; *483* [84]; *483* [85]; *483* [86]; *483* [87]; 489, *593* [79]; *593* [80]
Seshadri, R. 178, *201* [94]
Setti, A.G. 478, *482* [63]
Shaker, A.W. 528, *592* [64]
Shi, J. 488, 489, 542, 543, 565, 582, *593* [86]; *593* [87]
Shivaji, R. 451, *481* [17]
Siegel, D. 469, *483* [88]
Simon, J. 443, *483* [89]
Simon, L. 79, *116* [80]; 488, *591* [16]
Simons, J. 79, *116* [81]
Sirakov, B. 4, *57* [42]
Skan, S.W. 121, 176, *199* [37]
Smets, D. 30, *59* [104]
Solynin, A.Yu. 6, 15, 16, 18, 19, 24, 29, 32, 36, 53, *57* [37]; *59* [105]
Souplet, P. 444, *482* [64]
Sparrow, E.M. 178, 180, *201* [95]
Sperb, R.P. 359, 424, 443, 470, *480* [6]; *483* [90]; *483* [91]
Spruck, J. 444, *481* [38]
Srivastava, R.C. 146, *200* [60]
Stakgold, I. 424, 458, *480* [5]; *480* [6]
Stampacchia, G. 421, *483* [92]
Steiner, J. 6, 19, *59* [106]
Stewartson, K. 190, 192, *201* [96]
Strauss, W. 38, *58* [61]
Struwe, M. 3, 33, *59* [107]
Stuart, C.A. 488, *593* [88]; *593* [89]
Swanson, C.A. 541, *592* [63]
Sweers, G. 38, *59* [108]
Szegö, G. 6, 19, 27, *59* [96]

Takáč, P. 376, *481* [22]
Takhar, H.S. 157, 176, 181, 187, *200* [64]; *200* [65]; *201* [89]; *201* [97]; *201* [98]
Talenti, G. 6, 19, 27, 31–34, 36, 48, *56* [12]; *59* [109]; *59* [110]; *59* [111]; *59* [112]; *59* [113]; *59* [114]; 419, *483* [93]
Taliafero, S. 146, *201* [81]
Taous, K. 126, 132, 134, 137, 150, 177, 194, *198* [12]; *198* [13]; *198* [14]
Tarantello, G. 504, *590* [2]
Tartar, L. 488, 554, 564, *591* [34]
Taylor, B.A. 15, 17, 23–25, *56* [19]
Temam, R. 304, 310, *353* [19]
Terman, D. 168, 169, *199* [21]
Teugels, J.L. 515, *590* [13]
Thiel, U. 429, *481* [29]
Tkachev, V.G. 112, *116* [82]
Tolksdorf, P. 87, 90, *116* [83]; 376, *483* [94]
Trombetti, G. 6, 28, 33, 36, *56* [3]; *56* [4]; *56* [5]; *56* [6]; *56* [7]; *56* [8]; *56* [9]; *56* [10]; *56* [11]; *59* [115]
Trudinger, N.S. 64, 81, 94, 100, 102, 104, 105, 109, *114* [36]; *116* [84]; 358, 359, 361–364, 367, 368, 375, 376, 390, 392–394, 400, 401, 403, 405, 414, 418–421, 429, 462, 474, *481* [40]; 380, 385, 477, *483* [95]; *483* [96]; 491, 503, 543, 544, 548, 561, 580, 586, 589, *592* [54]

Ural'tseva, N.N. 87, *115* [50]; *482* [48]
Ushiyama, K. 196, *201* [99]

Vajravelu, K. 157, *202* [100]
Valdinoci, E. 98, *116* [85]
van Schaftingen, J. 11, 19, 24, 36, 51, *59* [116]; *59* [117]; *60* [118]; *60* [119]; *60* [120]
Varberg, D.E. 248, 273, *298* [17]
Vargas, J.V.C. 176, 178–180, 185, 187–189, *198* [10]
Vázquez, J.L. 48, *60* [121]; 424, 429, 431, 432, 438, *483* [97]; 490, *591* [42]
Vega, J.M. 488, *592* [58]; *592* [59]
Véron, L. 43, *60* [122]; 121, 132, *200* [50]; 451; 470, *481* [30]; *482* [65]; 488, 490, 506, 517, *593* [74]; *593* [75]; *593* [83]; *593* [90]
Vicsek, M. 165, *200* [61]
Volosov, K.A. 196, *199* [34]

Walter, W. 359, 368, *483* [98]
Warner, F.W. 564, *592* [61]

Watanabe, T. 157, *202* [101]
Weinberger, H.F. 65, 100, 103, *115* [70]; *116* [79]; 359, 362, 366, 368, 371, 474, *482* [66]
Weyl, H. 146, 153, *202* [102]
Wheeler, L.T. 329, *353* [9]
Widman, K. 386, *483* [99]
Willem, M. 11, 19, 30, *59* [104]; *60* [120]
Willmore, T.J. 478, 479, *483* [100]
Wolontis, V. 15, *60* [123]
Wong, J.S.W. 154, *202* [103]; 541, 552, *593* [91]
Wood, A.W. 490, *592* [65]
Wooding, R.A. 144, 145, 179, *202* [104]

Yao, M. 488, 489, 542, 543, 565, 582, *593* [86]; *593* [87]
Yau, S.T. 112, *113* [15]
Yu, J. 566, 567, *593* [94]
Yürüsoy, M. 178, *201* [85]

Zalgaller, V.A. 6, 19, *57* [39]
Zaturska, M.B. 120, 126, 177, 186, *198* [7]
Zeidler, E. 38, 43, *60* [124]; 213, 223, 291, *298* [18]; *298* [19]
Zhang, X.X. 146, 148, 152, *202* [105]
Zhang, Z. 559, 566, 567, 585, *593* [92]; *593* [93]; *593* [94]
Zheng, L.C. 146, 148, 152, *202* [105]
Zhou, H.-S. 488, *593* [89]
Ziemer, W.P. 3, 27, *57* [38]
Zou, H. 4, 42, 48, *59* [98]; *59* [103]; 425, 438–440, 444, 451, *483* [74]; *483* [86]; *483* [87]
Zweibel, J.A. 11, *57* [48]

This page intentionally left blank

Subject Index

A
a-priori estimates for elliptic equations, 33
aiding flow, 176, 179, 187
aiding/opposing parameter, 187
Allen–Cahn equation, 84, 95
ambient temperature, 179
analytical and numerical solutions, 148
analyticity of solution, 153
antisymmetric (vector filed), 307
apparent viscosity, 118
approximate
– non-similarity solution, 178
– numerical solution, 178
approximated solution, 171
asymptotic
– analysis, 486
– behavior, 121, 127, 128, 170, 515, 580
– solution, 193
autonomous third order non-linear differential equations, 125

B
Bernstein
– problem, 63, 79
– theorem, 75, 79, 80, 83, 109, 112
best constants in embedding theorems, 31
bifurcation, 485, 486, 489, 552, 566
– phenomena, 132
– problem, 541
Blasius
– method, 119
– equation, 137, 150, 153
blow-up, 121, 131, 163
– boundary solutions, 485, 489
– condition, 163
– profile, 162
– solution, 157
bootstrap, 559, 586, 589
boundary
– approximations, 158
– blow-up, 486
– condition, 120, 126
– layer, 118
– – approximation, 120, 145, 179
– – concept, 146
– – equation, 122, 132, 146, 176
– – flow, 126
– – system, 157
– – theory, 118, 176
– – thickness, 146, 147, 161
– obstacle problem, 280
– point lemma, 383, 435
– value problem, 125
bounded entire radial solution, 530
Boussinesq approximation, 145
Brouwer's fixed-point theorem, 223
buoyancy force, 179

C
Caccioppoli inequality, 92
cap symmetrization, 20
capillarity, 452
– equation, 110
Cauchy integral formula, 63
Cauchy–Riemann equation, 64, 80, 82
characteristic length, 145
classical, 127, 129, 130
– Blasius problem, 146
– solution, 121, 128
coefficient of thermal expansion, 179
compact
– support, 151
– – principle, 422
compactly supported, 147, 161
comparison principle, 365, 370, 376, 402, 404, 406, 408
concave solution, 142, 153
conformal
– Riemannian metric, 74
– type, 63, 73
constant speed, 144
continuity equation, 120
continuous stretching surface, 125
convection term, 485, 564, 565

convex, 217
coordinates system, 177
Crocco
– transformation, 148
– variables, 154, 166
– – approach, 153
Crocco-like transformation, 154

D
Darcy–Boussinesq, 179
Darcy-modified local Rayleigh, 180
Darcy's law, 144, 146
De Giorgi conjecture, 63, 83–85, 90, 93, 97, 98
dead core, 458, 467, 488
– lemma, 426
– with bursts, 469
deceleration, 132
decomposition technique, 176
degenerate, 121, 127, 148
– boundary layer equations, 117
demiclosed, 206
demicontinuous, 206
density, 145
dilatant, 119
dimensional analysis, 178
dimensionless
– characteristic number, 161
– non-linear PDE, 177
– stream function, 122
– temperature, 152, 159
– variable, 145
Dirichlet boundary operator, 507
distribution solutions, 391
divergence structure inequalities, 376, 380
divergent channel, 164
domain, 206, 361
– of attraction, 198
doubling property, 104
drag, 128
duality mapping, 215
dynamic scaling, 165

E
effective thermal diffusivity, 179
eigensolution, 187–189
eigenvalue, 399
– problem, 160
electric conductivity, 158
electrically conducting, 157
elliptic
– equation, 366, 368
– operator, 93
– regularity, 491, 505, 507, 532
– solution, 366
Emden–Fowler, 154
– equation, 552
energy, 133, 144
– function, 191
– method, 197
entire
– solution, 527
– – large, 490, 497, 499, 528, 533, 540
– – radial, 528, 533–535
equilibrium point, 196
Euler–Lagrange equation, 360, 405
exact
– differential, 120
– or explicit solution, 121
– or similarity solution, 143
– solution, 178, 180, 181
existence, 125, 127, 132, 148, 154, 157, 160, 174, 188
– of solution, 153
explicit
– construction, 160
– formula, 146
exponent, 144
exponential type, 183
exterior
– cone condition, 504
– Dirichlet problem, 426
external
– flow, 178
– velocity, 121, 122, 178
extremal flow, 178

F
Falkner–Skan (FS)
– method, 119, 121
– equation, 122, 190
– wedge flow, 164
fast and the slow orbits, 169
fast orbit, 168, 169
finite
– propagation, 146, 147
– speed of propagation, 121
first level of truncation, 184, 185
– local similarity technique, 178
flat plate, 157, 159
fluid
– mechanical parameter, 132
– temperature, 179
fluid-saturated porous medium, 126, 179
flux condition, 174
foam, 118
forced convection, 181

free
– boundary, 146
– convection, 126, 181
– – along a vertical, 194
– stream, 188
– – velocity, 120, 176, 179, 194
fully nonlinear equation, 364
fundamental solution, 146

G
Gauss curvature, 73
Gaussian curvature, 74, 487
Gelfand transformation, 490
general
– ordinary differential equation, 165
– outflow condition, 302
generalized
– Blasius, 188
– – equation, 125, 147
– – problem, 146
– F–KPP (Fischer–Kolmogorov Petrovskii Piskunov), 195
– Green's formula, 214
– Nusselt number, 161
global
– behavior of solution, 137
– existence, 144
– minimizer, 3
– positive solution, 142
– solution, 137, 139
– structure of solution, 160
– unbounded solution, 139
Görtler transformation, 181, 186
gradient obstacle problem, 280
graph, 206
gravitational acceleration, 145, 179
Green's function, 547, 580
Gronwall's inequality, 501, 531

H
Hadamard three-circles theorem, 65, 66
harmonic function, 63–65, 67, 68, 81, 98, 99
Harnack inequality, 99–102, 104, 106, 380
– in $\mathbb{R}^2$, 470
heat
– flux normal, 144
– transfer, 144, 152, 157, 161
Hilbert's nineteenth problem, 100
Hölder
– continuous, 174
– inequality, 550
homoclinic orbit, 197
homogeneous elliptic inequalities, 393

Hopf
– boundary point lemma, 387
– maximum principle, 386, 556, 573, 587
– strong maximum principle, 562
hydrodynamical problem, 132

I
impermeable vertical flat plate, 126
inclined angle ϕ, 179
incompressible flow, 176
independent mathematical object, 155
index of regular variation, 515
inequalities for symmetrizations, 24
infinite
– number
– – of global unbounded solution, 134
– – of solution, 126, 176
infinitesimal transformation, 178
initial
– boundary condition, 132
– value problem, 155
injection, 126, 127, 157, 165
– velocity, 165
inner product, 361
integrability condition, 163
integral equation, 168, 172
interior obstacle problem, 279
invariance property, 122, 127, 165, 177
invariant or similarity solution, 122
inverse, 206
inverse-linear temperature, 164
isoperimetric
– inequalities for eigenvalues, 32
– inequality
– – in $\mathbb{R}^N$, 32
– – on the sphere, 32

J
Jensen's inequality, 72–74
Jörgens theorem, 83

K
Karamata regular variation theory, 486, 515, 521
Kato's inequality, 73
Keller–Osserman condition, 69–71, 495, 504, 505, 533
kinematic viscosity, 146, 179

L
laminar
– boundary layer, 144
– flow, 146, 176
– incompressible, 159

– mixed convection, 181
– non-Newtonian fluid, 119, 120
Lane–Emden equation, 67, 91
Lane–Emden–Fowler equation, 485, 487, 541, 564
Laplace
– equation, 68
– operator, 66, 67, 101–103, 107, 121, 132, 547, 549
Laplace–Beltrami operator, 66, 92, 104
large
– η-behavior, 125, 137, 148, 150
– Rayleigh number, 144
– solution, 487, 490, 495, 504, 506, 507, 515, 517, 519, 521, 536
– suction parameter, 165
large-x behavior, 123
LaSalle invariance, 198
lateral suction, 165
leading edge, 144
lemma of choice, 209
Leray–Fujita's inequality, 312
Leray
– inequality, 312
– problem, 301
l'Hôpital's rule, 151, 169
Lie-group, 122
– method, 178
linear solution, 137
linearization, 196
lines of flow, 120
Liouville
– property, 99, 107
– theorem, 63, 67, 81, 82, 440
Liouville-type
– properties, 104
– result, 85
– theorem, 61, 63, 67, 71, 75, 76, 79, 80, 89, 90, 93, 108
local
– maximum, 132, 135
– minimizer, 40, 52
– non-similarity
– – method (LNSM), 178
– – technique, 178, 187
– Nusselt number, 161, 189
– Peclet number, 180, 189
– similarity method, 176, 178
– solution, 128, 130, 133, 148, 150, 160
– symmetry, 5, 47
logistic equation, 487, 503
longitudinal diffusivity, 144
Lyapunov function, 132

M
m-capacity, 107
m-Laplace operator, 98, 107
m-Laplace–Beltrami operator, 107
m-parabolic, 107, 108
magnetic
– field, 157, 158, 167
– parameter, 159, 161, 168, 187
magnetohydrodynamic (MHD), 157, 187
mass
– balance, 146
– transfer parameter, 126, 157
mass-conservation equation, 120
maximal
– interval, 174
– – of existence, 128
– monotone, 215
– solution, 168, 172–174, 561, 563
– – positive, 510
maximum principle, 66, 100, 102, 362, 363, 371, 372, 394, 395, 397, 410, 411, 413, 417, 418, 486, 492, 493, 504–507, 510, 515, 521, 526, 535, 536, 546, 561, 564
– for Riemannian manifolds, 478
– for thin sets, 397
mean curvature, 75, 110, 112
– Dirichlet problem, 400
– operator, 435, 437, 443, 445, 471
Merk–Chao series, 184
method
– of Clarkson and Krustal, 178
– of Goldstein, 178
– of Mangler, 178
MHD
– flow, 157
– problem, 157
micropolar fluid, 181
min–max lemma, 206
minimal
– graph, 63, 79, 81, 109, 112
– – equation, 79
– hypercone, 79
– solution, 563, 564
– solution
– – large, 506
– – positive, 510
– speed, 166, 168
– submanifold, 79
– surface, 75, 81
– – equation, 75, 77, 79, 80, 108, 109
– – system, 80
– – operator, 84, 98, 108
missing solution, 164, 165
mixed convection, 176, 177, 180, 185

– flow
– – from a wedge, 178
– – over a vertical surface in porous media, 178
– on a wedge, 176, 179
– parameter, 176, 180, 181, 185
modified
– Grashof number, 181
– local Rayleigh, 159
– – number, 159
– local similarity technique, 183
– of the first level of truncation, 183
– thermal conductivity, 144, 161
mollification, 392
momentum, 144
Monge–Ampère equation, 81, 83
– in $\mathbb{R}^2$, 367
monotone, 215
monotonic
– decreasing, 139, 167
– increasing, 130, 132, 134, 135, 149
Moreau–Yosida approximation, 268
Moser iteration method, 104
moving surface, 152
multiple
– eigensolutions, 188, 192
– solution, 126, 132, 134, 143, 165, 196
– – ordering, 172
– – physical, 136, 143
– – unbounded, 126, 127, 136
multiplicity (similarity) solution, 179

N
Navier–Stokes, 118
– equation, 132
negative exponent, 154
Neumann boundary operator, 507
Newtonian, 119, 137
– case, 121, 126, 176, 178
– cooling, 507
– fluid, 118, 121, 157, 164
– law, 118
Nicholson–Strang theorem, 378
non-dimensionless suction, 165
non-existence, 126, 131, 144, 157, 160, 188
non-global solution, 142, 143
non-homogeneous elliptic inequalities, 409
non-linearity, 132
non-Newtonian, 118, 119, 137
– case, 177
– fluid, 127, 488
non-similarity
– solution, 176, 177, 180
– technique, 178
– transformation, 176
– variable, 176
non-trivial solution, 126
non-uniqueness, 127, 188
nondecreasing rearrangement, 35
nonlinear
– equation
– – elliptic, 486
– – singular elliptic, 488
normalization, 122
numerical
– or theoretical exact solutions, 176
– results, 188
– solution, 126, 157
– – similarity, 127, 178
Nusselt number, 161, 162

O
opposing flow, 176, 179, 187
optimal
– regularity, 121
– system, 178
oscillatory
– solution, 154
– TW, 196
Ostwald–de Waele
– model, 120
– power-law model, 119
outer
– flow, 146
– temperature, 144
outflow condition, 301, 336
outlet, 301
overdetermined boundary value problems, 451

P
p-Laplace
– inequality, 430
– operator, 377, 400, 403, 413, 418, 443, 471
p-Laplacian, 3, 42
p-regular solutions, 391
paint, 118
parabolic, 66, 92, 107, 112
– coordinates, 178
permeability, 145, 179
permeable surface, 126
phase plane, 168, 169
physical
– meaning, 126, 128, 177
– problem, 164
– solution, 134–136
plane, 177
Poincaré's inequality, 420

point-mechanical analogy, 165, 196
Poiseuille flow, 309
Poisson equation, 67, 68, 72
polymer melt, 118
porous
– media, 144
– medium, 144, 176–179
– – equations, 146
positive solution, 148, 155
power
– function, 158
– law, 182, 183, 186
– – case, 183
– – exponent, 144, 176
– – fluid, 144, 152
– – index, 119
– – viscosity, 119, 120
Prandtl
– equations, 118
– number, 145, 152
– velocity profile, 121
prescribed
– external velocity, 124
prescribed, mean curvature359
pressure, 145
principle of unique continuation, 37
problem
– with mass constraint, 282
– with non-local constraint, 294
profile
– θ, 165
– f, 122
– function, 125
property
– of compact support, 541
– (R), 256
pseudo-monotone, 220
pseudo-plastic fluid, 119
pseudosimilarity, 118
– reduction, 179, 181
– solution, 117, 157, 162, 165, 168, 177, 178
– variable, 177, 180, 181, 184
pure
– convection
– – forced, 180
– – free, 180

Q
quasi-variational inequality, 286
quasilinear
– elliptic inequalities, 368, 371
– equation of second order, 366

R
radial
– large solution, 537
– symmetry, 49, 444
range, 206
rate of strain, 118
rearrangement, 7
reference
– density, 145
– temperature, 145
regular
– set, 368, 401
– variation theory, 515
regularity theory, 580
regularized approach, 172
resolvent, 257
Reynolds number, 118, 177
Riemann surface, 63, 73
Robin boundary operator, 507

S
saturated porous media, 144
scales, 124
scaling, 146
– relation, 125
– transformation, 122
Schrödinger equation, 88
Schwarz symmetrization, 19
second level
– of truncation, 187
– – local non-similarity technique, 178
secondary bifurcation, 132
self-adjoint operator, 504
self-similarity solution, 146
self-symmetric outlet, 303
semi-infinite
– channel, 301, 308
– flat plate, 119
– – vertical, 120
semi-maximum principle, 417
semi-monotone, 219
semilinear elliptic systems, 527
separatrix cycle, 197
shape function, 122
shear stress, 118
shear-thickening, 119
shooting
– argument, 126, 144
– method, 128, 136, 148, 187, 189
– parameter, 128, 155
similarity, 118, 124, 146, 178
– reduction, 122, 125, 144, 179
– solution, 117, 121–123, 127, 145, 157, 161, 176–179, 184

– transformations, 158
– variable, 122, 177
singular, 121, 127
– elliptic inequalities, 401
– Lane–Emden–Fowler equation, 489
– non-linear boundary value problem, 153
– nonlinearities, 485
– set, 368, 401
– solution, 163
singularity, 486, 582
slow orbit, 168–170
slowly varying function, 515
Sobolev's inequality, 419
stationary Navier–Stokes equations, 301
steady
– flow, 119, 176
– mixed convection, 179
– non-Newtonian, 146
Steiner symmetrization, 19
straight channel, 301
stream, 144
– free velocity, 179
– function, 120, 121, 123, 125, 127, 145, 158, 165, 176, 177, 180
streamline, 118, 120, 184
stretching
– permeable, 157
– – surfaces, 126
– velocity, 125, 157
– wall, 157
stringent outflow condition, 302
strong
– Liouville property, 64
– maximum principle, 48, 66, 95, 96, 363, 380, 422, 423, 435, 436
strongly degenerate operators, 405
structure of the set of solutions, 160
structured elliptic inequalities, 416
sub-harmonic function, 65
subdifferential, 217
subgradient, 217
sublinear singular elliptic problems, 552
subsonic gas dynamics, 360
suction, 126, 127, 157, 165
suction/injection
– parameter, 126, 187
– velocity, 157
sufficient condition, 132
sufficiently large suction parameter, 167
super-harmonic function, 65, 66, 549
super-m-harmonic function, 107
surface
– velocity, 132, 181
– – parameter, 186
sweeping principle, 375
symmetric
– decreasing rearrangement, 8
– vector filed, 307
system of elliptic PDE, 43

T
T-dependent density, 145
tangency
– principle, 364, 368, 369, 381
– theorems, 379
temperature distribution, 144
thermal
– conductivity, 181
– diffusivity, 145, 176
– expansion coefficient, 145
torsion, 452
transport property, 144
transversal of the phase-flow, 198
traveling
– wave, 121, 166
– – solution, 165
two-dimensional flow, 176
two-dimensional stationary heat convection, 144
two-point rearrangement, 15
type M, 220

U
unbounded, 129
– solution, 162
– trajectories, 163
uniform
– magnetic, 157
– – field, 157
– power law flow, 144
uniformly elliptic operator, 67, 99, 101, 102
unique, 132
– solution, 132, 155, 165, 168
– – global bounded, 162
– – global unbounded, 128
– – positive, 155
– – similarity, 175
uniqueness, 125–127, 144, 148, 153, 154
– of solutions, 160
– of the Dirichlet problem, 366, 382
– of the singular Dirichlet problem, 415
– results, 153
unsteady
– flow, 119
– mixed convection, 185
upper semicontinuous, 206

V
variational problem, 4, 38, 43, 52

velocity, 146
– component, 120, 144, 165, 179
– parallel, 122
– ratio parameter, 144, 152
vertical
– continuously moving plate, 144
– flat plate, 144
viscosity, 145
viscous fluids, 488, 541

W
w-limit, 198
wall temperature distribution, 158
weak
– maximum principle, 387, 543
– solution, 302, 309
wedge, 176, 177

Y
Yosida-approximation, 257